Computing the Limit Points of the Quasi-component of a Regular Chain in Dimension One

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Abstract. For a regular chain R in dimension one, we propose an algorithm which computes the (non-trivial) limit points of the quasi-component of R, that is, the set $\overline{W(R)} \setminus W(R)$. Our procedure relies on Puiseux series expansions and does not require to compute a system of generators of the saturated ideal of R. We provide experimental results illustrating the benefits of our algorithms.

1 Introduction

The theory of regular chains, since its introduction by J.F. Ritt [22], has been applied successfully in many areas including differential systems [8, 2, 13], difference systems [12], unmixed decompositions [14] and primary decomposition [23] of polynomial ideals, intersection multiplicity calculations [17], cylindrical algebraic decomposition [7], parametric [28] and non-parametric [4] semi-algebraic systems. Today, regular chains are at the core of algorithms computing triangular decomposition of polynomial systems and which are available in several software packages [16, 26, 27]. Moreover, those algorithms provide back-engines for computer algebra system front-end solvers, such as Maple's `solve` command.

This paper deals with a notorious issue raised in all types of triangular decompositions, the *Ritt problem*, stated as follows. Given two regular chains (algebraic or differential) R and S, whose saturated ideals $\mathrm{sat}(R)$ and $\mathrm{sat}(S)$ are radical, check whether the inclusion $\mathrm{sat}(R) \subseteq \mathrm{sat}(S)$ holds or not. In the algebraic case, the challenge is to test such inclusion without computing a system of generators of $\mathrm{sat}(R)$. This question would be answered if one would have a procedure with the following specification: for the regular chain R compute regular chains $R_1, \ldots, R_e$ such that $\overline{W(R)} = W(R_1) \cup \cdots \cup W(R_e)$ holds, where $W(R)$ is the quasi-component of R and $\overline{W(R)}$ is the Zariski closure of $W(R)$.

We propose a solution to this algorithmic quest, in the algebraic case. To be precise, our procedure computes the *non-trivial limit points* of the quasi-component $W(R)$, that is, the set $\lim(W(R)) := \overline{W(R)} \setminus W(R)$ as a finite union of quasi-components of some other regular chains, see Theorem 7 in Section 7. We focus on the case where $\mathrm{sat}(R)$ has dimension one.

When the regular chain R consists of a single polynomial r, primitive w.r.t. its main variable, one can easily check that $\lim(W(R)) = V(r, h_r)$ holds, where

h_r is the initial of r. Unfortunately, there is no generalization of this result when R consists of several polynomials, unless R enjoys remarkable properties, such as being a *primitive regular chain* [15]. To overcome this difficulty, it becomes necessary to view R as a "parametric representation" of the quasi-component $W(R)$. In this setting, the points of $\lim(W(R))$ can be computed as limits (in the usual sense of the Euclidean topology [1]) of sequences of points along "branches" (in the sense of the theory of algebraic curves) of $W(R)$. It turns out that these limits can be obtained as constant terms of convergent Puiseux series defining the "branches" of $W(R)$ in the neighborhood of the points of interest.

Here comes the main technical difficulty of this approach. When computing a particular point of $\lim(W(R))$, one needs to follow one branch per defining equation of R. Following a branch means computing a truncated Puiseux expansion about a point. Since the equation of R defining a given variable, say X_j, depends on the equations of R defining the variables $X_{j-1}, X_{j-2}, \ldots$, the truncated Puiseux expansion for X_j is defined by an equation whose coefficients involve the truncated Puiseux expansions for $X_{j-1}, X_{j-2}, \ldots$.

From Sections 3 to 7, we show that this principle indeed computes the desired limit points. In particular, we introduce the notion of a *system of Puiseux parametrizations of a regular chain*, see Section 3. This allows us to state in Theorem 3 a concise formula for $\lim(W(R))$ in terms of this latter notion. Then, we estimate to which accuracy one needs to effectively compute such Puiseux parametrizations in order to deduce $\lim(W(R))$, see Theorem 6 in Section 6.

In Section 8, we report on a preliminary implementation of the algorithms presented in this paper. We evaluate our code by applying it to the question of removing redundant components in Kalkbrener's decompositions and observe the benefits of this strategy. Section 2 briefly reviews notions from the theories of regular chains and algebraic curves. We conclude this section with an example.

Consider the regular chain $R = \{r_1, r_2\} \subset \mathbf{k}[X_1, X_2, X_3]$ with $r_1 = X_1X_2^2 + X_2 + 1, r_2 = (X_1+2)X_1X_3^2 + (X_2+1)(X_3+1)$. We have $W(R) = V(R) \setminus V(h_R)$ with $h_R = X_1^2(X_1 + 2)$. To determine $\lim(W(R))$, we compute Puiseux series expansions of r_1 at $X_1 = 0$ and $X_1 = -2$. For such calculation, we use MAPLE's command algcurves[puiseux] [24]. The Puiseux expansions of r_1 at $X_1 = 0$ are:

$$[X_1 = T, X_2 = -1 - T + O(T^2)], [X_1 = T, X_2 = -1/T + 1 + T + O(T^2)].$$

Clearly, the second expansion does not yield a limit point. After substituting the first expansion into r_2, we have:

$$r_2' = r_2(X_1 = T, X_2 = -1 - T + O(T^2), X_3) = (T+2)TX_3^2 + (-T + O(T^2))(X_3+1).$$

Now, we compute Puiseux series expansions of r_2' which are

$$[T = T, X_3 = 1 - 1/3\, T + O(T^2)], [T = T, X_3 = -1/2 + 1/12\, T + O(T^2)].$$

So the regular chains $\{X_1, X_2 + 1, X_3 - 1\}$ and $\{X_1, X_2 + 1, X_3 + 1/2\}$ give the limit points of $W(R)$ at $X_1 = 0$. Similarly, $\{X_1 + 2, X_2 - 1, X_3 + 1\}$ and $\{X_1 + 2, X_2 + 1/2, X_3 + 1\}$ give the limit points of $W(R)$ at $X_1 = -2$.

[1] The closures of $W(R)$ in Zariski and the Euclidean topologies are equal when $\mathbf{k} = \mathbb{C}$.

2 Preliminaries

This section is a review of various notions from the theories of regular chains, algebraic curves and topology. For these latter subjects, our references are the textbooks of R.J. Walker [25], G. Fischer [11] and J. R. Munkres [20]. Notations and hypotheses introduced in this section are used throughout the paper.

Multivariate polynomials. Let $\mathbf{k}$ be a field which is algebraically closed. Let $X_1 < \cdots < X_s$ be $s \geq 1$ ordered variables. We denote by $\mathbf{k}[X_1, \ldots, X_s]$ the ring of polynomials in the variables $X_1, \ldots, X_s$ and with coefficients in $\mathbf{k}$. For a non-constant polynomial $p \in \mathbf{k}[X_1, \ldots, X_s]$, the greatest variable in p is called *main variable* of p, denoted by $\mathrm{mvar}(p)$, and the leading coefficient of p w.r.t. $\mathrm{mvar}(p)$ is called *initial* of p, denoted by $\mathrm{init}(p)$.

Zariski topology. We denote by $\mathbb{A}^s$ the *affine s-space* over $\mathbf{k}$. An *affine variety* of $\mathbb{A}^s$ is the set of common zeroes of a collection $F \subseteq \mathbf{k}[X_1, \ldots, X_s]$ of polynomials. The *Zariski topology* on $\mathbb{A}^s$ is the topology whose closed sets are the affine varieties of $\mathbb{A}^s$. The *Zariski closure* of a subset $W \subseteq \mathbb{A}^s$ is the intersection of all affine varieties containing W.

Relation between Zariski topology and the Euclidean topology. When $\mathbf{k} = \mathbb{C}$, the affine space $\mathbb{A}^s$ is endowed with both Zariski topology and the Euclidean topology. While Zariski topology is coarser than the Euclidean topology, we have the following (Corollary 1 in I.10 of [19]) key result. Let $V \subseteq \mathbb{A}^s$ be an irreducible affine variety and $U \subseteq V$ be open in the Zariski topology induced on V. Then, the closure of U in Zariski topology and the closure of U in the Euclidean topology are both equal to V.

Limit points. Let (X, τ) be a topological space. Let $S \subseteq X$ be a subset. A point $p \in X$ is a *limit point* of S if every neighborhood of p contains at least one point of S different from p itself. If the space X is a metric space, the point p is a limit point of S if and only if there exists a sequence $(x_n, n \in \mathbb{N})$ of points of $S \setminus \{p\}$ with p as limit. In practice, the "interesting" limit points of S are those which do not belong to S. For this reason, we call such limit points *non-trivial* and we denote by $\lim(S)$ the set of non-trivial limit points of S.

Regular chain. A set R of non-constant polynomials in $\mathbf{k}[X_1, \ldots, X_s]$ is called a *triangular set*, if for all $p, q \in R$ with $p \neq q$ we have $\mathrm{mvar}(p) \neq \mathrm{mvar}(q)$. A variable X_i is said *free* w.r.t. R if there exists no $p \in R$ such that $\mathrm{mvar}(p) = X_i$. For a nonempty triangular set R, we define the *saturated ideal* $\mathrm{sat}(R)$ of R to be the ideal $\langle R \rangle : h_R^\infty$, where h_R is the product of the initials of the polynomials in R. The saturated ideal of the empty triangular set is defined as the trivial ideal $\langle 0 \rangle$. The ideal $\mathrm{sat}(R)$ has several properties, in particular it is unmixed [3]. We denote its height, that is the number of polynomials in R, by e, thus $\mathrm{sat}(R)$ has dimension $s - e$. Let $X_{i_1} < \cdots < X_{i_e}$ be the main variables of the polynomials in R. We denote by r_j the polynomial of R whose main variable is X_{i_j} and by h_j the initial of r_j. We say that R is a *regular chain* whenever R is empty or $\{r_1, \ldots, r_{e-1}\}$ is a regular chain and h_e is regular modulo the saturated ideal $\mathrm{sat}(\{r_1, \ldots, r_{e-1}\})$. The regular chain R is said *strongly normalized* whenever each of the main variables of the polynomials of R (that is, $X_{i_1} < \cdots < X_{i_e}$) does not appear in h_R.

Limit points of the quasi-component of a regular chain. We denote by $W(R) := V(R) \setminus V(h_R)$ the *quasi-component* of R, that is, the common zeros of R that do not cancel h_R. The above discussion implies that the closure of $W(R)$ in Zariski topology and the closure of $W(R)$ in the Euclidean topology are both equal to $V(\mathrm{sat}(R))$, that is, the affine variety of $\mathrm{sat}(R)$. We denote by $\overline{W(R)}$ this common closure and $\lim(W(R))$ this common set of limit points.

Rings of formal power series. Recall that $\mathbf{k}$ is an algebraically closed field. We denote by $\mathbf{k}[[X_1, \ldots, X_s]]$ and $\mathbf{k}\langle X_1, \ldots, X_s \rangle$ the rings of formal and convergent power series in $X_1, \ldots, X_s$ with coefficients in $\mathbf{k}$. When $s = 1$, we write T instead of X_1. For $f \in \mathbf{k}[[X_1, \ldots, X_s]]$, its *order* is defined by $\min\{d \mid f_{(d)} \neq 0\}$ if $f \neq 0$ and by ∞ otherwise, where $f_{(d)}$ is the *homogeneous part* of f in degree d. We denote by $\mathcal{M}_s$ the only maximal ideal of $\mathbf{k}[[X_1, \ldots, X_s]]$, that is, $\mathcal{M}_s = \{f \in \mathbf{k}[[X_1, \ldots, X_s]] \mid \mathrm{ord}(f) \geq 1\}$. Let $f \in \mathbf{k}[[X_1, \ldots, X_s]]$ with $f \neq 0$. Let $k \in \mathbb{N}$. We say that f is (1) *general* in X_s if $f \not\equiv 0 \mod \mathcal{M}_{s-1}$, (2) *general* in X_s of order k if we have $\mathrm{ord}(f \mod \mathcal{M}_{s-1}) = k$.

Formal Puiseux series. We denote by $\mathbf{k}[[T^*]] = \bigcup_{n=1}^{\infty} \mathbf{k}[[T^{\frac{1}{n}}]]$ the ring of *formal Puiseux series*. For a fixed $\varphi \in \mathbf{k}[[T^*]]$, there is an $n \in \mathbb{N}_{>0}$ such that $\varphi \in \mathbf{k}[[T^{\frac{1}{n}}]]$. Hence $\varphi = \sum_{m=0}^{\infty} a_m T^{\frac{m}{n}}$, where $a_m \in \mathbf{k}$. We call *order of* φ the rational number defined by $\mathrm{ord}(\varphi) = \min\{\frac{m}{n} \mid a_m \neq 0\} \geq 0$. We denote by $\mathbf{k}((T^*))$ the quotient field of $\mathbf{k}[[T^*]]$.

Convergent Puiseux series. Let $\varphi \in \mathbb{C}[[T^*]]$ and $n \in \mathbb{N}$ such that $\varphi = f(T^{\frac{1}{n}})$ holds for some $f \in \mathbb{C}[[T]]$. We say that the Puiseux series φ is *convergent* if we have $f \in \mathbb{C}\langle T \rangle$. Convergent Puiseux series form an integral domain denoted by $\mathbb{C}\langle T^* \rangle$; its quotient field is denoted by $\mathbb{C}(\langle T^* \rangle)$. For every $\varphi \in \mathbb{C}((T^*))$, there exist $n \in \mathbb{Z}$, $r \in \mathbb{N}_{>0}$ and a sequence of complex numbers $a_n, a_{n+1}, a_{n+2}, \ldots$ such that we have $\varphi = \sum_{m=n}^{\infty} a_m T^{\frac{m}{r}}$ and $a_n \neq 0$. Then, we define $\mathrm{ord}(\varphi) = \frac{n}{r}$.

Puiseux Theorem. If $\mathbf{k}$ has characteristic zero, the field $\mathbf{k}((T^*))$ is the algebraic closure of the field of formal Laurent series over $\mathbf{k}$. Moreover, if $\mathbf{k} = \mathbb{C}$, the field $\mathbb{C}(\langle T^* \rangle)$ is algebraically closed as well. From now on, we assume $\mathbf{k} = \mathbb{C}$.

Puiseux expansion. Let $\mathbb{B} = \mathbb{C}((X^*))$ or $\mathbb{C}(\langle X^* \rangle)$. Let $f \in \mathbb{B}[Y]$, where $d := \deg(f, Y) > 0$. Let $h := \mathrm{lc}(f, Y)$. According to Puiseux Theorem, there exists $\varphi_i \in \mathbb{B}$, $i = 1, \ldots, d$, such that $\frac{f}{h} = (Y - \varphi_1) \cdots (Y - \varphi_d)$. We call $\varphi_1, \ldots, \varphi_d$ the *Puiseux expansions* of f at the origin.

Puiseux parametrization. Let $f \in \mathbb{C}\langle X \rangle[Y]$. A *Puiseux parametrization* of f is a pair $(\psi(T), \varphi(T))$ of elements of $\mathbb{C}\langle T \rangle$ for some new variable T, such that (1) $\psi(T) = T^{\varsigma}$, for some $\varsigma \in \mathbb{N}_{>0}$; (2) $f(X = \psi(T), Y = \varphi(T)) = 0$ holds in $\mathbb{C}\langle T \rangle$, and (3) there is no integer $k > 1$ such that both $\psi(T)$ and $\varphi(T)$ are in $\mathbb{C}\langle T^k \rangle$. The index ς is called the *ramification index* of the parametrization $(T^{\varsigma}, \varphi(T))$. It is intrinsic to f and $\varsigma \leq \deg(f, Y)$. Let $z_1, \ldots, z_{\varsigma}$ denote the distinct roots of unity of order ς in $\mathbb{C}$. Then $\varphi(z_i X^{1/\varsigma})$, for $i = 1, \ldots, \varsigma$, are ς Puiseux expansions of f. For a Puiseux expansion φ of f, let c minimum such that both $\varphi = g(T^{1/c})$ and $g \in \mathbb{C}\langle T \rangle$ holds. Then $(T^c, g(T))$ is a Puiseux parametrization of f.

3 Puiseux expansions of a regular chain

In this section, we introduce the notion of Puiseux expansions of a regular chain, motivated by the work of [18, 1] on Puiseux expansions of space curves. Throughout this section, let $R = \{r_1, \ldots, r_{s-1}\} \subset \mathbb{C}[X_1 < \cdots < X_s]$ be a strongly normalized regular chain whose saturated ideal has dimension one and assume that X_1 is free w.r.t. R.

Lemma 1 *Let $h_R(X_1)$ be the product of the initials of the polynomials in R. Let $\rho > 0$ be small enough such that the set $U_\rho := \{x = (x_1, \ldots, x_s) \in \mathbb{C}^s \mid 0 < |x_1| < \rho\}$ does not contain any zeros of h_R. Define $V_\rho(R) := V(R) \cap U_\rho$. Then, we have $W(R) \cap U_\rho = V_\rho(R)$.*

Proof. It follows from the observation that $U_\rho \cap V(h_R) = \emptyset$.

Notation 1 *Let $W \subseteq \mathbb{C}^s$. Denote* $\lim_0(W) := \{x = (x_1, \ldots, x_s) \in \mathbb{C}^s \mid x \in \lim(W) \text{ and } x_1 = 0\}$.

Lemma 2 *We have* $\lim_0(W(R)) = \lim_0(V_\rho(R))$.

Proof. By Lemma 1, we have $W(R) \cap U_\rho = V_\rho(R)$. Meanwhile, $\lim_0(W(R)) = \lim_0(W(R) \cap U_\rho)$ holds. Thus $\lim_0(W(R)) = \lim_0(V_\rho(R))$ holds.

Lemma 3 *For $1 \le i \le s-1$, let $d_i := \deg(r_i, X_{i+1})$. Then R generates a zero-dimensional ideal in $\mathbb{C}(\langle X_1^* \rangle)[X_2, \ldots, X_s]$. Let $V^*(R)$ be the zero set of R in $\mathbb{C}(\langle X_1^* \rangle)^{s-1}$. Then $V^*(R)$ has exactly $\prod_{i=1}^{s-1} d_i$ points, counting multiplicities.*

Proof. It follows directly from the definition of regular chain, and the fact that $\mathbb{C}(\langle X_1^* \rangle)$ is an algebraically closed field.

Definition 1 *We use the notations of Lemma 3. Each point in $V^*(R)$ is called a* Puiseux expansion *of R.*

Notation 2 *Let $m = |V^*(R)|$. Write $V^*(R) = \{\Phi_1, \ldots, \Phi_m\}$. For $i = 1, \ldots, m$, write $\Phi_i = (\Phi_i^1(X_1), \ldots, \Phi_i^{s-1}(X_1))$. Let $\rho > 0$ be small enough such that for $1 \le i \le m$, $1 \le j \le s-1$, each $\Phi_i^j(X_1)$ converges in $0 < |X_1| < \rho$. We define $V_\rho^*(R) := \cup_{i=1}^m \{x \in \mathbb{C}^s \mid 0 < |x_1| < \rho, x_{j+1} = \Phi_i^j(x_1), j = 1, \ldots, s-1\}$.*

Theorem 1 *We have $V_\rho^*(R) = V_\rho(R)$.*

Proof. We prove this by induction on s. For $i = 1, \ldots, s-1$, recall that h_i is the initial of r_i. If $s = 2$, we have $r_1(X_1, X_2) = h_1(X_1) \prod_{i=1}^{d_1} (X_2 - \Phi_i^1(X_1))$. So $V_\rho^*(R) = V_\rho(R)$ clearly holds.

Now we consider $s > 2$. Write $R' = \{r_1, \ldots, r_{s-2}\}$, $R = R' \cup \{r_{s-1}\}$, $X' = X_2, \ldots, X_{s-1}$, $X = (X_1, X', X_s)$, $x' = x_2, \ldots, x_{s-1}$, $x = (x_1, x', x_s)$, and $m' = |V^*(R')|$. For $i = 1, \ldots, m$, let $\Phi_i = (\Phi_i', \Phi_i^{s-1})$, where Φ_i' stands for $\Phi_i^1, \ldots, \Phi_i^{s-2}$.

Assume the theorem holds for R', that is $V^*_\rho(R') = V_\rho(R')$. For any $i = 1, \ldots, m'$, there exist $i_1, \ldots, i_{d_{s-1}} \in \{1, \ldots, m\}$ such that we have

$$r_{s-1}(X_1, X' = \Phi'_i, X_s) = h_{s-1}(X_1) \prod_{k=1}^{d_{s-1}} (X_s - \Phi^{s-1}_{i_k}(X_1)). \qquad (1)$$

Note that $V^*(R) = \cup_{i=1}^{m'} \cup_{k=1}^{d_{s-1}} \{(X' = \Phi'_i, X_s = \Phi^{s-1}_{i_k})\}$. Therefore, by induction hypothesis and Equation (1), we have

$$\begin{aligned} V^*_\rho(R) &= \cup_{i=1}^{m'} \cup_{k=1}^{d_{s-1}} \{x \mid x \in U_\rho, x' = \Phi'_i(x_1), x_s = \Phi^{s-1}_{i_k}(x_1)\} \\ &= \cup_{k=1}^{d_{s-1}} \{x \mid (x_1, x') \in V^*_\rho(R'), x_s = \Phi^{s-1}_{i_k}(x_1)\} \\ &= \{x \mid (x_1, x') \in V^*_\rho(R'), r_{s-1}(x_1, x', x_s) = 0\} \\ &= \{x \mid (x_1, x') \in V_\rho(R'), r_{s-1}(x_1, x', x_s) = 0\} \\ &= V_\rho(R). \end{aligned}$$

Theorem 2 *Let* $V^*_{\geq 0}(R) := \{\Phi = (\Phi^1, \ldots, \Phi^{s-1}) \in V^*(R) \mid \mathrm{ord}(\Phi^j) \geq 0, j = 1, \ldots, s-1\}$. *Then we have* $\lim_0(W(R)) = \cup_{\Phi \in V^*_{\geq 0}(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}$.

Proof. By definition of $V^*_{\geq 0}(R)$, we immediately have

$$\lim_0(V^*_\rho(R)) = \cup_{\Phi \in V^*_{\geq 0}(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

Next, by Theorem 1, we have $V^*_\rho(R) = V_\rho(R)$. Thus, we have $\lim_0(V^*_\rho(R)) = \lim_0(V_\rho(R))$. Besides, with Lemma 2, we have $\lim_0(W(R)) = \lim_0(V_\rho(R))$. Thus the theorem holds.

Definition 2 *Let* $V^*_{\geq 0}(R)$ *be as defined in Theorem 2. Let* $M = |V^*_{\geq 0}(R)|$. *For each* $\Phi_i = (\Phi^1_i, \ldots, \Phi^{s-1}_i) \in V^*_{\geq 0}(R)$, $1 \leq i \leq M$, *we know that* $\Phi^j_i \in \mathbb{C}(\langle X_1^* \rangle)$. *Moreover, by Equation (1), we know that for* $j = 1, \ldots, s-1$, Φ^j_i *is a Puiseux expansion of* $r_j(X_1, X_2 = \Phi^1_i, \ldots, X_j = \Phi^{j-1}_i, X_{j+1})$. *Let* $\varsigma_{i,j}$ *be the ramification index of* Φ^j_i *and* $(T^{\varsigma_{i,j}}, X_{j+1} = \varphi^j_i(T))$, *where* $\varphi^j_i \in \mathbb{C}\langle T \rangle$, *be the corresponding Puiseux parametrization of* Φ^j_i. *Let* ς_i *be the least common multiple of* $\{\varsigma_{i,1}, \ldots, \varsigma_{i,s-1}\}$. *Let* $g^j_i = \varphi^j_i(T = T^{\varsigma_i/\varsigma_{i,j}})$. *We call the set* $\mathfrak{G}_R := \{(X_1 = T^{\varsigma_i}, X_2 = g^1_i(T), \ldots, X_s = g^{s-1}_i(T)), i = 1, \ldots, M\}$ *a* system of Puiseux parametrizations *of* R.

Theorem 3 *We have* $\lim_0(W(R)) = \mathfrak{G}_R(T = 0)$.

Proof. It follows directly from Theorem 2 and Definition 2.

Remark 1 *The limit points of* $W(R)$ *at* $X_1 = \alpha \neq 0$ *can be reduced to the computation of* $\lim_0(W(R))$ *by a coordinate transformation* $X_1 = X_1 + \alpha$. *Given an arbitrary one-dimensional regular chain* R, *the set* $\lim(W(R))$ *can be computed in the following manner. Compute a regular chain* N *which is strongly normalized and such that* $\mathrm{sat}(R) = \mathrm{sat}(N)$ *and* $V(h_N) = V(\widehat{h_R})$ *both hold, where* $\widehat{h_R}$ *is the iterated resultant of* h_R *w.r.t* R. *See [6]. Let* $X_i := \mathrm{mvar}(h_R)$. *Note that* N *is still a regular chain w.r.t. the new order* $X_i < \{X_1, \ldots, X_n\} \setminus \{X_i\}$. *Observe that* $\lim(W(R)) \subseteq \lim(W(N))$ *holds. Thus we have* $\lim(W(R)) = \lim(W(N)) \setminus W(R)$.

4 Puiseux parametrization in finite accuracy

In this section, we define the Puiseux parametrizations of a polynomial $f \in \mathbb{C}\langle X\rangle[Y]$ in finite accuracy, see Definition 4. For $f \in \mathbb{C}\langle X\rangle[Y]$, we also define the approximation of f for a given accuracy, see Definition 3. This approximation of f is a polynomial in $\mathbb{C}[X, Y]$. In Sections 5 and 6, we prove that to compute a Puiseux parametrization of f of a given accuracy, it suffices to compute a Puiseux parametrization of an approximation of f of some finite accuracy.

In this section, we review and adapt the classical Newton-Puiseux algorithm to compute Puiseux parametrizations of a polynomial $f \in \mathbb{C}[X, Y]$ of a given accuracy. Since we do not need to compute the singular part of Puiseux parametrizations, the usual requirement $\mathrm{discrim}(f, Y) \neq 0$ is dropped.

Definition 3 *Let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathbb{C}[[X]]$. For any $\tau \in \mathbb{N}$, we call $f^{(\tau)} := \sum_{i=0}^{\tau} a_i X^i$ the* polynomial part *of f* of accuracy *$\tau+1$. Let $f = \sum_{i=0}^{d} a_i(X) Y^i \in \mathbb{C}\langle X\rangle[Y]$. For any $\tau \in \mathbb{N}$, we call $\widetilde{f}^{(\tau)} := \sum_{i=0}^{d} a_i^{(\tau)} Y^i$ the* approximation of f of accuracy *$\tau + 1$.*

Definition 4 *Let $f \in \mathbb{C}\langle X\rangle[Y]$, with $\deg(f, Y) > 0$. Let $\sigma, \tau \in \mathbb{N}_{>0}$ and $g(T) = \sum_{k=0}^{\tau-1} b_k T^k$. Let $\{T^{k_1}, \ldots, T^{k_m}\}$ be the support of $g(T)$. The pair $(T^\sigma, g(T))$ is called* a Puiseux parametrization of f of accuracy τ *if there exists a Puiseux parametrization $(T^\varsigma, \varphi(T))$ of f such that: (i) σ divides ς; (ii) $\gcd(\sigma, k_1, \ldots, k_m) = 1$; and (iii) $g(T^{\varsigma/\sigma})$ is the polynomial part of $\varphi(T)$ of accuracy $(\varsigma/\sigma)(\tau - 1) + 1$. Note that if $\sigma = \varsigma$, then $g(T)$ is the polynomial part of $\varphi(T)$ of accuracy τ.*

Definition 5 ([10]) *A $\mathbb{C}$-term[2] is defined as a triple $t = (q, p, \beta)$, where q and p are coprime integers, $q > 0$ and $\beta \in \mathbb{C}$ is non-zero. A $\mathbb{C}$-expansion is a sequence $\pi = (t_1, t_2, \ldots)$ of $\mathbb{C}$-terms, where $t_i = (q_i, p_i, \beta_i)$. We say that π is* finite *if there are only finitely many elements in π.*

Definition 6 *Let $\pi = (t_1, \ldots, t_N)$ be a finite $\mathbb{C}$-expansion. We define a pair $(T^\sigma, g(T))$ of polynomials in $\mathbb{C}[T]$ in the following manner: (i) if $N = 1$, set $\sigma = 1$, $g(T) = 0$ and $\delta_N = 0$; (ii) otherwise, let $a := \prod_{i=1}^{N} q_i$, $c_i := \sum_{j=1}^{i} \left(p_j \prod_{k=j+1}^{N} q_k\right)$ $(1 \leq i \leq N)$, $\delta_i := c_i / \gcd(a, c_1, \ldots, c_N)$ $(1 \leq i \leq N)$. Set $\sigma := a/\gcd(a, c_1, \ldots, c_N)$ and $g(T) := \sum_{i=1}^{N} \beta_i T^{\delta_i}$. We call the pair $(T^\sigma, g(T))$* the Puiseux parametrization of π of accuracy $\delta_N + 1$. *Denote by* ConstructParametrization *an algorithm to compute $(T^\sigma, g(T))$ from π.*

Definition 7 *Let $f \in \mathbb{C}\langle X\rangle[Y]$ and write f as $f(X, Y) := \sum_{i=0}^{d} \left(\sum_{j=0}^{\infty} a_{i,j} X^j\right) Y^i$. The* Newton Polygon *of f is defined as the lower part of the convex hull of the set of points (i, j) in the plane such that $a_{i,j} \neq 0$.*

Let $f \in \mathbb{C}\langle X\rangle[Y]$. We denote by NewtonPolygon(f, I) an algorithm to compute the segments in the Newton Polygon of f, where I is a flag controlling

[2] It is a simplified version of Duval's definition.

the algorithm specification as follows. If $I = 1$, only segments with non-positive slopes are computed. If $I = 2$, only segments with negative slopes are computed. Such an algorithm can be found in [25]. Next we introduce some notations which are necessary to present Algorithm 2.

Let $f \in \mathbb{C}[X,Y]$, $t = (q,p,\beta)$ be a $\mathbb{C}$-term and $\ell \in \mathbb{N}$ s.t. $\mathsf{NewPoly}(f,t,\ell) := X^{-\ell} f(X^q, X^p(\beta + Y)) \in \mathbb{C}[X,Y]$. Let $f = \sum_{i=0}^{d} \sum_{j=0}^{m} a_{i,j} X^j Y^i \in \mathbb{C}[X,Y]$ and let Δ be a segment of the Newton Polygon of f. Denote $\mathsf{SegmentPoly}(f,\Delta) := (q,p,\ell,\phi)$ such that the following holds: (1) $q,p,\ell \in \mathbb{N}$; $\phi \in \mathbb{C}[Z]$; q and p are coprime, $q > 0$; (2) for any $(i,j) \in \Delta$, we have $qj + pi = \ell$; and (3) letting $i_0 := \min(\{i \mid (i,j) \in \Delta\})$, we have $\phi = \sum_{(i,j)\in\Delta} a_{i,j} Z^{(i-i_0)/q}$.

Theorem 4 *Algorithm 2 terminates and is correct.*

Proof. It directly follows from the proof of the Newton-Puiseux algorithm in Walker's book [25], the relation between $\mathbb{C}$-expansion and Puiseux parametrization discussed in Duval's paper [10], and Definitions 6 and 4.

Algorithm 1: $\mathsf{NonzeroTerm}(f, I)$

Input: $f \in \mathbb{C}[X,Y]$; $I = 1$ or 2.
Output: A finite set of pairs (t,ℓ), where t is a $\mathbb{C}$-term, and $\ell \in \mathbb{N}$.

$S := \emptyset$;
for *each* $\Delta \in \mathsf{NewtonPolygon}(f,I)$ **do**
  $(q,p,\ell,\phi) := \mathsf{SegmentPoly}(f,\Delta)$;
  for *each root* ξ *of* ϕ *in* $\mathbb{C}$ **do**
    for *each root* β *of* $U^q - \xi$ *in* $\mathbb{C}$ **do** $\{t := (q,p,\beta); S := S \cup \{(t,\ell)\}\}$
return S

Algorithm 2: $\mathsf{NewtonPuiseux}$

Input: $f \in \mathbb{C}[X,Y]$; a given accuracy $\tau > 0 \in \mathbb{N}$.
Output: All the Puiseux parametrizations of f of accuracy τ.

$\pi := (\ )$; $S := \{(\pi, f)\}$; $P := \emptyset$;
while $S \neq \emptyset$ **do**
  let $(\pi^*, f^*) \in S$; $S := S \setminus \{(\pi^*, f^*)\}$; **if** $\pi^* = (\ )$ **then** $I := 1$ **else** $I := 2$;
  $(T^\sigma, g(T)) := \mathsf{ConstructParametrization}(\pi^*)$;
  if $\deg(g(T),T) + 1 < \tau$ **then**
    $C := \mathsf{NonzeroTerm}(f^*, I)$;
    if $C = \emptyset$ **then**
      $P := P \cup \{(T^\sigma, g(T))\}$
    else
      for *each* $(t = (p,q,\beta),\ell) \in C$ **do**
        $\pi^{**} := \pi^* \cup (t)$; $f^{**} := \mathsf{NewPoly}(f^*, t, \ell)$; $S := S \cup \{(\pi^{**}, f^{**})\}$
  else
    $P := P \cup \{(T^\sigma, g(T))\}$
return P

5 Computing in finite accuracy

Let $f \in \mathbb{C}\langle X\rangle[Y]$. In this section, we consider the following problems: (a) Is it possible to use an approximation of f of some finite accuracy m in order to compute a Puiseux parametrization of f of a prescribed finite accuracy τ? (b) If yes, how to calculate m from f and τ? (c) Provide an upper bound on m. Theorem 5 provides the answers to (a) and (b) while Lemma 6 answers (c).

In the rest of this paper, the proof of a lemma is omitted if it is a routine.

Lemma 4 *Let $f \in \mathbb{C}\langle X\rangle[Y]$. Let $d := \deg(f, Y) > 0$. Let $q \in \mathbb{N}_{>0}$, $p, \ell \in \mathbb{N}$ and assume that q and p are coprime. Let $\beta \neq 0 \in \mathbb{C}$. Assume that q, p, ℓ define the segment $qj+pi = \ell$ of the Newton Polygon of f. Let $f_1 := X_1^{-\ell} f(X_1^q, X_1^p(\beta+Y_1))$.*

Then, we have the following results: (i) for any given $m_1 \in \mathbb{N}$, there exists a number $m \in \mathbb{N}$ such that the approximation of f_1 of accuracy m_1 can be computed from the approximation of f of accuracy m; (ii) moreover, it suffices to take $m = \lfloor \frac{m_1+\ell}{q} \rfloor$.

Theorem 5 *Let $f \in \mathbb{C}\langle X\rangle[Y]$. Let $\tau \in \mathbb{N}_{>0}$. Let $\sigma \in \mathbb{N}_{>0}$ and $g(T) = \sum_{k=0}^{\tau-1} b_k T^k$. Assume that $(T^\sigma, g(T))$ is a Puiseux parametrization of f of accuracy τ. Then one can compute a number $m \in \mathbb{N}$ such that $(T^\sigma, g(T))$ is a Puiseux parametrization of accuracy τ of $\widetilde{f}^{m-1}$, where $\widetilde{f}^{m-1}$ is the approximation of f of accuracy m. We denote by* AccuracyEstimate *an algorithm to compute m from f and τ.*

Proof. By Lemma 4 and the construction of the Newton-Puiseux algorithm, we conclude that there exists a number $m \in \mathbb{N}$ such that $(T^\sigma, g(T))$ is a Puiseux parametrization of accuracy τ of the approximation of f of accuracy m.

Next we show that there is an algorithm to compute m. We initially set $m' := \tau$. Let $f_0 := \sum_{i=0}^{d} \left(\sum_{j=0}^{m'} a_{i,j} X^j\right) Y^i$. That is, f_0 is the approximation of f of accuracy $m'+1$. We run the Newton-Puiseux algorithm to check whether the terms $a_{k,m'} X^{m'} Y^k$, $0 \le k \le d$, make any contributions in constructing the Newton Polygons of all f_i. If at least one of them make contributions, we increase the value of m' and restart the Newton-Puiseux algorithm until none of the terms $a_{k,m'} X^{m'} Y^k$, $0 \le k \le d$, makes any contributions in constructing the Newton Polygons of all f_i. We set $m := m'$.

Lemma 5 *Let $d, \tau \in \mathbb{N}_{>0}$. Let $a_{i,j}$, $0 \le i \le d$, $0 \le j < \tau$, and b_k, $0 \le k < \tau$ be symbols. Write $\mathbf{a} = (a_{0,0}, \ldots, a_{0,\tau-1}, \ldots, a_{d,0}, \ldots, a_{d,\tau-1})$ and $\mathbf{b} = (b_0, \ldots, b_{\tau-1})$. Let $f(\mathbf{a}, X, Y) = \sum_{i=0}^{d} \left(\sum_{j=0}^{\tau-1} a_{i,j} X^j\right) Y^i \in \mathbb{C}[\mathbf{a}][X, Y]$ and let $g(\mathbf{b}, X) = \sum_{k=0}^{\tau-1} b_k X^k \in \mathbb{C}[\mathbf{b}][X]$. Let $p := f(\mathbf{a}, X, Y = g(\mathbf{b}, X))$. Let $F_k :=$ $\mathrm{coeff}(p, X^k)$, $0 \le k < \tau - 1$, and $F := \{F_0, \ldots, F_{\tau-1}\}$. Then under the order $\mathbf{a} < \mathbf{b}$ and $b_0 < b_1 < \cdots < b_{\tau-1}$, F forms a zero-dimensional regular chain in $\mathbb{C}(\mathbf{a})[\mathbf{b}]$ with main variables $(b_0, b_1, \ldots, b_{\tau-1})$ and main degrees $(d, 1, \ldots, 1)$. In addition, we have (i) $F_0 = \sum_{i=0}^{d} a_{i,0} b_0^i$, and (ii) $\mathrm{init}(F_1) = \cdots = \mathrm{init}(F_{\tau-1}) =$ $\mathrm{der}(F_0, b_0) = \sum_{i=1}^{d} i \cdot a_{i,0} b_0^{i-1}$.*

Proof. Write $p = \sum_{i=0}^{d} \left(\sum_{j=0}^{\tau-1} a_{i,j} X^j\right) \left(\sum_{k=0}^{\tau-1} b_k X^k\right)^i$ as a univariate polynomial in X. Observe that $F_0 = \sum_{i=0}^{d} a_{i,0} b_0^i$. Therefore F_0 is irreducible in $\mathbb{C}(\mathbf{a})[\mathbf{b}]$. Moreover, we have $\mathrm{mvar}(F_0) = b_0$ and $\mathrm{mdeg}(F_0) = d$.

Since $d > 0$, we know that $a_{1,0} \left(\sum_{k=0}^{\tau-1} b_k X^k\right)$ appears in p. Thus, for $0 \le k < \tau$, b_k appears in F_k. Moreover, for any $k \ge 1$ and $i < k$, b_k can not appear in F_i since b_k and X^k are always raised to the same power. For the same reason, for any $i > 1$, b_k^i cannot appear in F_k, for $1 \le k < \tau$. Thus $\{F_0, \ldots, F_{\tau-1}\}$ is a triangular set with main variables $(b_0, b_1, \ldots, b_{\tau-1})$ and main degrees $(d, 1, \ldots, 1)$.

Moreover, we have $\mathrm{init}(F_1) = \cdots = \mathrm{init}(F_{\tau-1}) = \sum_{i=1}^{d} i \cdot a_{i,0} b_0^{i-1}$, which is coprime with F_0. Thus $F = \{F_0, \ldots, F_{\tau-1}\}$ is a regular chain.

As a direct corollary, we have the following lemma.

Lemma 6 *Let* $f = \sum_{i=0}^{d} \left(\sum_{j=0}^{\infty} a_{i,j} X^j\right) Y^i \in \mathbb{C}[[X]][Y]$. *Assume that* $d = \deg(f, Y) > 0$ *and* f *is general in* Y. *Let* $\varphi(X) = \sum_{k=0}^{\infty} b_k X^k \in \mathbb{C}[[X]]$ *such that* $f(X, \varphi(X)) = 0$ *holds. Let* $\tau > 0 \in \mathbb{N}$. *Then all coefficients* b_i, *for* $0 \le i < \tau$, *can be completely determined by* $\{a_{i,j} \mid 0 \le i \le d, 0 \le j < \tau\}$ *if and only if* b_0 *is a simple zero of* $f(0, Y)$. *Therefore, "generically", all coefficients* b_i, *for* $0 \le i < \tau$, *can be completely determined by the approximation of* f *of accuracy* τ.

6 Accuracy estimates

Let $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\} \subset \mathbb{C}[X_1 < \cdots < X_s]$ be a strongly normalized regular chain. In this section, we show that to compute the limit points of $W(R)$, it suffices to compute the Puiseux parametrizations of R of some accuracy. Moreover, we provide accuracy estimates in Theorem 6.

Lemma 7 *Let* $f = a_d(X)Y^d + \cdots + a_0(X) \in \mathbb{C}\langle X\rangle[Y]$, *where* $d = \deg(f, Y) > 0$. *For* $0 \le i \le d$, *let* $\delta_i := \mathrm{ord}(a_i)$. *Let* $k := \min(\delta_0, \ldots, \delta_d)$. *Let* $\widetilde{f} := X^{-k} f$. *Then we have* $\widetilde{f} \in \mathbb{C}\langle X\rangle[Y]$ *and* $\widetilde{f}$ *is general in* Y. *This operation of producing* $\widetilde{f}$ *from* f *is called* "making f general" *and we denote it by* MakeGeneral.

The following lemma shows that computing limit points reduces to making a polynomial f general.

Lemma 8 *Let* $f \in \mathbb{C}\langle X\rangle[Y]$, *where* $\deg(f, Y) > 0$, *be general in* Y. *Let* $\rho > 0$ *be small enough such that* f *converges in* $|X| < \rho$. *Let* $V_\rho(f) := \{(x, y) \in \mathbb{C}^2 \mid 0 < |x| < \rho, f(x, y) = 0\}$. *Then* $\lim_0(V_\rho(f)) = \{(0, y) \in \mathbb{C}^2 \mid f(0, y) = 0\}$ *holds.*

Proof. With $1 \le i \le c$, for some c such that $1 \le c \le \deg(f, Y)$, let $(X = T^{\varsigma_i}, Y = \varphi_i(T))$ be the distinct Puiseux parametrizations of f. By Lemma 1 and Theorem 3, we have $\lim_0(V_\rho(f)) = \cup_{i=1}^{c} \{(0, y) \in \mathbb{C}^2 \mid y = \varphi_i(0)\}$. Let $(X = T^{\sigma_i}, g_i(T))$, $i = 1, \ldots, c$, be the corresponding Puiseux parametrizations of f of accuracy 1. By Theorem 5, there exists an approximation $\widetilde{f}$ of f of some finite accuracy such that $(X = T^{\sigma_i}, g_i(T))$, $i = 1, \ldots, c$, are also Puiseux parametrizations of $\widetilde{f}$ of accuracy 1. Thus, we have $\varphi_i(0) = g_i(0)$, $i = 1, \ldots, c$. Since $\widetilde{f}$ is general in Y, by Theorem 2.3 in [25], we have $\cup_{i=1}^{c}\{(0, y) \in \mathbb{C}^2 \mid y = g_i(0)\} = \{(0, y) \in \mathbb{C}^2 \mid \widetilde{f}(0, y) = 0\}$. Since $\widetilde{f}(0, y) = f(0, y)$, the lemma holds.

Lemma 9 *Let $a(X_1, \ldots, X_s) \in \mathbb{C}[X_1, \ldots, X_s]$. Let $g_i = \sum_{j=0}^{\infty} c_{ij} T^j \in \mathbb{C}\langle T\rangle$, for $i = 1 \cdots s$. We write $a(g_1, \ldots, g_s)$ as $\sum_{k=0}^{\infty} b_k T^k$. To compute a given coefficient b_k, one only needs to know the coefficients of the polynomial a and the coefficients $c_{i,j}$ for $1 \le i \le s, 0 \le j \le k$.*

Lemma 10 *Let $f = a_d(X)Y^d + \cdots + a_0(X) \in \mathbb{C}\langle X\rangle[Y]$, where $d = \deg(f, Y) > 0$. Let $\delta := \mathrm{ord}(a_d(X))$. Then "generically", a Puiseux parametrization of f of accuracy τ can be computed from an approximation of f of accuracy $\tau + \delta$.*

Proof. Let $\widetilde{f} := \mathsf{MakeGeneral}(f)$. Observe that f and $\widetilde{f}$ have the same system of Puiseux parametrizations. Then the conclusion follows from Lemma 7 and 6.

Let $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\} \subset \mathbb{C}[X_1 < \cdots < X_s]$ be a strongly normalized regular chain. For $1 \le i \le s-1$, let $h_i := \mathrm{init}(r_i)$, $d_i := \deg(r_i, X_{i+1})$ and $\delta_i := \mathrm{ord}(h_i)$. We define f_i, ς_i, T_i, $\varphi_i(T_i)$, $1 \le i \le s-1$, as follows. Let $f_1 := r_1$. Let $(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1))$ be a Puiseux parametrization of f_1. For $i = 2, \ldots, s-1$ do

(i) Let $f_i := r_i(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1), \ldots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1})$.

(ii) Let $(T_{i-1} = T_i^{\varsigma_i}, X_{i+1} = \varphi_i(T_i))$ be a Puiseux parametrization of f_i.

Before stating our main result on the bound, we first present several lemmas.

Lemma 11 *For $0 \le i \le s-2$, define $g_i(T_{s-2}) := T_{s-2}^{\prod_{k=i+1}^{s-2} \varsigma_k}$. Let $T_0 := X_1$. Then we have $T_i = g_i(T_{s-2})$, $0 \le i \le s-2$.*

Proof. We prove it by induction. Clearly it holds for $i = s-2$. Suppose it holds for i. Then we have $T_{i-1} = T_i^{\varsigma_i} = \left(T_{s-2}^{\prod_{k=i+1}^{s-2} \varsigma_k}\right)^{\varsigma_i} = \left(T_{s-2}^{\prod_{k=i}^{s-2} \varsigma_k}\right)$. Therefore it also holds for $i-1$. So it holds for all $0 \le i \le s-2$.

Lemma 12 *There exist numbers $\tau_1, \ldots, \tau_{s-2} \in \mathbb{N}$ such that in order to make f_{s-1} general in X_s, it suffices to compute the polynomial parts of φ_i of accuracy τ_i, $1 \le i \le s-2$. Moreover, if we write the algorithm* $\mathsf{AccuracyEstimate}$ *for short as θ, the accuracies τ_i can be computed in the following manner: $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k)\delta_{s-1} + 1$, $\tau_{i-1} := \max(\theta(f_i, \tau_i), (\prod_{k=1}^{i-1} \varsigma_k)\delta_{s-1} + 1)$, for $2 \le i \le s-2$.*

Proof. By Lemma 11, we have $g_0(T_{s-2}) = T_{s-2}^{\prod_{k=1}^{s-2} \varsigma_k}$. Since $\mathrm{ord}(h_{s-1}(X_1)) = \delta_{s-1}$, we have $\mathrm{ord}(h_{s-1}(X_1 = g_0(T_{s-2}))) = \left(\prod_{k=1}^{s-2} \varsigma_k\right) \delta_{s-1}$. Let $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k)\delta_{s-1} + 1$. By Lemma 7, to make f_{s-1} general in X_s, it suffices to compute the polynomial parts of the coefficients of f_{s-1} of accuracy τ_{s-2}.

By Lemma 9, we need to compute the polynomial parts of $\varphi_i(g_i(T_{s-2}))$, $1 \le i \le s-2$, of accuracy τ_{s-2}. Since $\mathrm{ord}(g_i(T_{s-2})) = \prod_{k=i+1}^{s-2} \varsigma_k$, to achieve this accuracy, it is enough to compute the polynomial parts of φ_i of accuracy $(\prod_{k=1}^{i} \varsigma_k)\delta_{s-1}+1$, for $1 \leq i \leq s-2$.

Since we have $f_i = r_i(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1), \ldots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1})$ and $(T_{i-1} = T_i^{\varsigma_i}, X_{i+1} = \varphi_i(T_i))$ is a Puiseux parametrization of f_i, by Theorem 5 and Lemma 9, to compute the polynomial part of φ_i of accuracy τ_i, we need the polynomial part of φ_{i-1} of accuracy $\theta(f_i, \tau_i)$. Thus, $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k)\delta_{s-1}+1$ and $\tau_{i-1} = \max(\theta(f_i,\tau_i), (\prod_{k=1}^{i-1} \varsigma_k)\delta_{s-1}+1)$ for $2 \leq i \leq s-2$ will guarantee f_{s-1} can be made general in X_s.

Theorem 6 *One can compute positive integer numbers $\tau_1, \ldots, \tau_{s-1}$ such that, in order to compute $\lim_0(W(R))$, it suffices to compute Puiseux parametrizations of f_i of accuracy τ_i, for $i = 1, \ldots, s-1$. Moreover, generically, one can choose $\tau_{s-1} := 1$, $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k)\delta_{s-1}+1$, $\tau_i = (\prod_{k=1}^{s-2} \varsigma_k)(\sum_{k=2}^{s-1} \delta_i)+1$, for $i = 1, \ldots, s-3$, and each index ς_k can be set to d_k, for $k = 1, \ldots, s-2$.*

Proof. By Lemma 12, we know that $\tau_1, \ldots, \tau_{s-1}$ can be computed. By Lemma 11, we have $X_1 = T_{i-1}^{\prod_{k=1}^{i-1} \varsigma_k}$. Since $\mathrm{ord}(h_i(X_1)) = \delta_i$, we have $\mathrm{ord}(h_i(X_1 = T_{i-1}^{\prod_{k=1}^{i-1} \varsigma_k})) = \left(\prod_{k=1}^{i-1} \varsigma_k\right)\delta_i$. By Lemma 10, generically a Puiseux parametrization of f_i of accuracy τ_i can be computed from an approximation of f_i of accuracy $\tau_i + \delta_i$. In Lemma 12, let $\theta(f_i, \tau_i) = \tau_i + (\prod_{k=1}^{i-1} \varsigma_k)\delta_i$, $2 \leq i \leq s-2$, which implies the bound in the theorem. Finally we observe that $\varsigma_k \leq d_k$ holds, for $1 \leq k \leq s-2$.

7 Algorithm

In this section, we provide a complete algorithm for computing the non-trivial limit points of the quasi-component of a one-dimensional strongly normalized regular chain based on the results of the previous sections.

Proposition 1 *Algorithm 4 is correct and terminates.*

Proof. This follows from Theorem 3, Theorem 5, Theorem 6 and Lemma 8.

Theorem 7 *Let $R \subset \mathbb{Q}[X_1, \ldots, X_n]$ be a regular chain such that $\dim(\mathrm{sat}(R)) = 1$. Then there exists an algorithm to compute regular chains $R_i \in \mathbb{Q}[X_1, \ldots, X_n]$, $i = 1, \ldots, e$, such that $\lim(W(R)) = \cup_{i=1}^{e} W(R_i)$.*

Proof. By Remark 1, we can assume that R is strongly normalized and X_1 is free w.r.t. R. By Proposition 1, there is an algorithm to compute $\lim(W(R))$. Thus, it suffices to prove that $\lim(W(R))$ can be represented by regular chains in $\mathbb{Q}[X_1, \ldots, X_n]$, whenever $R \subset \mathbb{Q}[X_1, \ldots, X_n]$ holds. By examining carefully Algorithms 1, 2, 3, 4, and their subroutines, one observes that only Algorithms 1 and 4 may introduce numbers that are in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and not in $\mathbb{Q}$ itself. In fact, for each $x = (x_1, \ldots, x_n) \in \lim(W(R))$, Algorithms 1 and 4 introduce a field extension $\mathbb{Q}(\xi_1, \ldots, \xi_m)$ such that we have $x_i \in \mathbb{Q}[\xi_1, \ldots, \xi_m]$. Let $Y_1, \ldots, Y_m$ be m new symbols. Let $G := \{g_1(Y_1), g_2(Y_1, Y_2), \ldots, g_m(Y_1, Y_2, \ldots, Y_m)\}$

Algorithm 3: LimitPointsAtZero

Input: A regular chain $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\}$.
Output: The non-trivial limit points of $W(R)$ whose X_1-coordinates are 0.

let $S := \{(T_0)\}$;
compute the accuracy estimates $\tau_1, \ldots, \tau_{s-2}$ by Theorem 6; let $\tau_{s-1} = 1$;
for i *from* 1 *to* $s-1$ **do**
  $S' := \emptyset$;
  for $\Phi \in S$ **do**
    $f_i := r_i(X_1 = \Phi_1, \ldots, X_i = \Phi_i, X_{i+1})$;
    if $i > 1$ **then**
      let $\delta := \mathrm{ord}(f_i, T_{i-1})$; let $f_i := f_i / T_{i-1}^{\delta}$;
    $E :=$ NewtonPuiseux(f_i, τ_i);
    for $(T_{i-1} = \phi(T_i), X_{i+1} = \varphi(T_i)) \in E$ **do**
      $S' := S' \cup \{\Phi(T_{i-1} = \phi(T_i)) \cup (\varphi(T_i))\}$
  $S := S'$
if $S = \emptyset$ **then** return $\emptyset$ **else** return $\mathrm{eval}(S, T_{s-1} = 0)$

Algorithm 4: LimitPoints

Input: A regular chain $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\}$.
Output: All the non-trivial limit points of $W(R)$.

let $h_R := \mathrm{init}(R)$; let L be the set of zeros of h_R in $\mathbb{C}$; $S := \emptyset$;
for $\alpha \in L$ **do**
  $R_\alpha := R(X_1 = X_1 + \alpha)$; $S_\alpha :=$ LimitPointsAtZero(R_α);
  update S_α by replacing the first coordinate of every point in S_α by α;
  $S := S \cup S_\alpha$
return S

be an irreducible regular chain (i.e. generating a maximal ideal over $\mathbb{Q}$) such that $G(Y_1 = \xi_1, \ldots, Y_m = \xi_m) = 0$ holds. Since $x_i \in \mathbb{Q}[\xi_1, \ldots, \xi_m]$, there exists $f_i \in \mathbb{Q}[Y_1, \ldots, Y_m]$, $i = 1, \ldots, n$, such that $x_i = f_i(Y_1 = \xi_1, \ldots, Y_m = \xi_m)$. Let $\mathcal{S}_x := \{X_1 = f_1(Y_1, \ldots, Y_m), \ldots, X_n = f_n(Y_1, \ldots, Y_m), G(Y_1, \ldots, Y_m) = 0\}$. The projection of the zero set of $\mathcal{S}_x$ on the $(X_1, \ldots, X_n)$-space is the zero set of an irregular chain $R_x \in \mathbb{Q}[X_1, \ldots, X_m]$ and we have $\lim(W(R)) = \cup_{x \in \lim(W(R))} W(R_x)$.

8 Experimentation

We have implemented Algorithm 4 of Section 7, which computes the limit points of the quasi-component of a one-dimensional strongly normalized regular chain. The implementation is based on the `RegularChains` library and the command algcurves[puiseux] [24] of Maple. The code is available at `http://www.orcca.on.ca/~cchen/ACM13/LimitPoints.mpl`. This preliminary implementation relies on algebraic factorization, whereas, as suggested in [10], applying the D5 principle [9], in the spirit of triangular decomposition algorithms [6], would be sufficient when computations need to split into different cases. This would certainly improve performance greatly and this enhancement is work in progress.

As pointed out in the introduction, the computation of the limit points of the quasi-component of a regular chain can be applied to removing redundant components in a Kalkbrener triangular decomposition. In Table 1, we report on experimental results of this application.

The polynomial systems listed in this table are one-dimensional polynomial systems selected from the literature [5, 6]. For each system, we first call the Triangularize command of the library RegularChains, with the option "'normalized='strongly', 'radical'='yes'". For the input system, this process computes a Kalkbrener triangular decomposition $\mathcal{R}$ where the regular chains are strongly normalized and their saturated ideals are radical. Next, for each one-dimensional regular chain R in the output, we compute the limit points $\lim(W(R))$, thus deducing a set of regular chains $R_1, \ldots, R_e$ such that the union of their quasi-components equals the Zariski closure $\overline{W(R)}$. The algorithm Difference [5] is then called to test whether or not there exists a pair R, R' of regular chains of $\mathcal{R}$ such that the inclusion $\overline{W(R)} \subseteq \overline{W(R')}$ holds. In Table 1, the columns T and #(T) denote respectively the timings spent by Triangularize and the number of regular chains returned by this command; the columns d-1 and d-0 denote respectively the number of 1-dimensional and 0-dimensional regular chains; the columns R and #(R) denote respectively the timings spent on removing redundant components in the output of Triangularize and the number of regular chains in the output irredundant decomposition. As we can see in the table, most of the decompositions are checked to be irredundant, which we could not do before this work by means of triangular decomposition algorithms. In addition, the three redundant 0-dimensional components in the Kalkbrener triangular decomposition of system f-744 are successfully removed in about 7 minutes, whereas we cannot draw this conclusion in more than one hour by a brute-force method computing the generators of the saturated ideals of regular chains. Therefore, we have verified experimentally the benefits provided by the proposed algorithms.

Table 1. Removing redundant components.

Sys	T	#(T)	d-1	d-0	R	#(R)
f-744	14.360	4	1	3	432.567	1
Liu-Lorenz	0.412	3	3	0	216.125	3
MontesS3	0.072	2	2	0	0.064	2
Neural	0.296	5	5	0	1.660	5
Solotareff-4a	0.632	7	7	0	32.362	7
Vermeer	1.172	2	2	0	75.332	2
Wang-1991c	3.084	13	13	0	6.280	13

9 Concluding remarks

In this paper, we proposed an algorithm for computing the limit points of the quasi-component of a regular chain in dimension one by means of Puiseux series expansions. In the future, we will investigate how to compute the limit points in higher dimension with the help of the Abhyankar-Jung theorem [21].

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