Memory Efficiency in Polynomial Multiplication

Daniel S. Roche

Symbolic Computation Group
School of Computer Science
University of Waterloo

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Preliminaries

We study algorithms for univariate polynomial multiplication:

The Problem

Given: A ring R, an integer n,
and $f, g \in \mathrm{R}[x]$ with degrees less than n

Compute: Their product $f \cdot g \in \mathrm{R}[x]$

The Model

- Ring operations have unit cost
- Random reads from input, random reads/writes to output
- Space complexity determined by size of auxiliary storage

Univariate Multiplication Algorithms

	Time Complexity	**Space Complexity**
Classical Method	$O(n^2)$	$O(1)$
Divide-and-Conquer Karatsuba/Ofman ’63	$O(n^{\log_2 3})$ or $O(n^{1.59})$	$O(n)$
FFT-based Schönhage/Strassen ’71 Cantor/Kaltofen ’91	$O(n \log n \log \log n)$	$O(n)$

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Previous Work

- Monagan 1993: Importance of space efficiency for multiplication over $\mathbb{Z}_p[x]$
- Maeder 1993: Bounds extra space for Karatsuba multiplication so that storage can be preallocated — about $2n$ extra memory cells required.
- Thomé 2002: Karatsuba multiplication for polynomials using n extra memory cells.
- Zimmerman & Brent 2008:
  "The efficiency of an implementation of Karatsuba's algorithm depends heavily on memory usage."

Present Contributions

- New Karatsuba-like algorithm with $O(\log n)$ space
- New FFT-based algorithm with $O(1)$ space
  under certain conditions
- Implementations in C over $\mathbb{Z}/p\mathbb{Z}$

Standard Karatsuba Algorithm

Idea: Reduce one degree-$2k$ multiplication to three of degree k.

- Originally noticed by Gauss (multiplying complex numbers), rediscovered and formalized by Karatsuba & Ofman

Input: $f, g \in \mathrm{R}[x]$ each with degree less than $2k$.

Write $f = f_0 + f_1 x^k$ and $g = g_0 + g_1 x^k$.

Low-Space Karatsuba Algorithms

Version “0”

Read-Only Input Space:

f01	f11

g0	g1

Read/Write Output Space:

(empty)	(empty)	(empty)	(empty)

To Compute: $f \cdot g$

Low-Space Karatsuba Algorithms

Version “1”

1. The low-order coefficients of the output are initialized as h, and the product $f \cdot g$ is added to this.

Read-Only Input Space:

f01	f11	g0	g1

Read/Write Output Space:

h0	h1	(empty)	(empty)

To Compute: $f \cdot g + h$

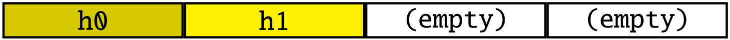
Low-Space Karatsuba Algorithms

Version “2”

1. The low-order coefficients of the output are initialized as h, and the product $f \cdot g$ is added to this.
2. The first polynomial f is given as a sum $f^{(0)} + f^{(1)}$.

Read-Only Input Space:

Read/Write Output Space:



To Compute: $(f^{(0)} + f^{(1)}) \cdot g + h$

Classical and modular arithmetic

Restrict modulus to 29 bits to allow for delayed reductions

In the Karatsuba step

- Only 4 values are added/subtracted in one position
- Delay reductions, perform two “corrections”

Classical algorithm

- Switch over at $n \leq 32$ (determined experimentally)
- Perform arithmetic in double-precision `long longs`; delay reductions (a la Monagan)

Problem: code explosion

3 “versions” of algorithms (based on extra constraints)

$\times$

Karatsuba or classical

$\times$

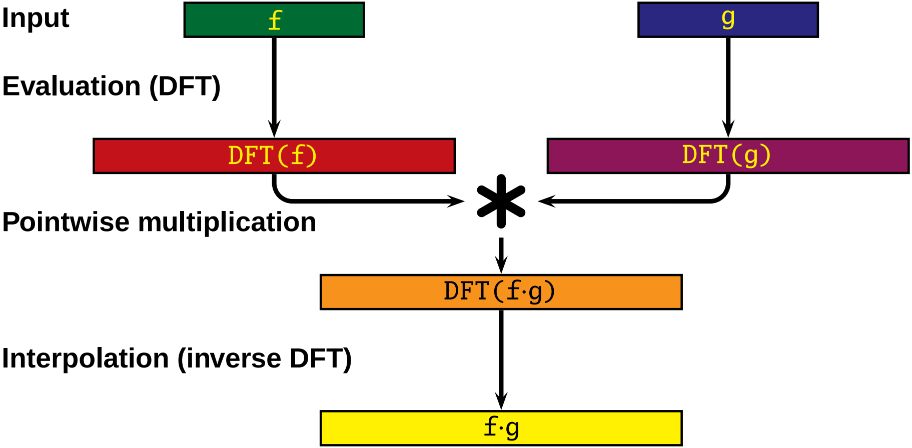
odd-sized or even-sized operands

$\times$

equal-sized operands or “one different”

Solution: Use “supermacros” in C:
Same file is included multiple times with some parameter values changed (crude form of code generation).

DFT-Based Multiplication



Simplifying Assumptions

From now on:

- $\deg f + \deg g < n = 2^k$ for some $k \in \mathbb{N}$
- The base ring R contains a 2^k-PRU ω

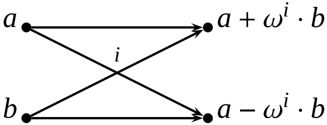
That is, assume “virtual roots of unity” have already been found, and optimize from there.

Usual Formulation of the FFT

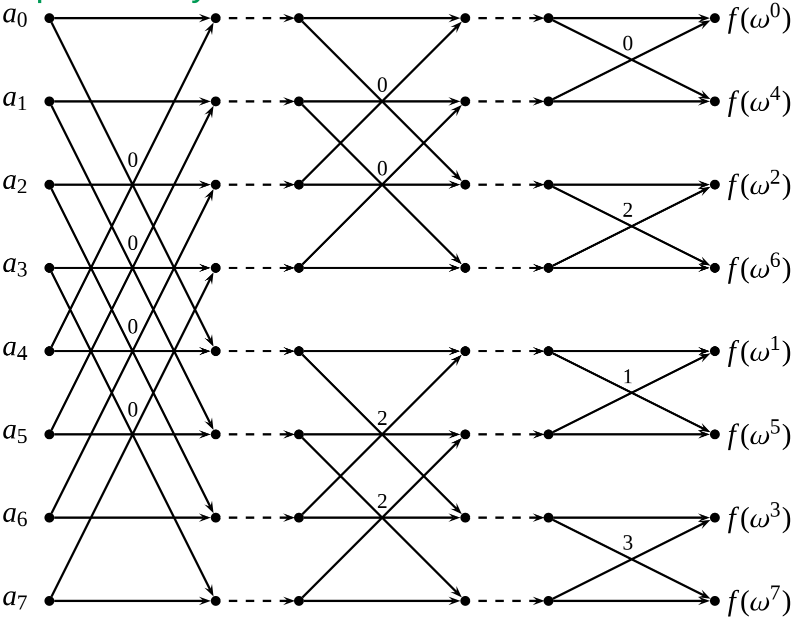
Perform two $\frac{n}{2}$-DFTs followed by $\frac{n}{2}$ 2-DFTs:

- Write $f(x) = f_{\mathrm{even}}(x^2) + x \cdot f_{\mathrm{odd}}(x^2)$
  (i.e. $\deg f_{\mathrm{even}}, \deg f_{\mathrm{odd}} < n/2$)
- Compute $\mathrm{DFT}_{\omega^2}(f_{\mathrm{even}})$ and $\mathrm{DFT}_{\omega^2}(f_{\mathrm{odd}})$
- Compute each $f(\omega^i) = f_{\mathrm{even}}(\omega^{2i}) + \omega \cdot f_{\mathrm{odd}}(\omega^{2i})$

Make use of “butterfly circuit” for each size-2 DFT:



Example: 8-Way FFT



Reverted Binary Ordering

In-Place FFT permutes the ordering into reverted binary:

$0=000_2$	$1=001_2$	$2=010_2$	$3=011_2$	$4=100_2$	$5=101_2$	$6=110_2$	$7=111_2$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$0=000_2$	$4=100_2$	$2=010_2$	$6=110_2$	$1=001_2$	$5=101_2$	$3=011_2$	$7=111_2$

Problem: Powers of ω are not accessed in order

Possible solutions:

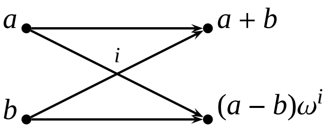
- Precompute all powers of ω — too much space
- Perform steps out of order — terrible for cache
- Permute input before computing — costly

Alternate Formulation of FFT

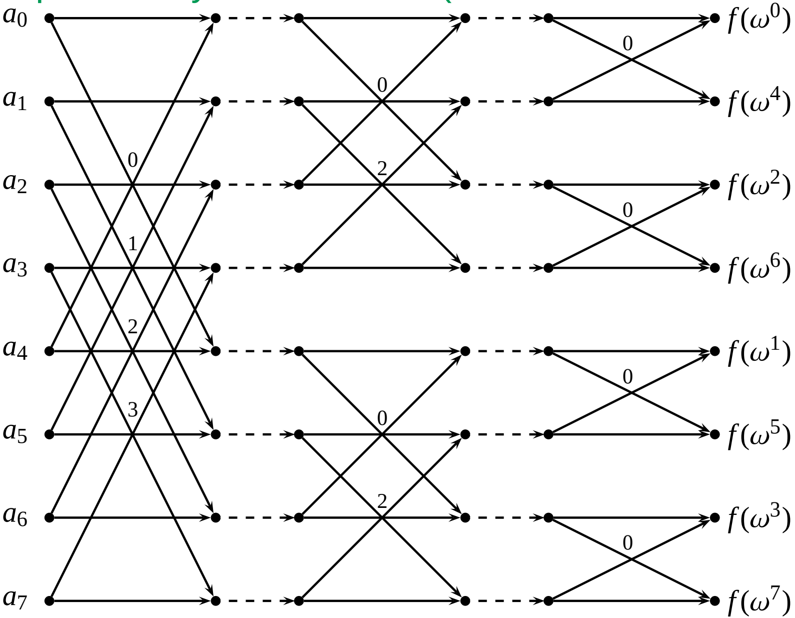
Perform $\frac{n}{2}$ 2-DFTs followed by two $\frac{n}{2}$-DFTs

- Write $f = f_{\text{low}} + x^{n/2} \cdot f_{\text{high}}$
- Compute $f_0 = f_{\text{low}} + f_{\text{high}}$ and $f_1 = f_{\text{low}}(\omega x) - f_{\text{high}}(\omega x)$
- Compute each $f(\omega^{2i}) = f_0(\omega^{2i})$ and $f(\omega^{2i+1}) = f_1(\omega^{2i})$

Modified "butterfly circuit":



Example: 8-Way In-Place FFT (Alternate Formulation)



Folded Polynomials

Recall the basis for the “alternate” FFT formulation:

$$\begin{aligned} f_0 &= f_{\mathrm{low}} + f_{\mathrm{high}} \\ f_1 &= f_{\mathrm{low}}(\omega x) - f_{\mathrm{high}}(\omega x) \end{aligned}$$

A generalization (recalling that $n = 2^k$):

Definition (Folded Polynomials)

$$f_i = f(\omega^{2^{i-1}} x) \quad \mathrm{rem}\, x^{2^{k-i}} - 1$$

Theorem

$$f\left(\omega^{2^i(2j+1)}\right) = f_{i+1}\left(\omega^{2^{i+1}j}\right)$$

So by computing each f_i at all powers of ω^i,
we get the values of f at all powers of ω.

Recursively Applying the Alternate Formulation

Example (Iterative Generation of Reverted Binary Ordering)

0

Recursively Applying the Alternate Formulation

Example (Iterative Generation of Reverted Binary Ordering)

0, 8

Recursively Applying the Alternate Formulation

Example (Iterative Generation of Reverted Binary Ordering)

0, 8, 4, 12

Recursively Applying the Alternate Formulation

Example (Iterative Generation of Reverted Binary Ordering)

0, 8, 4, 12, 2, 10, 6, 14

Recursively Applying the Alternate Formulation

Example (Iterative Generation of Reverted Binary Ordering)

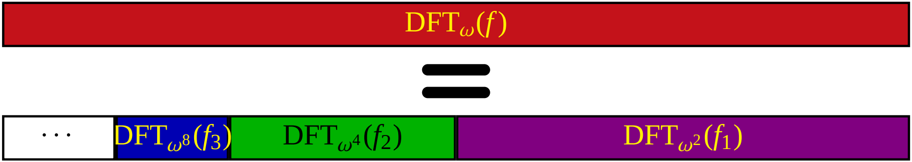
0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15

Recursively Applying the Alternate Formulation

Example (Iterative Generation of Reverted Binary Ordering)

0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15

$\mathrm{DFT}_\omega(f)$ in binary reversed order
can be computed by DFTs of f_is:



FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

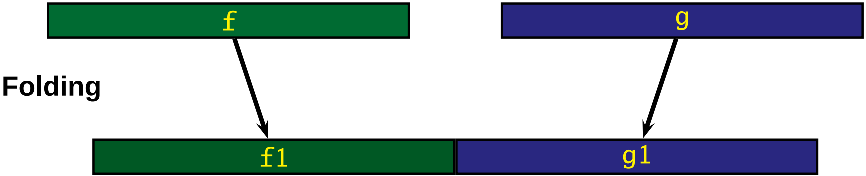
`f` `g`

Input

`(empty)`

FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration



FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

`f` `g`

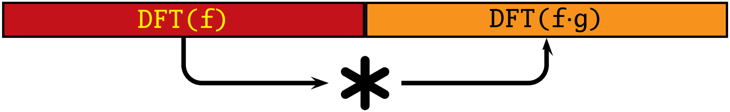
In-Place FFTs (alternate formulation)

`DFT(f1)` `DFT(g1)`

FFT-Based Multiplication without Extra Space

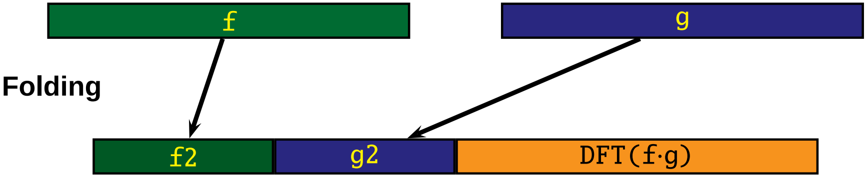
Idea: Solve half of remaining problem at each iteration

Pointwise Multiplication



FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration



FFT-Based Multiplication without Extra Space

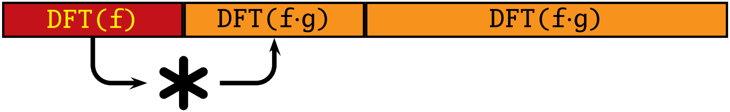
Idea: Solve half of remaining problem at each iteration

In-Place FFTs (alternate formulation)

FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

Pointwise Multiplication



FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

(k iterations)

FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

In-Place Reverse FFT (usual formulation)

Modular Arithmetic

Use floating-point Barrett reduction (from NTL):

- Pre-compute an approximation of $1/p$
- Given $a, b \in \mathbb{Z}_p$, compute an approximation of $q = \lfloor a \cdot b \cdot (1/p) \rfloor$
- Then $ab - qp$ equals $ab \operatorname{rem} p$ plus or minus p.

The cost of this method:

- 2 `double` multiplications
- 2 `int` multiplications
- 1 `int` subtraction
- 3 conversions between `int` and `double`
- 2 “correction” steps to get exact result
  $\hookrightarrow$ not necessary until the very end!

Implementation Benchmarking

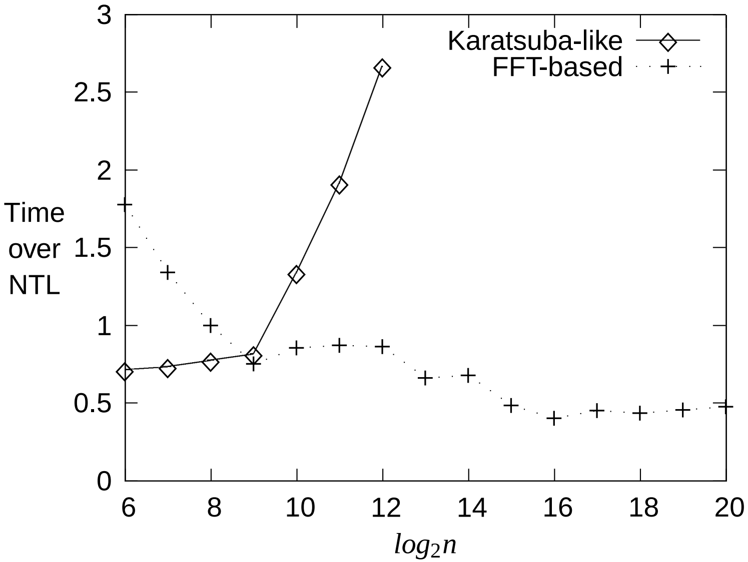
Details of tests:

- 2.5 GHz 64-bit Athalon, 256KB L1, 1MB L2, 2GB RAM
- $p = 167772161$ (28 bits)
- Comparing CPU time (in seconds) for the computation

Disclaimer

We are comparing apples to oranges.

Timing Benchmarks



Future Directions

- Efficient implementation over $\mathbb{Z}$ (GMP)
- Similar results for
  Toom-Cook 3-way or k-way
- What modulus bit restriction is "best"?
- Is completely in-place (overwriting input) possible?