# Balanced Dense Polynomial Multiplication on Multicores

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# Introduction

**Motivation:** Multicore-enabling parallel polynomial arithmetic in computer algebra

- Fast dense polynomial multiplication via FFT
- Multivariate polynomials over finite fields

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**Framework:**

- Assume 1-D FFT (in particular 1-D TFT) as a black box
- Rely on the `modpn` C library (shipped with Maple 13):
  – for 1-D FFT and 1-D TFT computations,
  – for integer modulo arithmetic (Montgomery trick)
- Implement in Cilk++ targeting multi-cores:
  – provably efficient work-stealing scheduling
  – ease-of-use and low-overhead parallel constructs:
  `cilk_for`, `cilk_spawn`, `cilk_sync`
  – Cilkscreen for data race detection and parallelism analysis

# FFT-based Multivariate Multiplication
## (Algorithm Review)

- Let $\mathbf{k}$ be a field and $f, g \in \mathbf{k}[x_1 < \cdots < x_n]$ be polynomials.
- Define $d_i = \deg(f, x_i)$ and $d'_i = \deg(g, x_i)$, for all $i$.
- Assume there exists a primitive $s_i$-th root unity $\omega_i \in \mathbf{k}$ for all $i$, where $s_i$ is a power of 2 satisfying $s_i \geq d_i + d'_i + 1$.

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Then $fg$ can be computed as follows.

*Step* 1. Evaluate $f$ and $g$ at each point $P$ (i.e. $f(P), g(P)$) of the $n$-dimensional grid $((\omega_1^{e_1}, \ldots, \omega_n^{e_n}), 0 \leq e_1 < s_1, \ldots, 0 \leq e_n < s_n)$ via $n$-D FFT.

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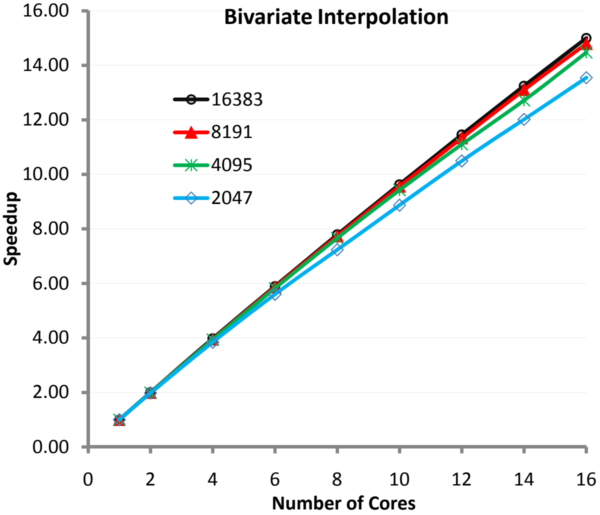
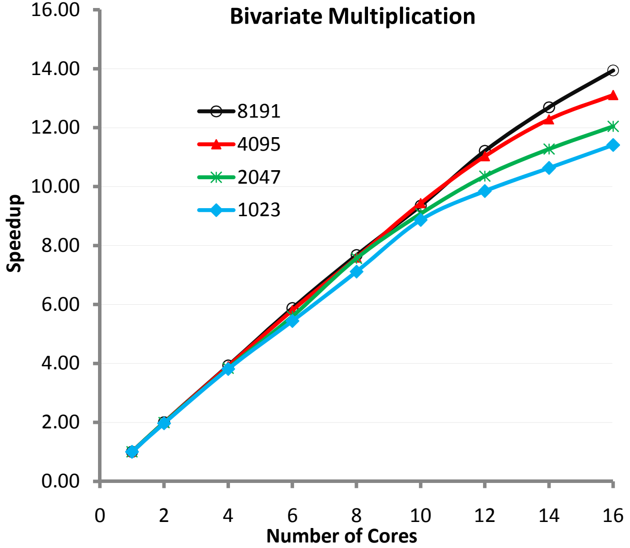
*Step* 3. Interpolate $fg$ (from its values on the grid) via $n$-D FFT.

# Performance of Bivariate Interpolation in Step 3 $(d_1 = d_2)$

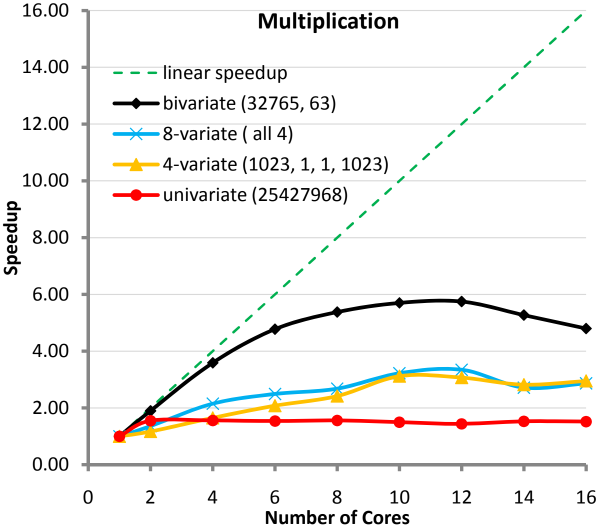



Thanks to Dr. Frigo for his cache-efficient code for matrix transposition!

# Performance of Bivariate Multiplication
## $(d_1 = d_2 = d_1' = d_2')$



# Challenges: Irregular Input Data




These unbalanced data pattern are common in symbolic computation.

# Performance Analysis by VTune

| No. | Size of Two Input Polynomials | Product Size |
|---|---|---|
| 1 | 8191×8191 | 268402689 |
| 2 | 259575×258 | 268401067 |
| 3 | 63×63×63×63 | 260144641 |
| 4 | 8 vars. of deg. 5 | 214358881 |

| No. | INST_ RETIRED. ANY×$10^9$ | Clocks per Instruction Retired | L2 Cache Miss Rate ($\times 10^{-3}$) | Modified Data Sharing Ratio ($\times 10^{-3}$) | Time on 8 Cores (s) |
|---|---|---|---|---|---|
| 1 | 659.555 | 0.810 | 0.333 | 0.078 | 16.15 |
| 2 | 713.882 | 0.890 | 0.735 | 0.192 | 19.52 |
| 3 | 714.153 | 0.854 | 1.096 | 0.635 | 22.44 |
| 4 | 1331.340 | 1.418 | 1.177 | 0.576 | 72.99 |

# Complexity Analysis (1/2)

- Let $s = s_1 \cdots s_n$. The number of operations in $\mathbf{k}$ for computing $fg$ via n-D FFT is

$$\frac{9}{2}\sum_{i=1}^{n}(\prod_{j\neq i} s_j)s_i \lg(s_i) + (n+1)s = \frac{9}{2}s\lg(s) + (n+1)s.$$

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- Under our 1-D FFT black box assumption, the span of *Step* 1 is

$$\tfrac{9}{2}\left(s_1 \lg(s_1) + \cdots + s_n \lg(s_n)\right),$$

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$$s/\max(s_1, \ldots, s_n). \tag{1}$$
- Let $L$ be the size of a cache line. For some constant $c > 0$, the number of cache misses of *Step* 1 is upper bounded by
$$n\frac{cs}{L} + cs\left(\frac{1}{s_1} + \cdots + \frac{1}{s_n}\right). \tag{2}$$

# Complexity Analysis (2/2)

- Let $Q(s_1, \ldots, s_n)$ denotes the total number of cache misses for the whole algorithm, for some constant $c$ we obtain

$$Q(s_1, \ldots, s_n) \leq cs\frac{n+1}{L} + cs(\frac{1}{s_1} + \cdots + \frac{1}{s_n}) \tag{3}$$

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**Remark 1:** For $n \geq 2$, Expr. (4) is minimized at $n = 2$ and $s_1 = s_2 = \sqrt{s}$. Moreover, when $n = 2$, under a fixed $s = s_1 s_2$, Expr. (1) is maximized at $s_1 = s_2 = \sqrt{s}$.

# Our Solutions

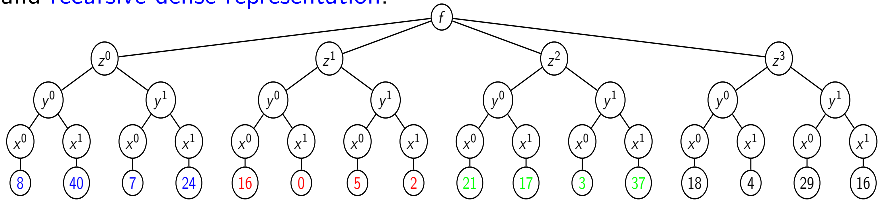
(1) Contraction to bivariate from multivariate

(2) Extension from univariate to bivariate

(3) Balanced multiplication by extension and contraction

# Solution 1: Contraction to Bivariate from Multivar.

**Example.** Let $f \in \mathbf{k}[x, y, z]$ where $\mathbf{k} = \mathbb{Z}/41\mathbb{Z}$, with $d_x = d_y = 1$, $d_z = 3$, and recursive dense representation:




- ⋆ The coefficients (not monomials) are stored in a contiguous array.
- ⋆ Index monomial $x^{e_1} y^{e_2} z^{e_3}$ by $e_1 + (d_x + 1)e_2 + (d_x + 1)(d_y + 1)e_3$.

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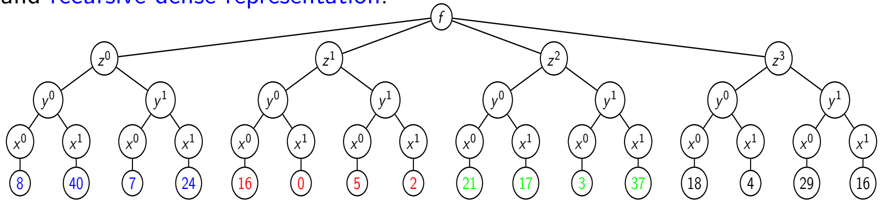


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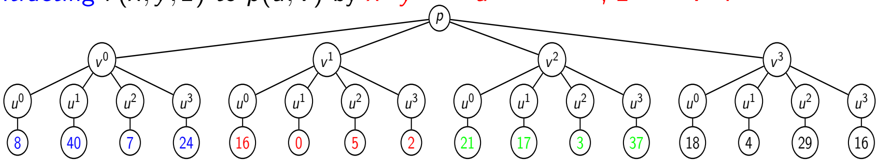
Contracting $f(x, y, z)$ to $p(u, v)$ by $x^{e_1} y^{e_2} \mapsto u^{e_1 + (d_x+1)e_2}$, $z^{e_3} \mapsto v^{e_3}$:



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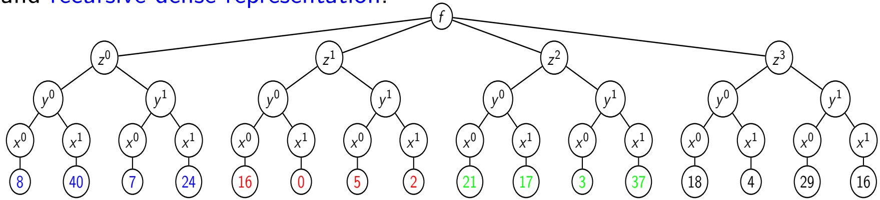


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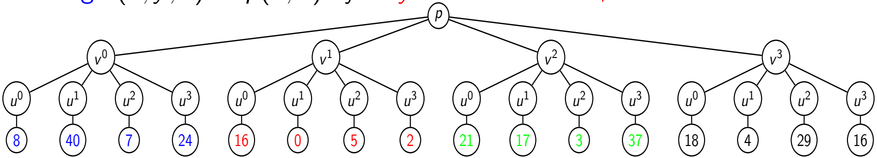
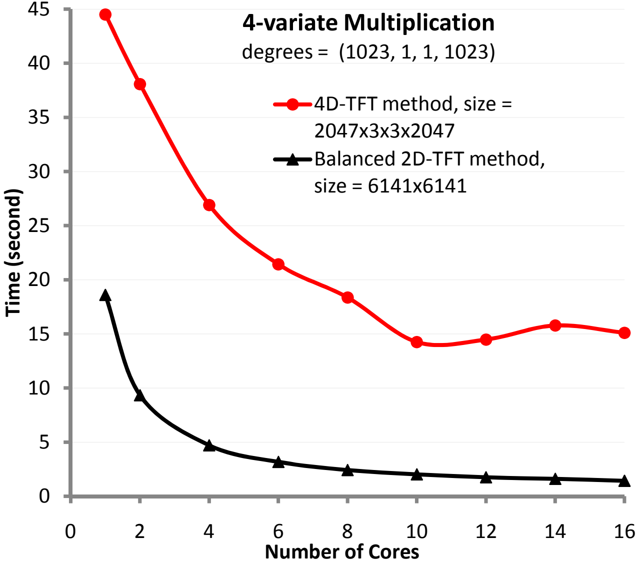
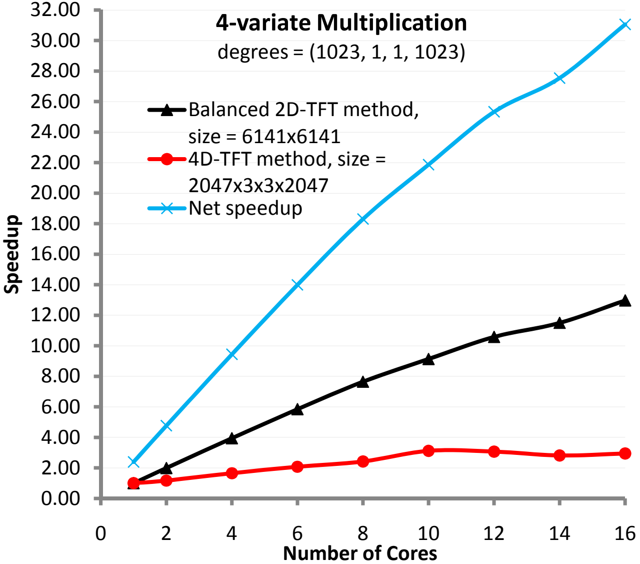



**Remark 2**: The coefficient array is "essentially" unchanged by contraction, which is a property of recursive dense representation.

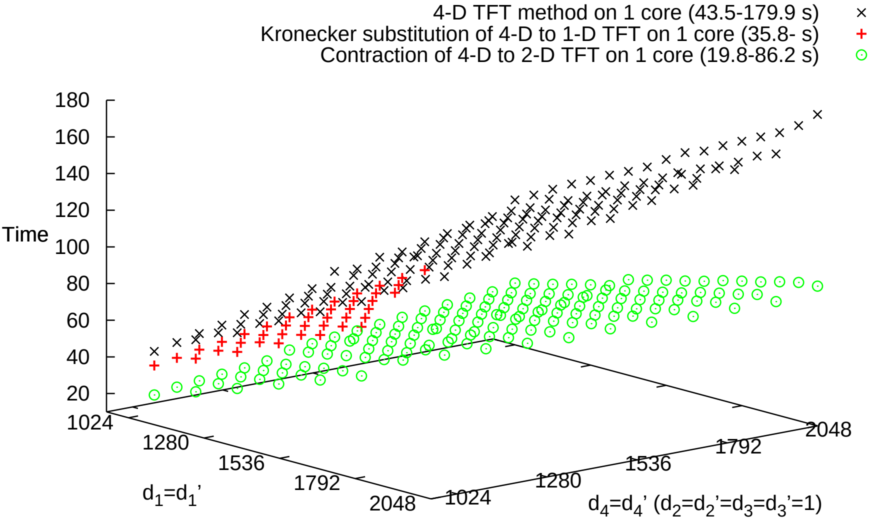
# Performance of Contraction (timing)



# Performance of Contraction (speedup)



# Performance of Contraction for a Large Range of Problems (timing on 1 processor)



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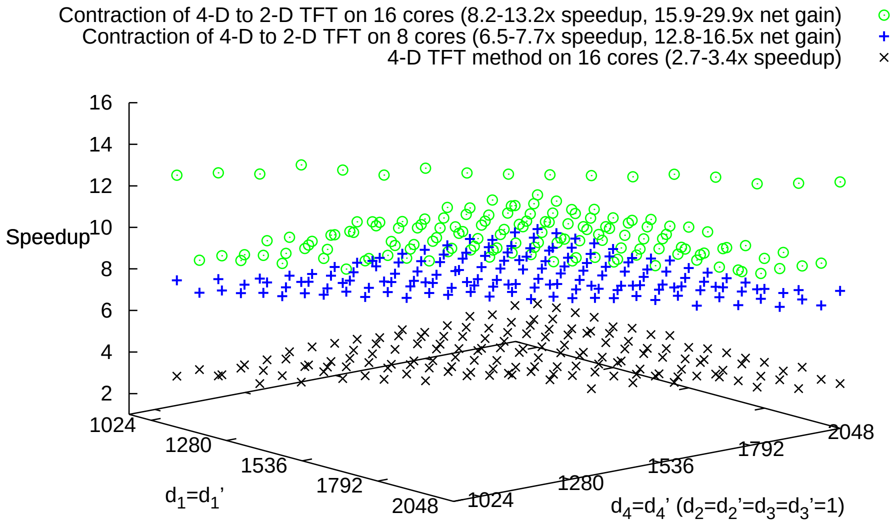

# Solution 2: Extension from Univariate to Bivariate

**Example**: Consider $f, g \in \mathbf{k}[x]$ univariate, with $\deg(f) = 7$ and $\deg(g) = 8$; $fg$ has "dense size" 16.

- We compute an integer $b$, such that $fg$ can be performed via $f_b g_b$ using "nearly square" 2-D FFTs, where $f_b := \Phi_b(f)$, $g_b := \Phi_b(g)$ and

$$\boldsymbol{\Phi_b} : \mathbf{x^e} \longmapsto \mathbf{u}^{\mathbf{e} \,\mathrm{rem}\, \mathbf{b}} \; \mathbf{v}^{\mathbf{e} \,\mathrm{quo}\, \mathbf{b}}.$$

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- ⋆ Here $b = 3$ works since $\deg(f_b g_b, u) = \deg(f_b g_b, v) = 4$; moreover the dense size of $f_b g_b$ is 25.

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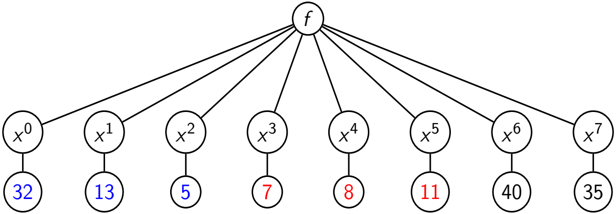
**Proposition**: For any non-constant $f, g \in \mathbf{k}[x]$, one can always compute $b$ such that $|deg(f_b g_b, u) - deg(f_b g_b, v)| \leq 2$ and the dense size of $f_b g_b$ is at most twice that of $fg$.

# Extension of $f(x)$ to $f_b(u, v)$ in Recursive Dense Representation



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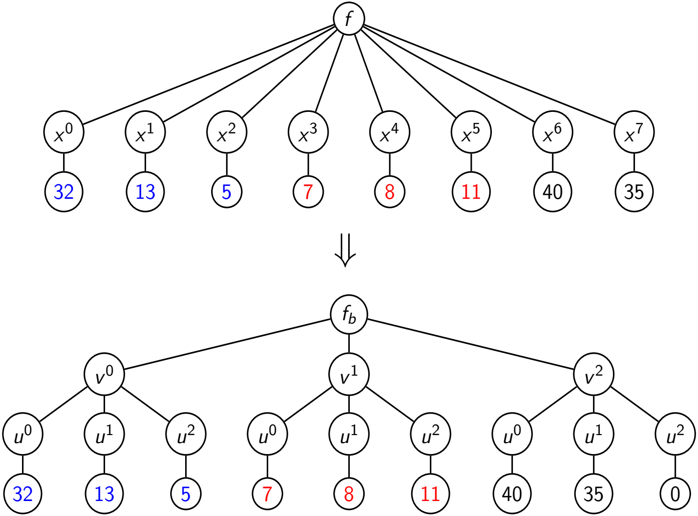
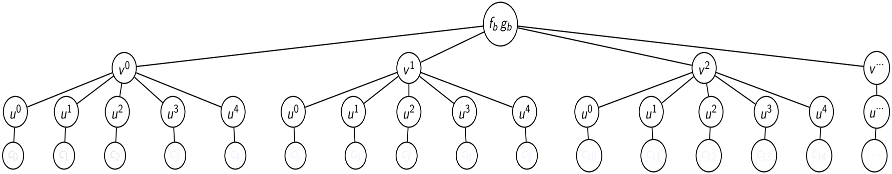


# Conversion to Univariate from the Bivariate Product

- The bivariate product: $deg(f_b g_b, u) = 4, deg(f_b g_b, v) = 4.$



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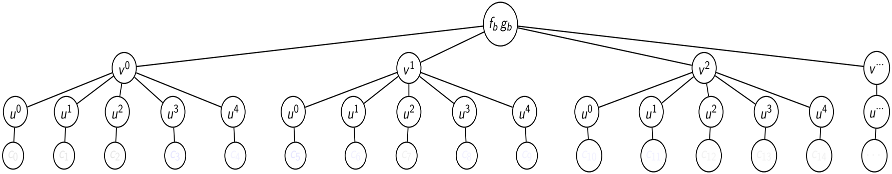


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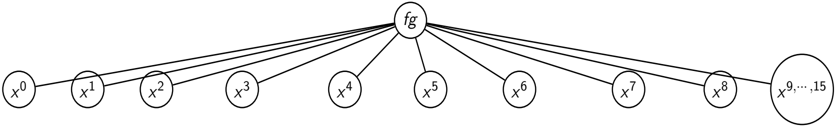

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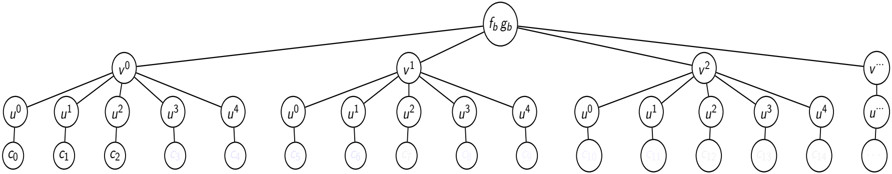


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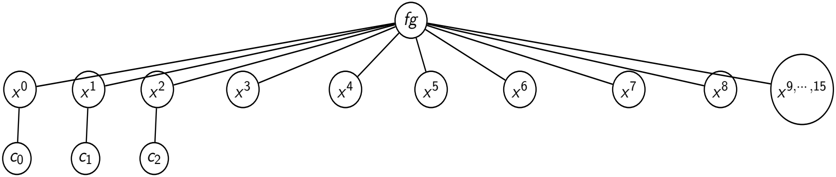

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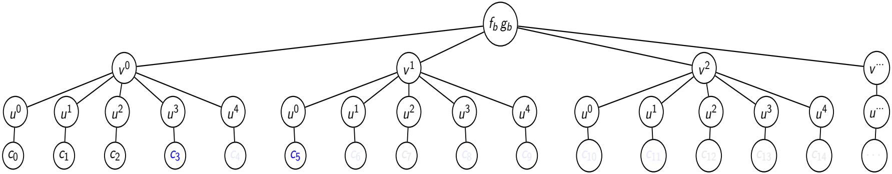


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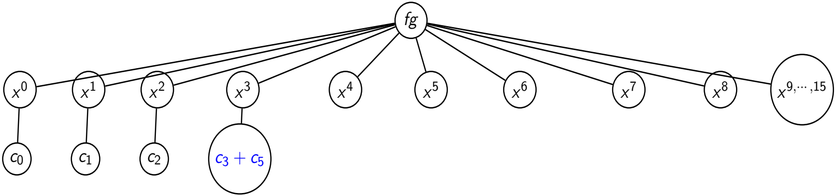

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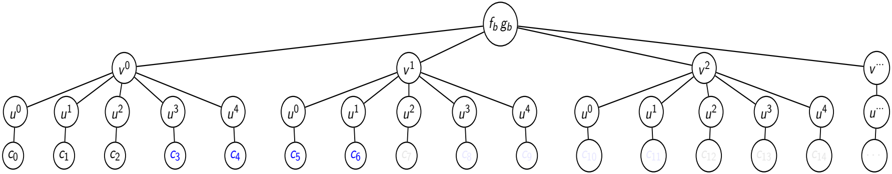


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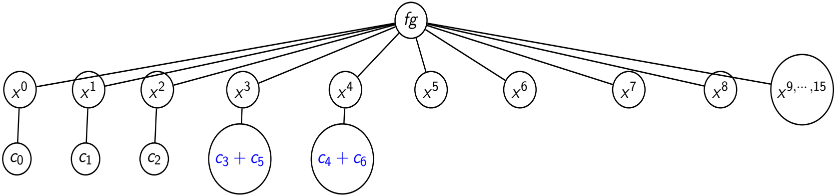

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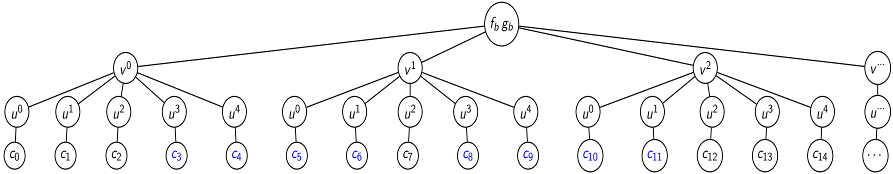


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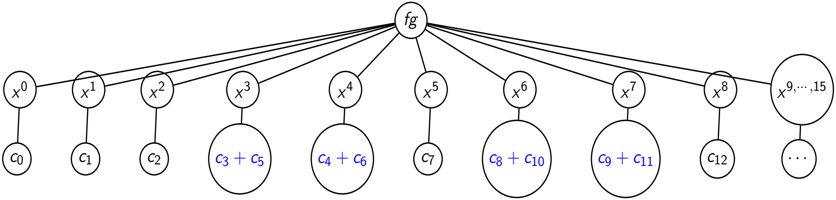

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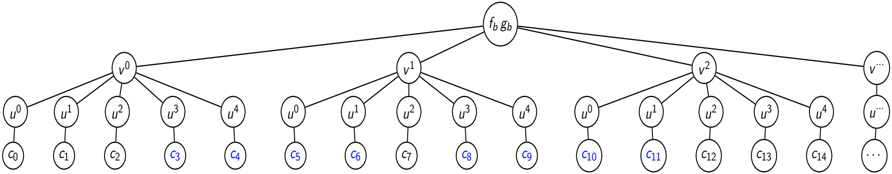


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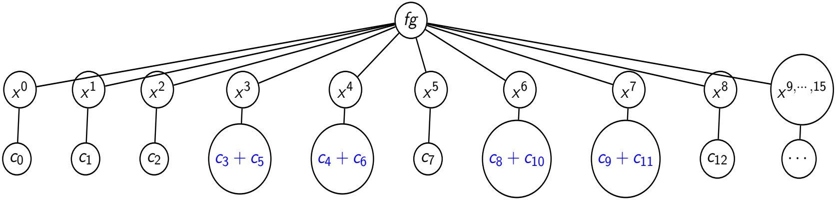
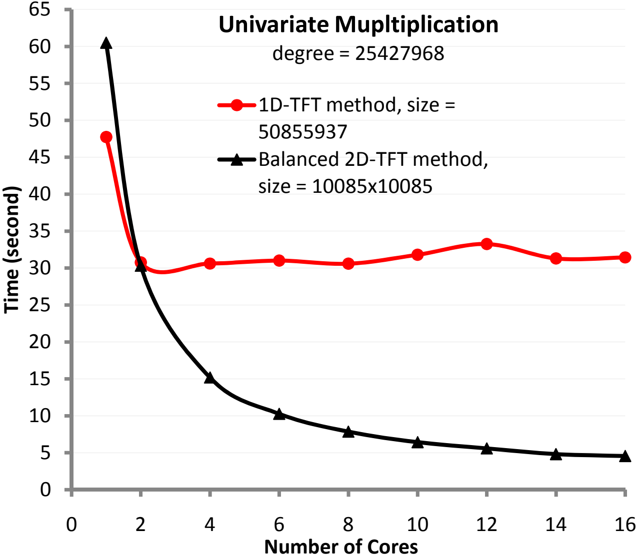



**Remark 4:** Converting back to $fg$ from $f_b g_b$ requires only to traverse the coefficient array once, and perform at most $\deg(fg, x)$ additions.

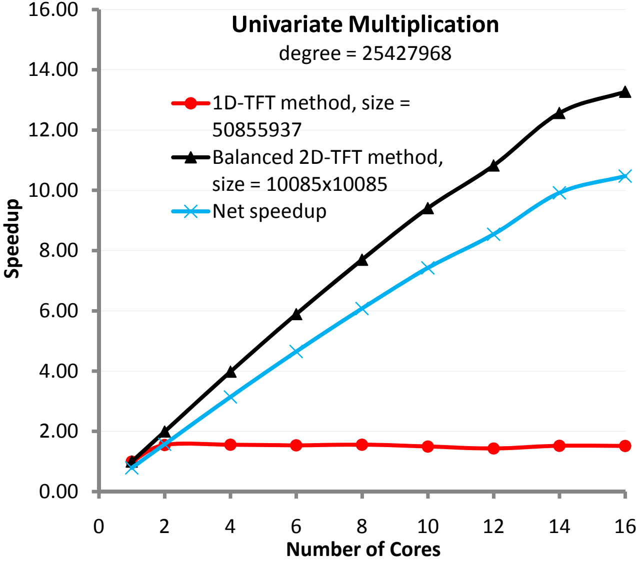
# Remark 3

Our extension technique provides an alternative to the Schönage -Strassen Algorithm for handling problems which sizes are too large for the available primitive roots of unity, a limitation with FFTs over finite fields.

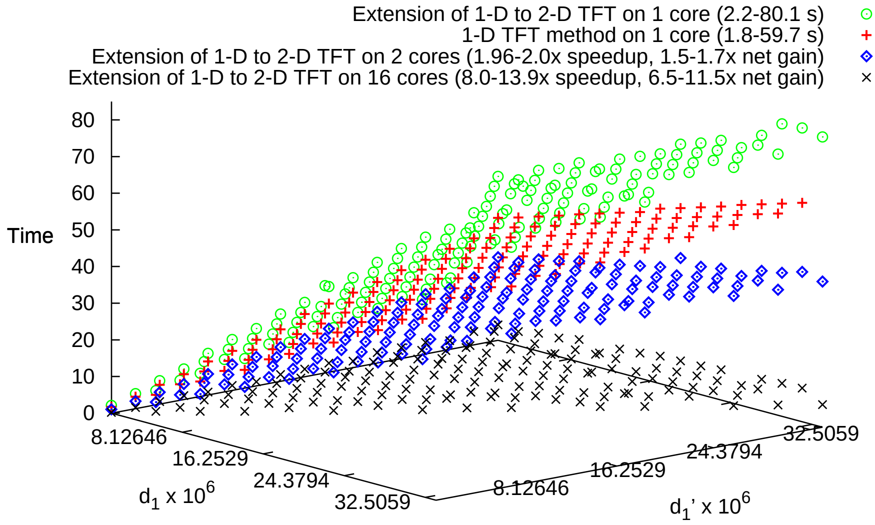
# Performance of Extension (timing)



# Performance of Extension (speedup)



# Performance of Extension for a Large Range of Problems (timing)



# Solution 3: Balanced Multiplication

**Definition**. A pair of bivariate polynomials $p, q \in \mathbf{k}[u, v]$ is *balanced* if $\deg(p, u) + \deg(q, u) = \deg(p, v) + \deg(q, v)$.

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**Algorithm**. Let $f, g \in \mathbf{k}[x_1 < \ldots < x_n]$. W.l.o.g. one can assume $d_1 >> d_i$ and $d_1{}' >> d_i$ for $2 \leq i \leq n$ (up to variable re-ordering and contraction). Then we obtain $fg$ by

*Step* 1. Extending $x_1$ to $\{u, v\}$.

*Step* 2. Contracting $\{v, x_2, \ldots, x_n\}$ to $v$.

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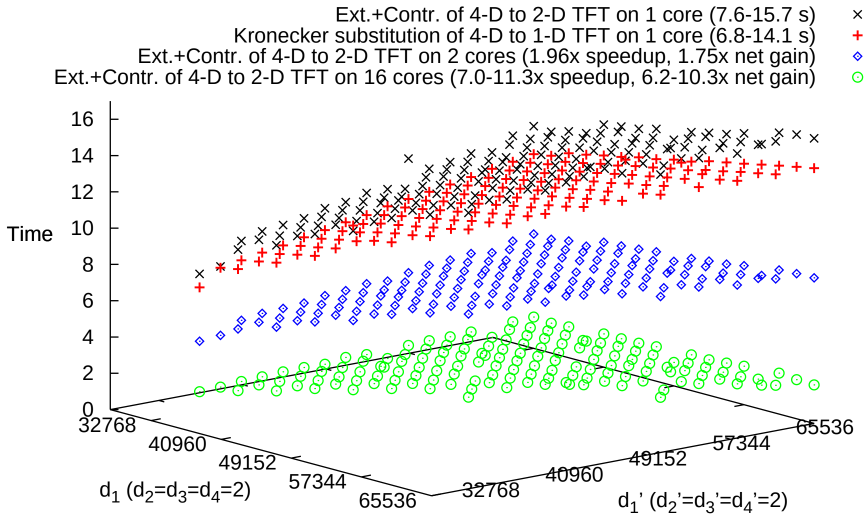
**Remark 5:** The above extension $\Phi_b$ can be determined such that $f_b, g_b$ is (nearly) a balanced pair and $f_b g_b$ has dense size at most twice that of $fg$.

# Performance of Balanced Multiplication for a Large Range of Problems (timing)



# Conclusion

**Summary and future work:**

- We have obtained efficient techniques and implementations for dense polynomial multiplication on multicores:
  - use balanced bivariate multiplication as a kernel,
  - contract mutivariate to bivariate,
  - extend univariate to bivariate,
  - combine contraction and extension.
- Our work-in-progress include normal form, GCD/resultant and a polynomial solver via triangular decompositions.

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