Sparse matrices in computer algebra when using distributed memory: theory and applications

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Outline of the talk

History

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Applications

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Algorithms

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Experiments

History: block-recursive matrix algorithms

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... but very impotant for sparse super large matrices on a supercomputer with distributed memory.

Generalisation of Strassen's inversion and algorithm for the solution of systems in commutative domains was discraibed at 1997-2006 ([7], [8], [10], [9]) with strong restriction: the leading minors should not be zero.

History: block-recursive matrix algorithms

Algorithms for solution of a system of linear equations of size n in an integral domain, which served as the basis for creating recursive algorithms:

(1983) Forward and backward algorithm ($\sim n^3$) [4].
(1989) One pass algorithm ($\sim \frac{2}{3}n^3$) [5].
(1995) Combined algoritm with upper left block of size r ($\sim \frac{7}{12}n^3$ for $r = \frac{n}{2}$) [6].

History: block-recursive matrix algorithms

Recursive algorithms for solution of a system of linear equations and for adjoint matrix computation in an integral domain without permutations:

(1997) Recursive algorithm for solution of a system of linear equations [7].
(2000) Adjoint matrix computation (with 6 levels) [8].
(2006) Adjoint matrix computation alternative algorithm (with 5 levels) [10].

History: block-recursive matrix algorithms

This restriction was removed at (2008- 2015):

Main recursive algorithms for sparse matrices:

(2008) The algorithm that computes the adjoint matrix, the echelon form, and the kernel of the matrix operator for the commutative domains was described in [11] .

(2010) The block-recursive algorithm for the Bruhat decomposition and the LEU decomposition for the matrix over the field was obtained in [12] ,

(2013, 2015) and these algorithms were generaized to the LDU and Bruhat decomposition for the matrices over commutative domains in [14], [15] .

New achivements:

It is proved that the LEU algorithm has the complexity $O(n^2 r^{\beta-2})$ for matrices of rank r. [19] (2013).

It is proved that the LEU algorithm has the complexity $O(n^2 s^{\beta-2})$ for quasiseparable matrix, if any it's submatrix which entirely below or above the main diagonal has small rank s [20] (2017) .

1) Applications: Calculation of electronic circuits.

The behavior of electronic circuits can be described by Kirchhoff's laws. The three basic approaches in this theory are direct current, constant frequency current and a current that varies with time. All these cases require the compilation and solution of sparse systems of equations (numerical, polynomial or differential). The solution of such differential equations by the Laplace method also leads to the solution of polynomial systems of equations [16].

2) Applications: Control systems.

In 1967 Howard H. Rosenbrock introduced a useful state-space representation and transfer function matrix form for control systems, which is known as the Rosenbrock System Matrix [17]. Since that time, the properties of the matrix of polynomials being intensively studied in the literature of linear control systems.

3). Applications: Computation of Groebner basis.

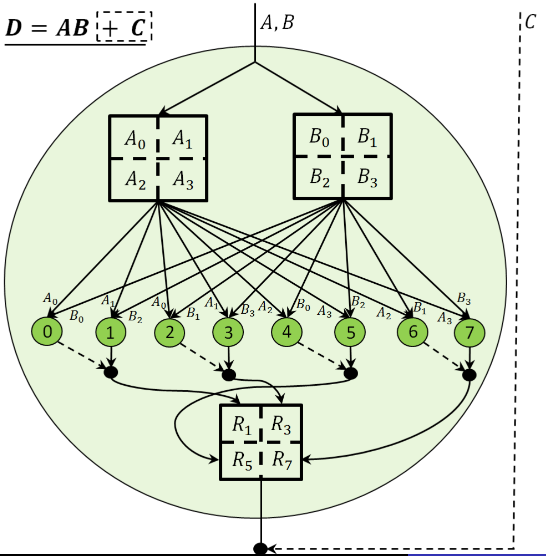
A matrix composed of Buchberger S-polynomials is a strongly sparse matrix. Reduction of the polynomial system is performed when calculating the echelon and diagonal forms of this matrix. The algorithm F4 [18] was the first such matrix algorithm.

4) Applications: Solving ODE's and PDEs.
Solving ODE's and PDE's is often based on solution of leanear systems with sparse matrices over numbers or over polynomials. One of the important class of sparse matrix is called quasiseparable. Any submatrix of quasiseparable matrix entirely below or above the main diagonal has small rank. These quasiseparable matrices arise naturally in solving PDEs for particle interaction with the Fast Multi-pole Method (FMM). The efficiency of application of the block-recursive algorithm of the Bruhat decomposition to the quasiseparable matrices is studied in [20].

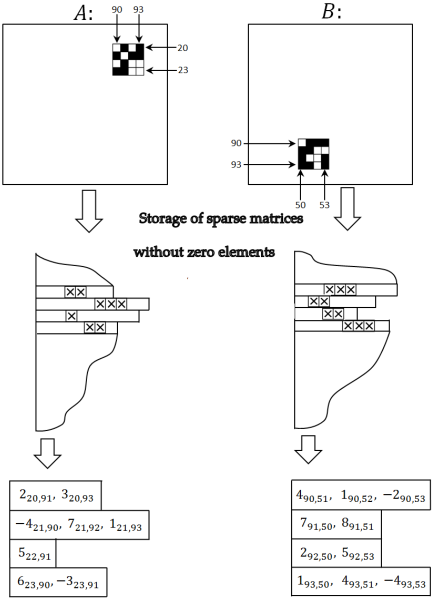
Recursive matrix multiplication for tree trunk and branches

$$\begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix} \times \begin{pmatrix} B_0 & B_1 \\ B_2 & B_3 \end{pmatrix} + \begin{pmatrix} C_0 & C_1 \\ C_2 & C_3 \end{pmatrix} = \begin{pmatrix} D_0 & D_1 \\ D_2 & D_3 \end{pmatrix}$$

$\boldsymbol{D = AB + C}$



Recursive sparse matrix multiplication on the leaf's block



Recursive Strassen matrix inversion

If $\mathcal{A} = \begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix}$, $\det(\mathcal{A}) \neq 0$ and $\det(A_0) \neq 0$ then

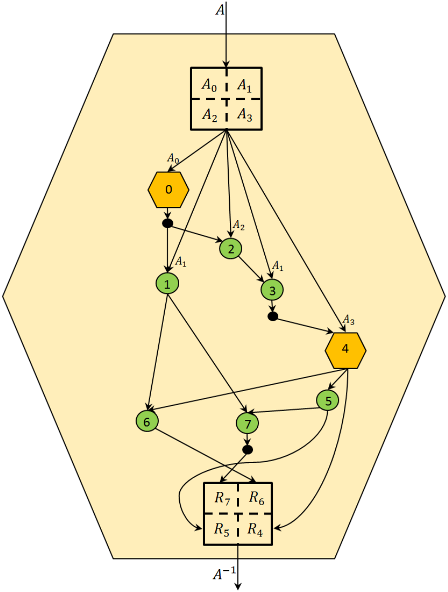
$$\mathcal{A}^{-1} = \begin{pmatrix} \mathbf{I} & -A_0^{-1}A_1 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ 0 & (A_3 - A_2A_0^{-1}A_1)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ -A_2 & \mathbf{I} \end{pmatrix} \begin{pmatrix} A_0^{-1} & 0 \\ 0 & \mathbf{I} \end{pmatrix}$$

$$= \begin{pmatrix} A_0^{-1} + A_0^{-1}A_1(A_3 - A_2A_0^{-1}A_1)^{-1}A_2A_0^{-1} & -A_0^{-1}A_1(A_3 - A_2A_0^{-1}A_1)^{-1} \\ -(A_3 - A_2A_0^{-1}A_1)^{-1}A_2A_0^{-1} & (A_3 - A_2A_0^{-1}A_1)^{-1} \end{pmatrix}.$$

If $M_0 = -A_0^{-1}$, $M_1 = M_0A_1$, $M_2 = A_2M_0$, $M_3 = M_2A_1$, $M_4 = (A_3 + M_3)^{-1}$, $M_5 = -M_4M_2$, then

$$\mathcal{A}^{-1} = \begin{pmatrix} M_1M_5 - M_0 & M_1M_4 \\ M_5 & M_4 \end{pmatrix}.$$

Recursive matrix inversion Strassen



Recursive computation of the adjoint and kernel: 1 of 2

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$A_{ext}(M_{11}, d_0) = (A_{11}, S_{11}, E_{11}, d_{11}).$$

$$M_{12}^1 = \frac{A_{11} M_{12}}{d_0}, M_{21}^1 = -\frac{M_{21} Y_{11}}{d_0}, M_{22}^1 = \frac{M_{22} d_{11} - M_{21} E_{11}^T M_{12}^1}{d_0}.$$

$$A_{ext}(\bar{I}_{11} M_{12}^1, d_{11}) = (A_{12}, S_{12}, E_{12}, d_{12}), A_{ext}(M_{21}^1, d_{11}) = (A_{21}, S_{21}, E_{21}, d_{21}).$$

$$M_{22}^2 = -\frac{A_{21} M_{22}^1 Y_{12}}{(d_{11})^2}, \ d_s = \frac{d_{21} d_{12}}{d_{11}}.$$

$$A_{ext}(\bar{I}_{21} M_{22}^2, d_s) = (A_{22}, S_{22}, E_{22}, d_{22}).$$

$$M_{11}^2 = -\frac{S_{11} Y_{21}}{d_{11}}, M_{12}^2 = \frac{\left(\frac{\frac{S_{11} E_{21}^T A_{21}}{d_{11}} M_{22}^1 - I_{11} M_{12}^1 d_{21}}{d_{11}} \right) Y_{12} + S_{12} d_{21}}{d_{11}}, M_{12}^3 = -\frac{M_{12}^2 Y_{22}}{d_s},$$

$$M_{22}^3 = S_{22} - \frac{I_{21} M_{22}^2 Y_{22}}{d_s},\ A^1 = A_{12} A_{11},\ A^2 = A_{22} A_{21},$$

$$L = \left(\frac{A^1 - \frac{I_{11} M_{12}^1 E_{12}^T A^1}{d_{11}}}{d_{11}} \right) d_{22}, \quad P = \frac{A^2 - \frac{I_{21} M_{22}^2 E_{22}^T A^2}{d_s}}{d_{21}},$$

$$F = -\frac{\left(\frac{S_{11} E_{21}^T A_{21}}{d_{11}} \right) d_{22} + \frac{M_{12}^2 E_{22}^T A^2}{d_s}}{d_{21}}, \quad G = -\frac{\left(\frac{M_{21} E_{11}^T A_{11}}{d_0} \right) d_{12} + \frac{M_{22}^1 E_{12}^T A^1}{d_{11}}}{d_{11}},$$

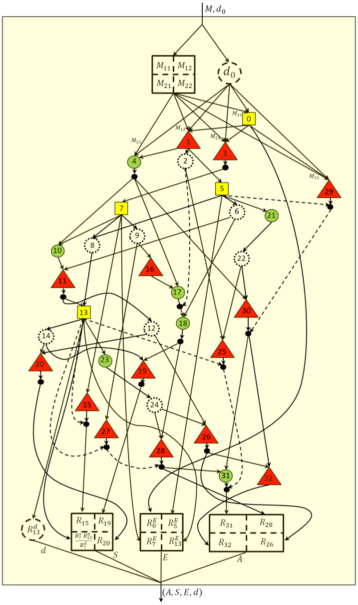
$$A = \begin{pmatrix} \frac{L+FG}{d_{12}} & F \\ \frac{PG}{d_{12}} & P \end{pmatrix}, S = \begin{pmatrix} \frac{M_{11}^2 d_{22}}{d_{21}} & M_{12}^3 \\ \frac{S_{21} d_{22}}{d_{21}} & M_{22}^3 \end{pmatrix}, E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix},\ d = d_{22}.$$

Then

$$A_{ext}(M, d_0) = (A, S, E, d_{22}).$$

$$I_{ij} = E_{ij} E_{ij}^T, \bar{I}_{ij} = \mathbf{I} - I_{ij}, Y_{ij} = E_{ij}^T S_{ij} - d_{ij} \mathbf{I}, \quad i, j \in 1, 2.$$

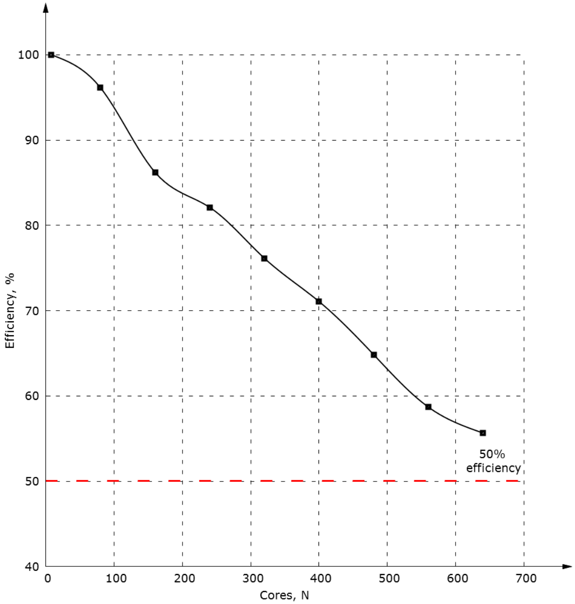
Recursive adjoint matrix computation



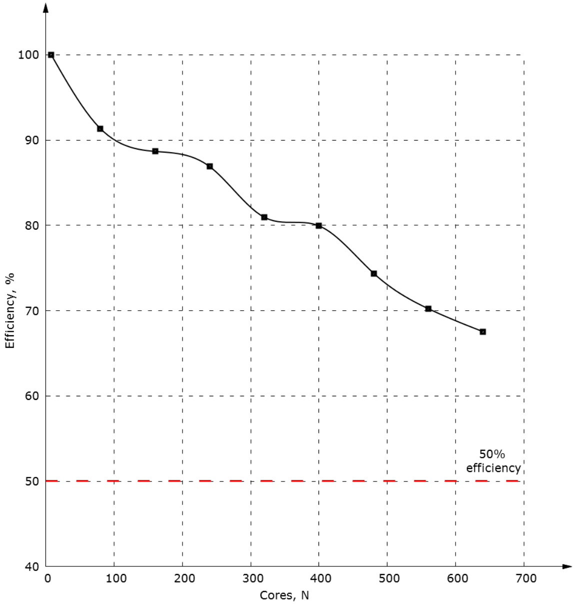
Sparse matrices when using distributed memory

The block-recursive matrix algorithms for sparse matrix require a special approachs to managing parallel programs. One approach to the cluster computations management is a scheme with one dispatcher (or one master).
We consider another scheme of cluster menagement. It is a scheme with multidispatching, when each involved computing module has its own dispatch thread and several processing threads [21], [22].
We demonstrate the results of experiments with parallel programms on the base of multidispatching.

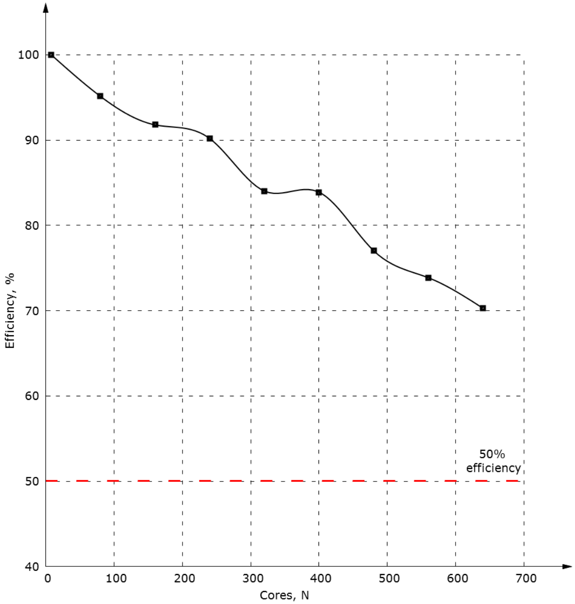
Recursive matrix multiplication (dence, n=8000, Z, 15b)



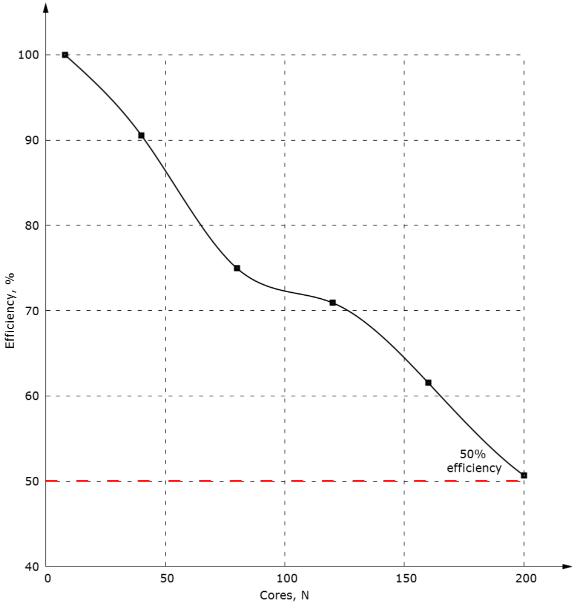
Recursive matrix multiplication (dence, n=12000,Z, 15b)



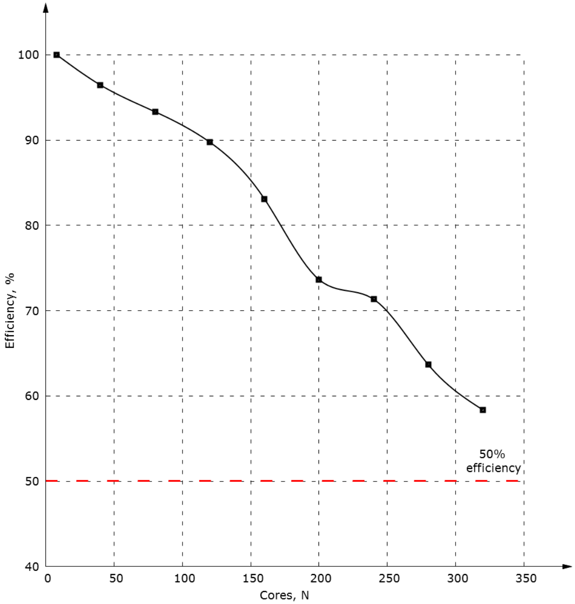
Recursive matrix multiplication (dence, n=14000, Z, 15b)



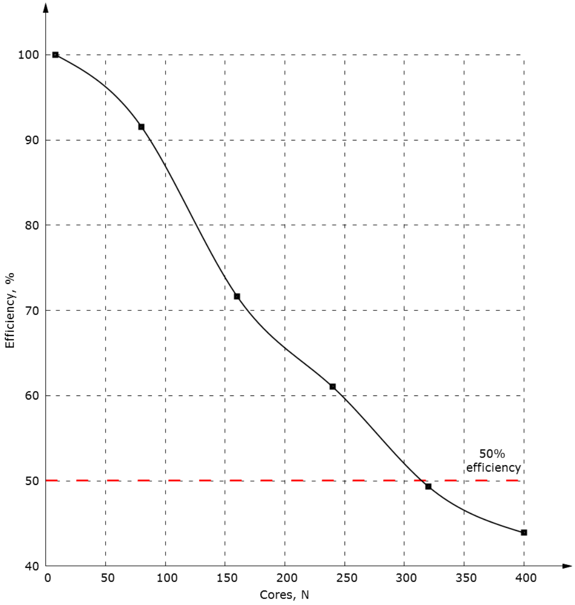
Recursive inversion Strassen (dence, n=8000, double)



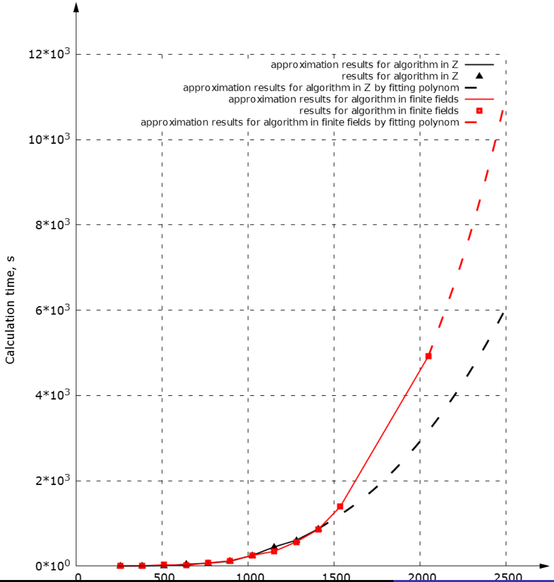
Recursive inversion Strassen (dence, n=16000, double)



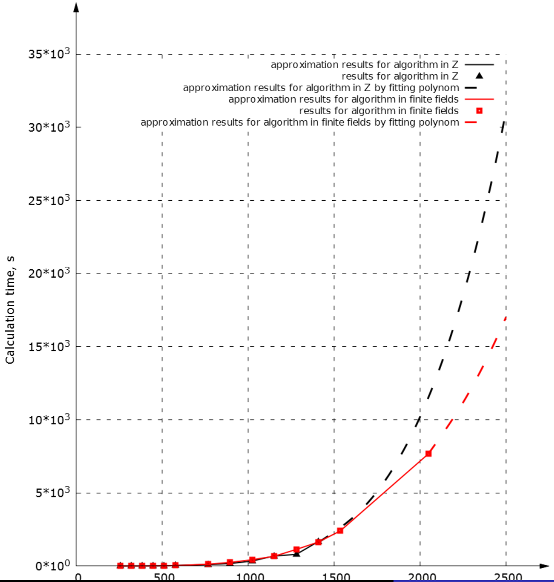
Recursive adjoint and kernel (dence, n=8000, Z)



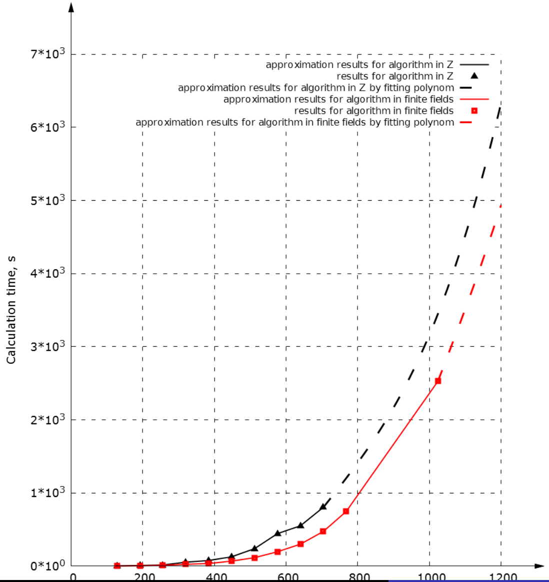
Recursive adjoint and kernel (d=100%, n=100, Z, 15b, CRT+P)



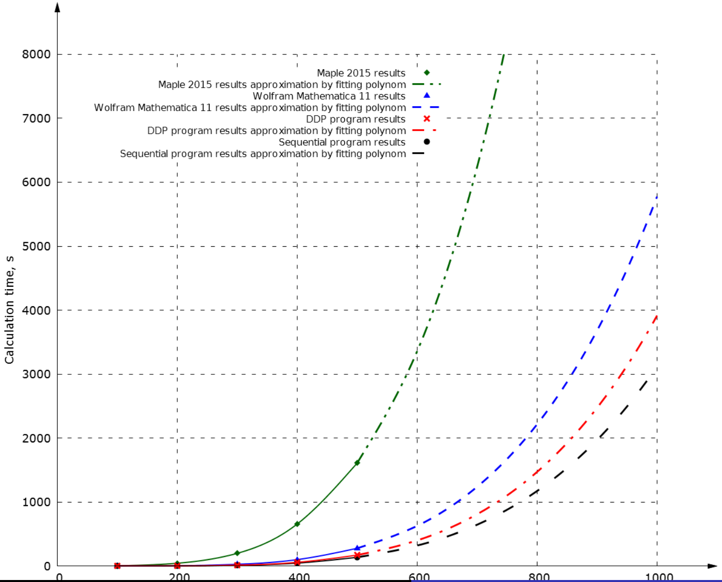
Recursive adjoint and kernel (d=1%, n=10000, Z, 15b, CRT+P)



Recursive adjoint and kernel (d=1%, n=10000, Z, CRT+P)



Comparing sequantional program with Mathematica and MAPLE



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