CS434a/541a: Pattern Recognition
Prof. Olga Veksler

Lecture 4

Outline

- Normal Random Variable
  - Properties
  - Discriminant functions

Why Normal Random Variables?

- Analytically tractable
- Works well when observation comes form a corrupted single prototype (μ)
- Is an optimal distribution of data for many classifiers used in practice

The Univariate Normal Density

- ***x*** is a scalar (has dimension 1)

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],$$

Where:

μ = mean (or expected value) of *x*

σ^2 = variance

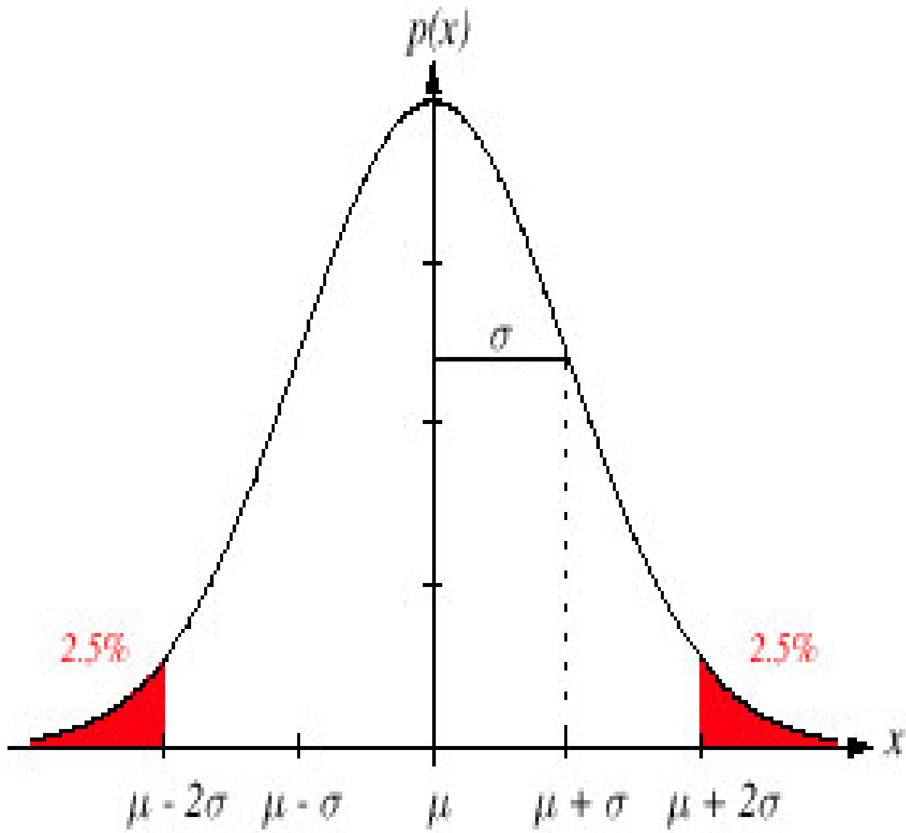


FIGURE 2.7. A univariate normal distribution has roughly 95% of its area in the range $|x - \mu| \leq 2\sigma$, as shown. The peak of the distribution has value $p(\mu) = 1/\sqrt{2\pi}\sigma$. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Several Features

- What if we have several features $\boldsymbol{x_1, x_2, \ldots, x_d}$
  - each normally distributed
  - may have different means
  - may have different variances
  - may be dependent or independent of each other
- How do we model their joint distribution?

The Multivariate Normal Density

- Multivariate normal density in ***d*** dimensions is:

inverse of Σ

$$p(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)^t \Sigma^{-1} (x-\mu)\right]$$

determinant of Σ

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{d1} & \cdots & \sigma_d^2 \end{bmatrix}$$

covariance of x_1 and x_d

$$x = [x_1, x_2, \ldots, x_d]^t$$

$$\mu = [\mu_1, \mu_2, \ldots, \mu_d]^t$$

- Each x_i is $N(\mu_i, \sigma_i^2)$
  - to prove this, integrate out all other features from the joint density

More on Σ

- $\Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{d1} & \cdots & \sigma_d^2 \end{bmatrix}$ plays role similar to the role that σ^2 plays in one dimension

- From Σ we can find out
  1. The individual variances of features $\boldsymbol{x_1, x_2, \ldots, x_d}$
  2. If features $\boldsymbol{x_i}$ ***and*** $\boldsymbol{x_j}$ are
     - independent $\sigma_{ij}=0$
     - have positive correlation $\sigma_{ij}>0$
     - have negative correlation $\sigma_{ij}<0$

The Multivariate Normal Density

- If Σ is diagonal $\begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$ then the features $\boldsymbol{x_i},\ldots,\boldsymbol{x_j}$ are independent, and

$$p(x) = \prod_{i=1}^{d} \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left[-\frac{(x_i - \mu_i)^2}{2\sigma_i^2} \right]$$

The Multivariate Normal Density

$$p(x) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2} (x-\mu)^t \Sigma^{-1} (x-\mu) \right]$$

$$p(x) = c \cdot \exp\left[-\frac{1}{2} \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 & x_3-\mu_3 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ x_3-\mu_3 \end{bmatrix} \right]$$

normalizing constant

scalar s (single number), the closer s to 0 the larger is p(x)

- Thus ***P***(***x***) is larger for smaller $(x-\mu)^t \Sigma^{-1} (x-\mu)$

$(\boldsymbol{x}-\mu)^t \Sigma^{-1} (\boldsymbol{x}-\mu)$

- Σ is positive semi definite ($\boldsymbol{x}^t \Sigma \boldsymbol{x} >= 0$)
- If $\boldsymbol{x}^t \Sigma \boldsymbol{x} = 0$ for nonzero x then det(Σ)=0. This case is not interesting, $\boldsymbol{p}(\boldsymbol{x})$ is not defined
  1. one feature vector is a constant (has zero variance)
  2. or two components are multiples of each other
- so we will assume Σ is positive definite ($\boldsymbol{x}^t \Sigma \boldsymbol{x} > 0$)
- If Σ is positive definite then so is Σ^{-1}

$(\boldsymbol{x}-\mu)^t \Sigma^{-1} (\boldsymbol{x}-\mu)$

- Positive definite matrix of size ***d*** by ***d*** has ***d*** distinct real eigenvalues and its ***d*** eigenvectors are orthogonal
- Thus if Φ is a matrix whose columns are normalized eigenvectors of Σ, then $\Phi^{-1} = \Phi^{t}$
- $\Sigma\Phi = \Phi\Lambda$ where Λ is a diagonal matrix with corresponding eigenvalues on the diagonal
- Thus $\Sigma = \Phi\Lambda\Phi^{-1}$ and $\Sigma^{-1} = \Phi\Lambda^{-1}\Phi^{-1}$
- Thus if $\Lambda^{-1/2}$ denotes matrix s.t. $\Lambda^{-1/2}\Lambda^{-1/2} = \Lambda^{-1}$

$$\Sigma^{-1} = \left(\Phi\Lambda^{-\frac{1}{2}}\right)\left(\Phi\Lambda^{-\frac{1}{2}}\right)^{t} = MM^{t}$$

$(x-\mu)^t \Sigma^{-1} (x-\mu)$

- Thus

$$(x-\mu)^t \Sigma^{-1} (x-\mu) = (x-\mu)^t M M^t (x-\mu) =$$

$$= \left(M^t (x-\mu)\right)^t \left(M^t (x-\mu)\right) = \left|M^t (x-\mu)\right|^2$$

- Thus $\boxed{(x-\mu)^t \Sigma^{-1} (x-\mu) = \left|M^t (x-\mu)\right|^2}$

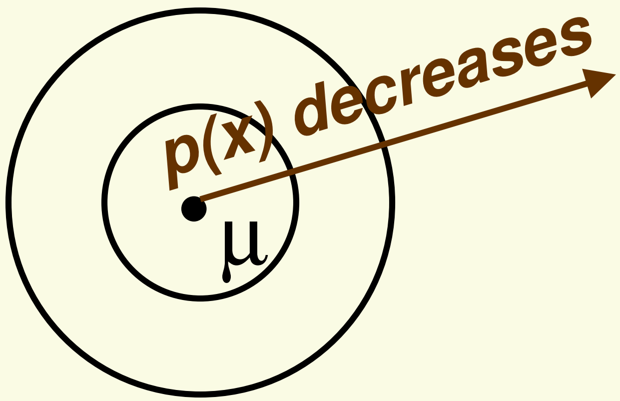
  where $M^t = \underbrace{\Lambda^{-\frac{1}{2}}}_{\text{scaling matrix}} \underbrace{\Phi^{-1}}_{\text{rotation matrix}}$

- Points x which satisfy $\left|M^t (x-\mu)\right|^2 = const$ lie on an ellipse

$$(\boldsymbol{x}-\mu)^t\,\Sigma^{-1}(\boldsymbol{x}-\mu)$$

$$(\boldsymbol{x}-\mu)^t(\boldsymbol{x}-\mu)$$

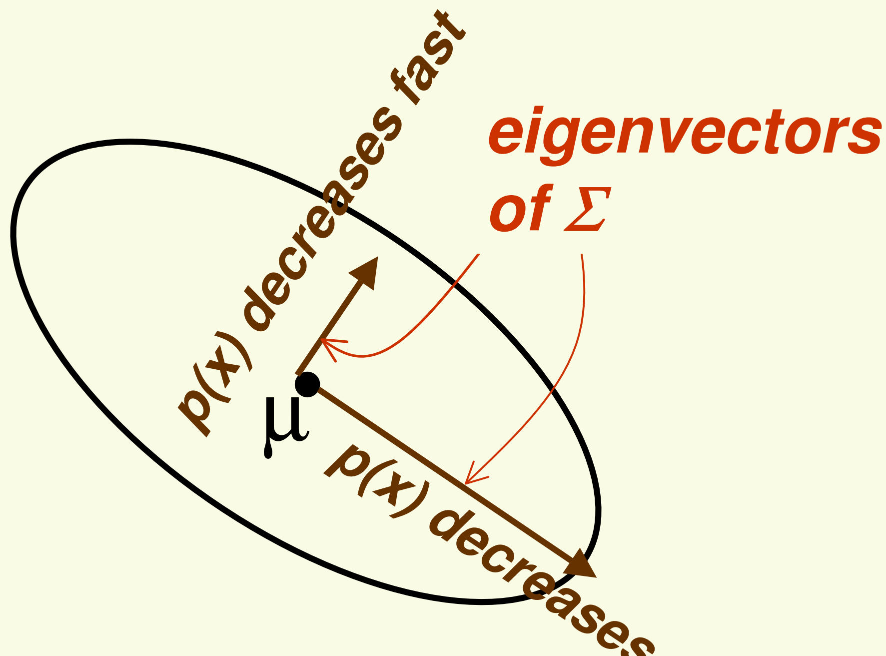
usual (Eucledian) distance between x and μ



points ***x*** at equal Eucledian distance from μ lie on a circle

$$(\boldsymbol{x}-\mu)^t\sum^{-1}(\boldsymbol{x}-\mu)$$

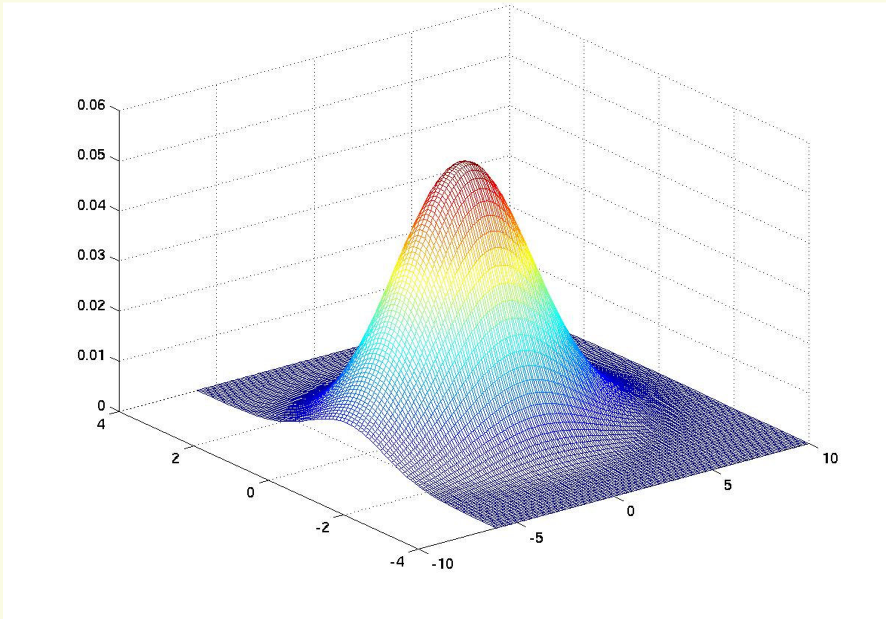
Mahalanobis distance between x and μ



points ***x*** at equal Mahalanobis distance from μ lie on an ellipse: Σ stretches circles to ellipses

2-d Multivariate Normal Density

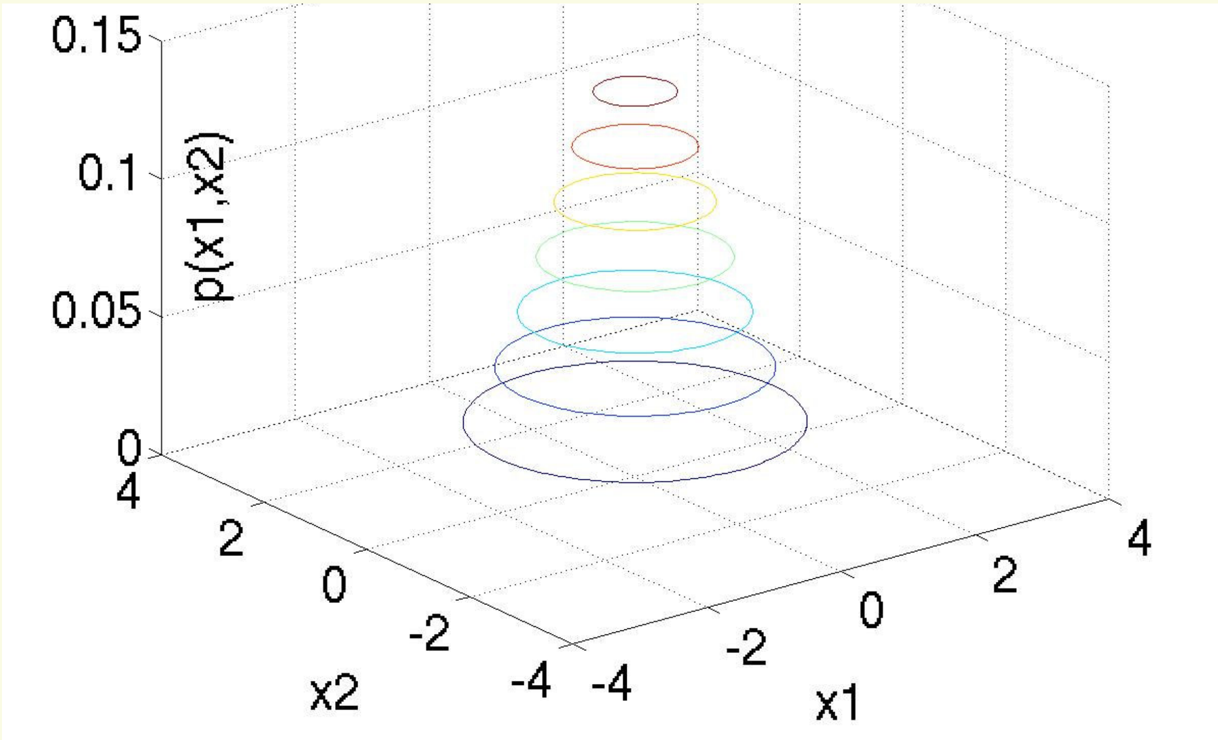
- Can you see much in this graph?



- At most you can see that the mean is around [0,0], but can't really tell if $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are correlated

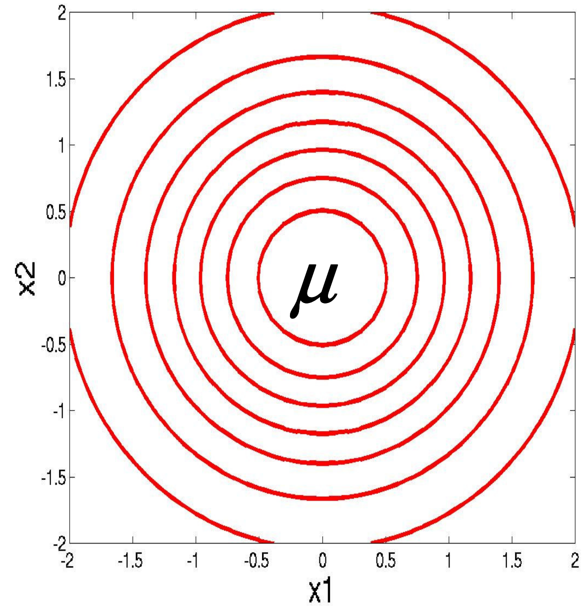
2-d Multivariate Normal Density

- How about this graph?



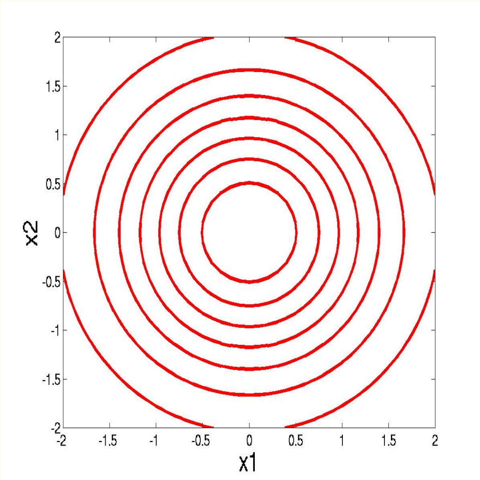
2-d Multivariate Normal Density

- Level curves graph
  - $p(x)$ is constant along each contour
  - topological map of 3-d surface



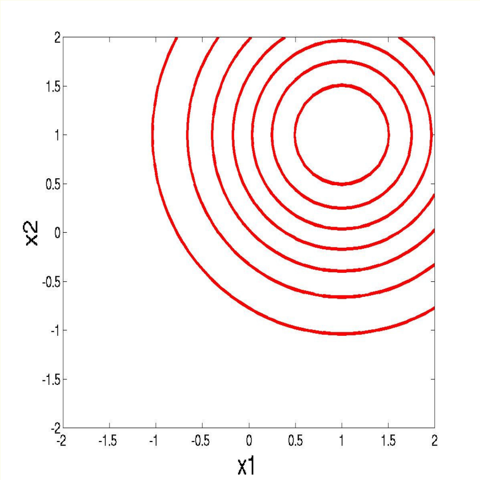
- Now we can see much more
  - x_1 and x_1 are independent
  - σ_1^2 and σ_2^2 are equal

2-d Multivariate Normal Density



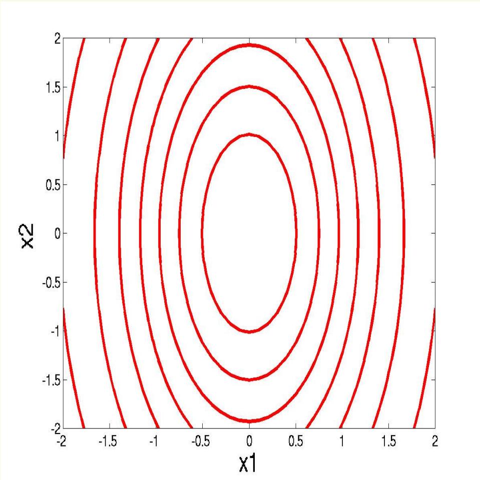
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mu = [0,0]$$



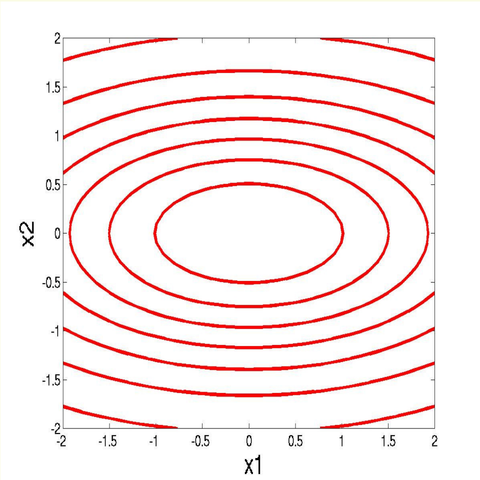
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mu = [1,1]$$



$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

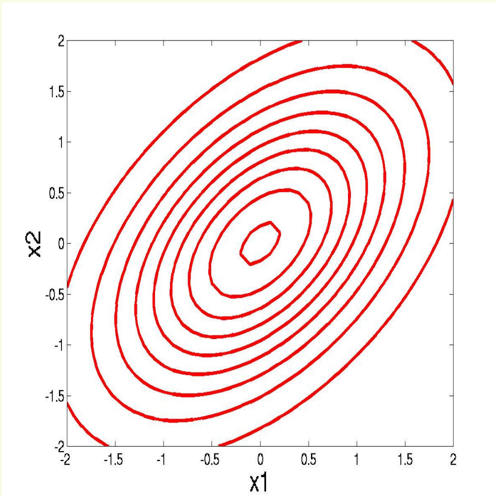
$$\mu = [0,0]$$



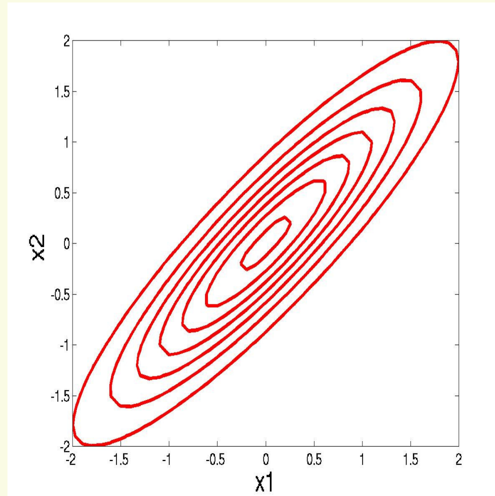
$$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mu = [0,0]$$

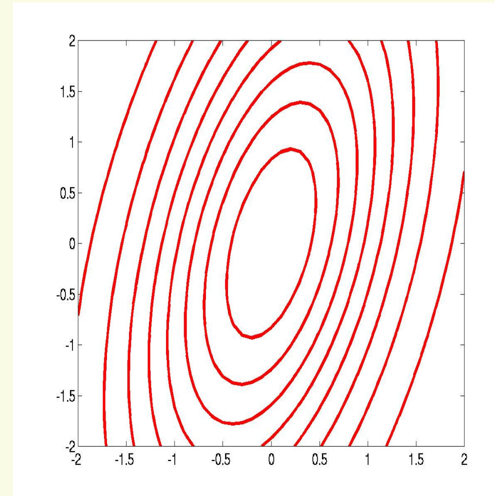
2-d Multivariate Normal Density $\mu = [0,0]$



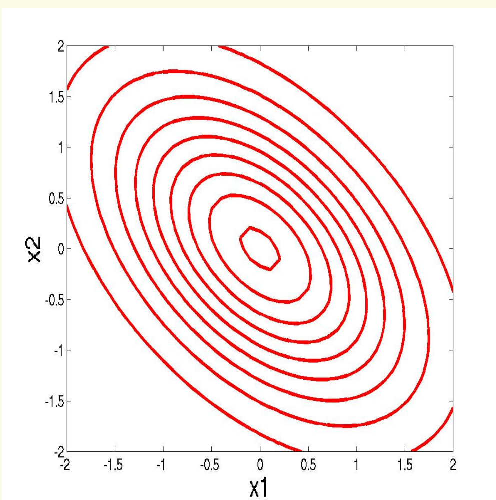
$$\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$



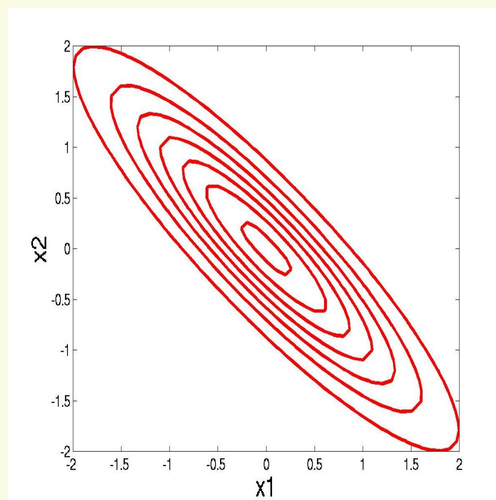
$$\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$$



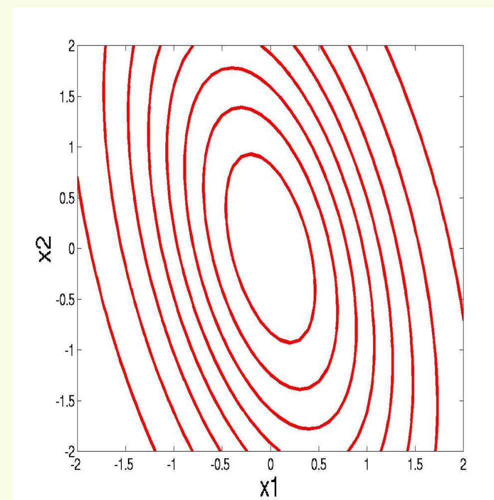
$$\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 4 \end{bmatrix}$$



$$\Sigma = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$



$$\Sigma = \begin{bmatrix} 1 & -0.9 \\ -0.9 & 1 \end{bmatrix}$$

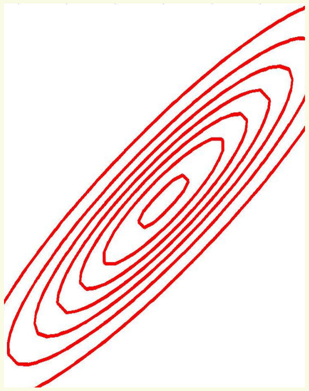


$$\Sigma = \begin{bmatrix} 1 & -0.9 \\ -0.9 & 4 \end{bmatrix}$$

The Multivariate Normal Density

- If X has density $N(\mu,\Sigma)$ then AX has density $N(A^t\mu, A^t\Sigma A)$
  - Thus X can be transformed into a spherical normal variable (covariance of spherical density is the identity matrix I) with whitening transform

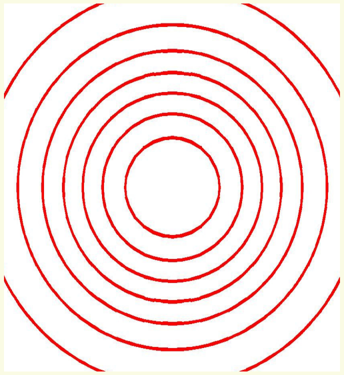
X



$$\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$$

$$A_w = \Phi\Lambda^{-\frac{1}{2}}$$

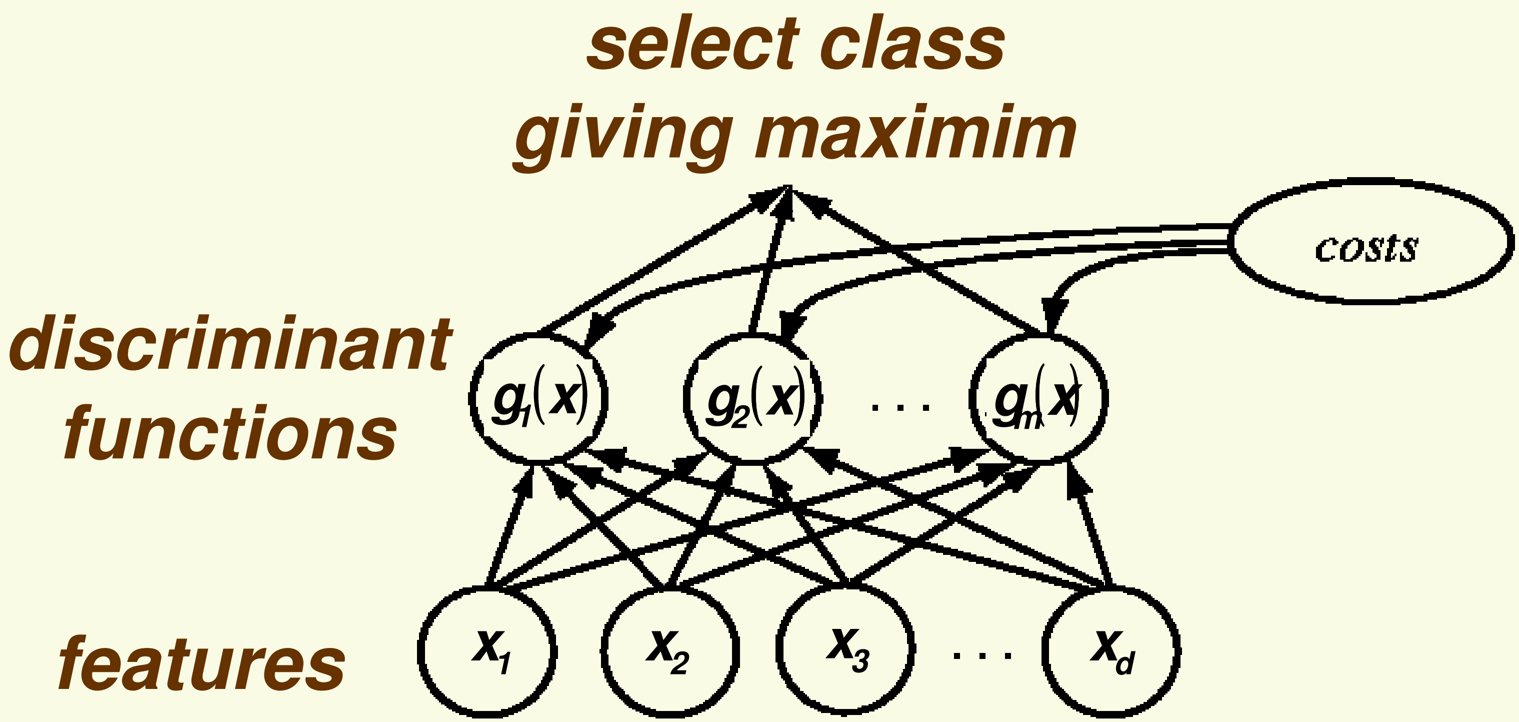
AX



$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Discriminant Functions

- Classifier can be viewed as network which computes ***m*** discriminant functions and selects category corresponding to the largest discriminant



- $g_i(x)$ can be replaced with any monotonically increasing function, the results will be unchanged

Discriminant Functions

- The minimum error-rate classification is achieved by the discriminant function

$$g_i(x) = P(c_i\,|x) = P(x|c_i)P(c_i)/P(x)$$

- Since the observation *x* is independent of the class, the equivalent discriminant function is

$$g_i(x) = P(x|c_i)P(c_i)$$

- For normal density, convinient to take logarithms. Since logarithm is a monotonically increasing function, the equivalent discriminant function is

$$g_i(x) = \ln P(x|c_i) + \ln P(c_i)$$

Discriminant Functions for the Normal Density

- Suppose we for class c_i its class conditional density $p(x|c_i)$ is $N(\mu_i,\Sigma_i)$

$$p(\boldsymbol{x} \mid c_i) = \frac{1}{\boxed{(2\pi)^{d/2}}\boxed{|\Sigma_i|^{1/2}}} \exp\boxed{\left[-\frac{1}{2}(\boldsymbol{x}-\mu_i)^t \Sigma_i^{-1}(\boldsymbol{x}-\mu_i)\right]}$$

- Discriminant function $g_i(x) = \ln P(x|c_i) + \boxed{\ln P(c_i)}$

- Plug in $p(x|c_i)$ and $P(c_i)$ get

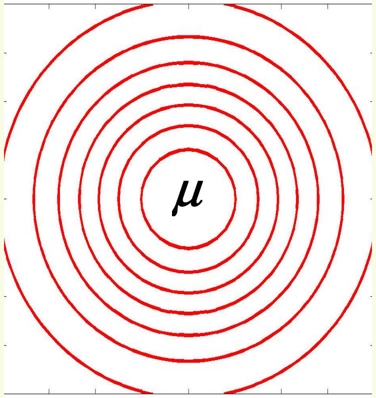
constant for all i

$$g_i(\boldsymbol{x}) = \boxed{-\frac{1}{2}(\boldsymbol{x}-\mu_i)^t \Sigma_i^{-1}(\boldsymbol{x}-\mu_i)} - \boxed{\cancel{\frac{d}{2}\ln 2\pi}} - \boxed{\frac{1}{2}\ln|\Sigma_i|} + \boxed{\ln P(c_i)}$$

$$\boxed{g_i(\boldsymbol{x}) = -\frac{1}{2}(\boldsymbol{x}-\mu_i)^t \Sigma_i^{-1}(\boldsymbol{x}-\mu_i) - \frac{1}{2}\ln|\Sigma_i| + \ln P(c_i)}$$

Case $\Sigma_i = \sigma^2 I$

- That is $\sum_i = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} = \sigma^2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- In this case, features x_1, $x_{2.}$,…, x_d are independent with different means and equal variances σ^2



Case $\Sigma_i = \sigma^2 I$

- Discriminant function

$$g_i(x) = -\frac{1}{2}(x-\mu_i)^t \sum{}^{-1}(x-\mu_i) - \frac{1}{2}\ln|\Sigma_i| + \ln P(c_i)$$

- Det(Σ_i)=σ^{2d} and $\Sigma_i^{-1}=(1/\sigma^2)I$ $= \begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 \\ 0 & \frac{1}{\sigma^2} & 0 \\ 0 & 0 & \frac{1}{\sigma^2} \end{bmatrix}$

- Can simplify discriminant function

$$g_i(x) = -\frac{1}{2}(x-\mu_i)^t \frac{I}{\sigma^2}(x-\mu_i) - \cancel{\frac{1}{2}\ln\left(\sigma^{2d}\right)} + \ln P(c_i)$$

constant for all i

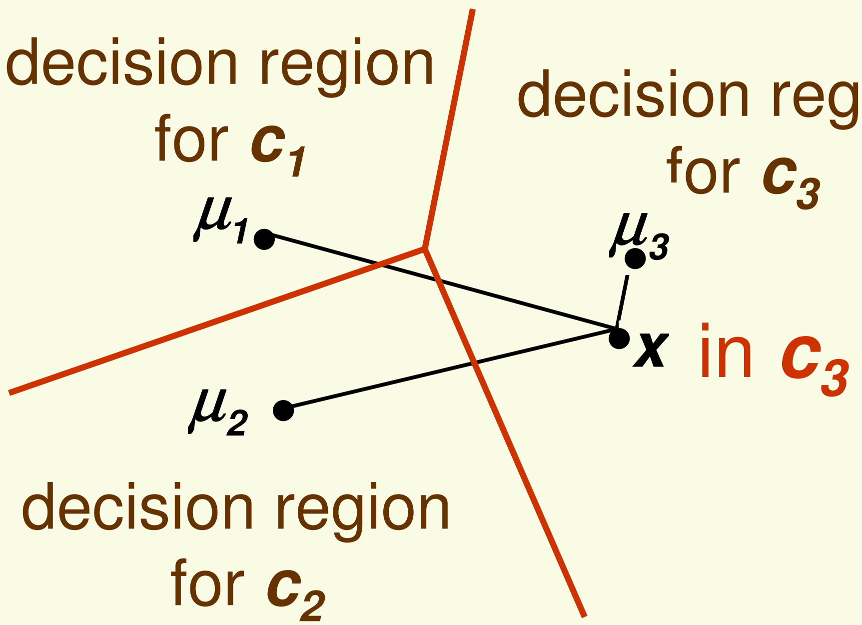
$$g_i(x) = -\frac{1}{2\sigma^2}(x-\mu_i)^t(x-\mu_i) + \ln P(c_i) =$$

$$= -\frac{1}{2\sigma^2}\left|x-\mu_i\right|^2 + \ln P(c_i)$$

Case $\Sigma_i = \sigma^2 I$ Geometric Interpretation

If $\ln P(c_i) = \ln P(c_j)$, **then**

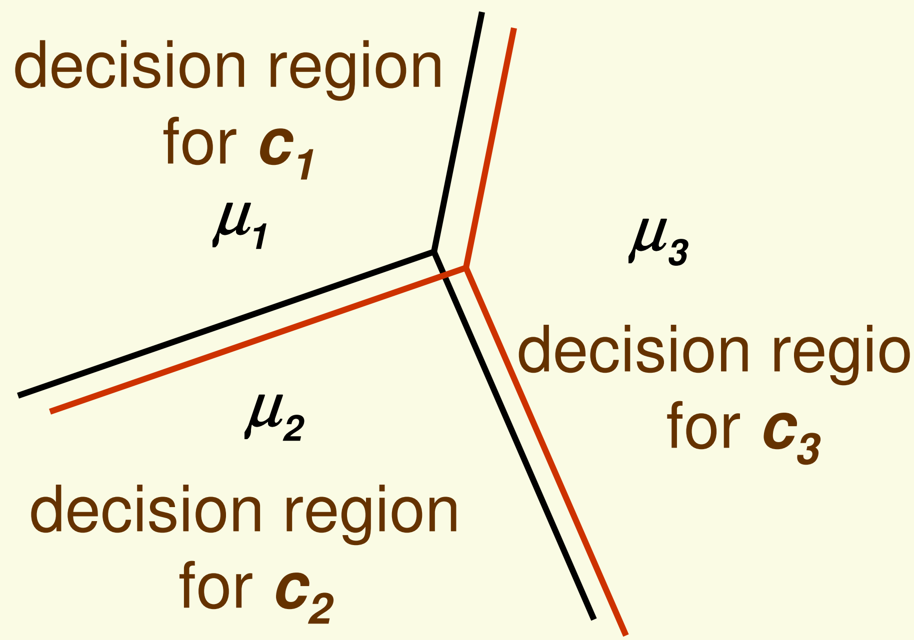
$$g_i(x) = -\left|x - \mu_i\right|^2$$



voronoi diagram: *points in each cell are closer to the mean in that cell than to any other mean*

If $\ln P(c_i) \neq \ln P(c_j)$, **then**

$$g_i(x) = -\frac{1}{2\sigma^2}\left|x - \mu_i\right|^2 + \ln P(c_i)$$



Case $\Sigma_i = \sigma^2 I$

$$g_i(x) = -\frac{1}{2\sigma^2}(x-\mu_i)^t(x-\mu_i) + \ln P(c_i) =$$

$$= -\frac{1}{2\sigma^2}(\cancel{x^t x} - \mu_i^t x - x^t \mu_i + \mu_i^t \mu_i) + \ln P(c_i)$$

constant
for all classes

$$g_i(x) = -\frac{1}{2\sigma^2}(-2\mu_i^t x + \mu_i^t \mu_i) + \ln P(c_i) = \frac{\mu_i^t}{\sigma^2} x + \left(-\frac{\mu_i^t \mu_i}{2\sigma^2} + \ln P(c_i)\right)$$

$$g_i(x) = w_i^t x + w_{i0}$$

discriminant function is linear

Case $\Sigma_i = \sigma^2 I$

constant in x

$$g_i(\mathbf{x}) = \mathbf{w}_i^t \mathbf{x} + w_{i0}$$

linear in x:

$$\mathbf{w}_i^t \mathbf{x} = \sum_{i=1}^{d} w_i x_i$$

- Thus discriminant function is linear,
- Therefore the decision boundaries $\boldsymbol{g_i(x)=g_j(x)}$ are linear
  - lines if $\boldsymbol{x}$ has dimension 2
  - planes if $\boldsymbol{x}$ has dimension 3
  - hyper-planes if $\boldsymbol{x}$ has dimension larger than 3

Case $\Sigma_i = \sigma^2 I$: Example

- 3 classes, each 2-dimensional Gaussian with

$$\mu_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \mu_3 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \quad \Sigma_1 = \Sigma_2 = \Sigma_3 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

- Priors $P(c_1) = P(c_2) = \frac{1}{4}$ and $P(c_3) = \frac{1}{2}$
- Discriminant function is $g_i(x) = \frac{\mu_i^t}{\sigma^2} x + \left(-\frac{\mu_i^t \mu_i}{2\sigma^2} + \ln P(c_i) \right)$
- Plug in parameters for each class

$$g_1(x) = \frac{[1\ 2]}{3} x + \left(-\frac{5}{6} - 1.38\right) \qquad g_2(x) = \frac{[4\ 6]}{3} x + \left(-\frac{52}{6} - 1.38\right)$$

$$g_3(x) = \frac{[-2\ 4]}{3} x + \left(-\frac{20}{6} - 0.69\right)$$

Case $\Sigma_i = \sigma^2 I$: Example

- Need to find out when $\boldsymbol{g_i}(\boldsymbol{x}) < \boldsymbol{g_j}(\boldsymbol{x})$ for i,j=1,2,3
- Can be done by solving $\boldsymbol{g_i}(\boldsymbol{x}) = \boldsymbol{g_j}(\boldsymbol{x})$ for i,j=1,2,3
- Let's take $\boldsymbol{g_1}(\boldsymbol{x}) = \boldsymbol{g_2}(\boldsymbol{x})$ first

$$\frac{[1\ 2]}{3}x+(-\frac{5}{6}-1.38)=\frac{[4\ 6]}{3}x+(-\frac{52}{6}-1.38)$$

- Simplifying, $\frac{[-3\ \ -4]}{3}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{47}{6}$

$$-x_1-\frac{4}{3}x_2=-\frac{47}{6}$$

line equation

Case $\Sigma_i = \sigma^2 I$: Example

- Next solve $g_2(\mathbf{x}) = g_3(\mathbf{x})$

$$2x_1 + \frac{2}{3}x_2 = 6.02$$

- Almost finally solve $g_1(\mathbf{x}) = g_3(\mathbf{x})$

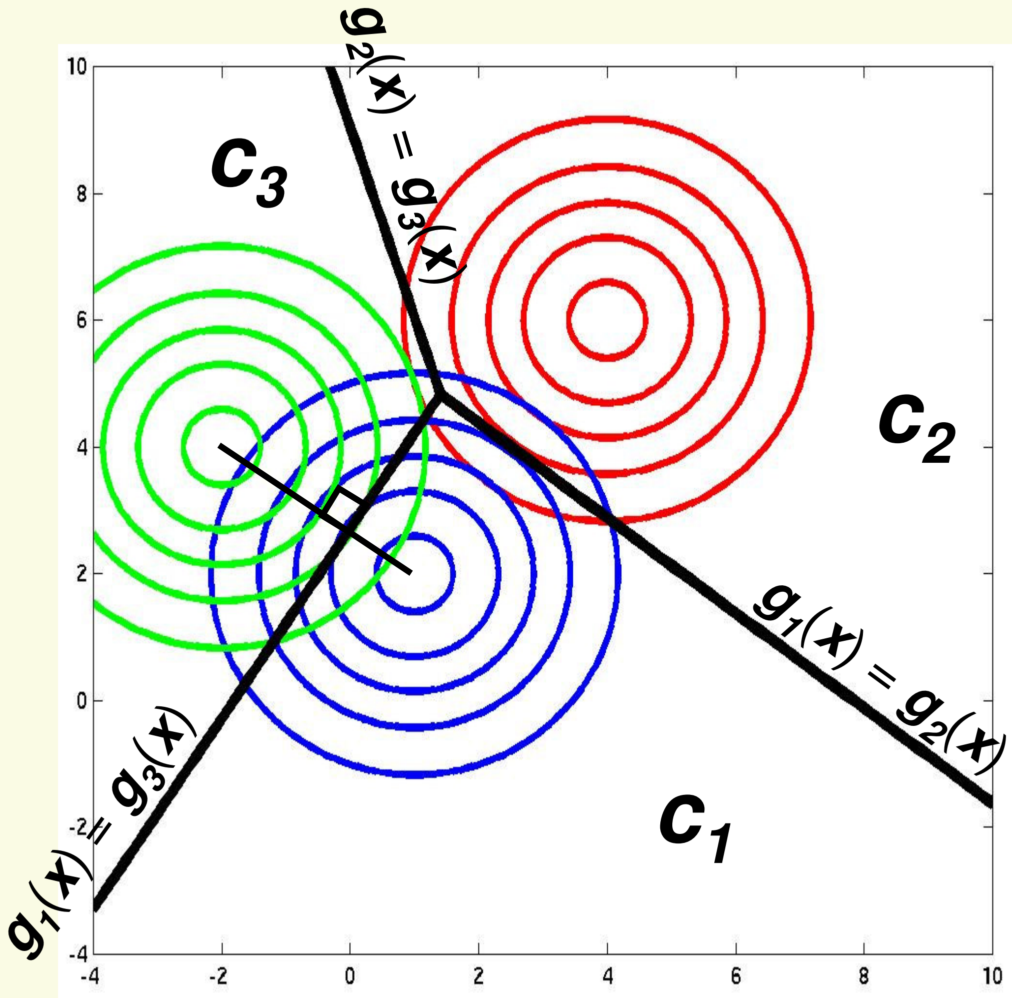
$$x_1 - \frac{2}{3}x_2 = -1.81$$

- And finally solve $g_1(\mathbf{x}) = g_2(\mathbf{x}) = g_3(\mathbf{x})$

$$x_1 = 1.4 \quad \text{and} \quad x_2 = 4.82$$

Case $\Sigma_i = \sigma^2 I$: Example

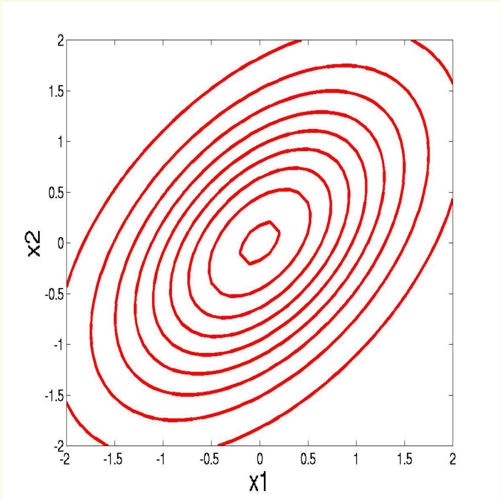
- Priors $P(c_1) = P(c_2) = \frac{1}{4}$ and $P(c_3) = \frac{1}{2}$



lines connecting means are perpendicular to decision boundaries

Case $\Sigma_i = \Sigma$

- Covariance matrices are equal but arbitrary
- In this case, features x_1, $x_{2.}$,…, x_d are not necessarily independent



$$\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Case $\Sigma_i = \Sigma$

- Discriminant function

$$g_i(x) = -\frac{1}{2}(x-\mu_i)^t \Sigma^{-1}(x-\mu_i) - \cancel{\frac{1}{2}\ln|\Sigma_i|} + \ln P(c_i)$$

constant for all classes

- Discriminant function becomes

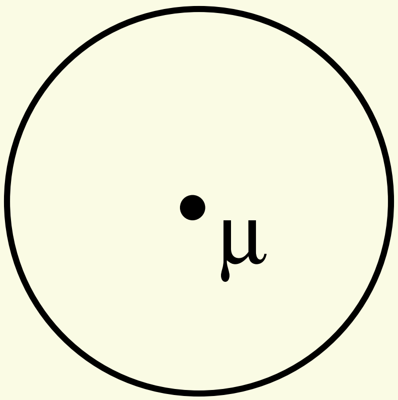
$$g_i(x) = -\frac{1}{2}\underbrace{(x-\mu_i)^t \Sigma^{-1}(x-\mu_i)}_{\text{squared Mahalanobis Distance}} + \ln P(c_i)$$

- Mahalanobis Distance $\|x-y\|^2_{\Sigma^{-1}} = (x-y)^t \Sigma^{-1}(x-y)$
- If $\Sigma = I$, Mahalanobis Distance becomes usual Eucledian distance

$$\|x-y\|^2_{I^{-1}} = (x-y)^t(x-y)$$

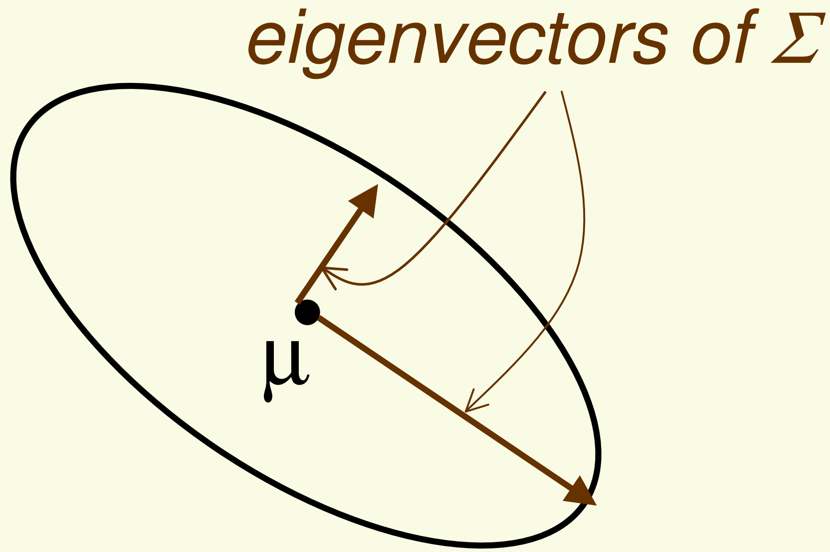
Eucledian vs. Mahalanobis Distances

$$\left| \boldsymbol{x} - \mu \right|^2 = (\boldsymbol{x} - \mu)^t (\boldsymbol{x} - \mu)$$



points $\boldsymbol{x}$ at equal Eucledian distance from μ lie on a circle

$$\left\| \boldsymbol{x} - \mu \right\|^2_{\Sigma^{-1}} = (\boldsymbol{x} - \mu)^t \sum{}^{-1} (\boldsymbol{x} - \mu)$$

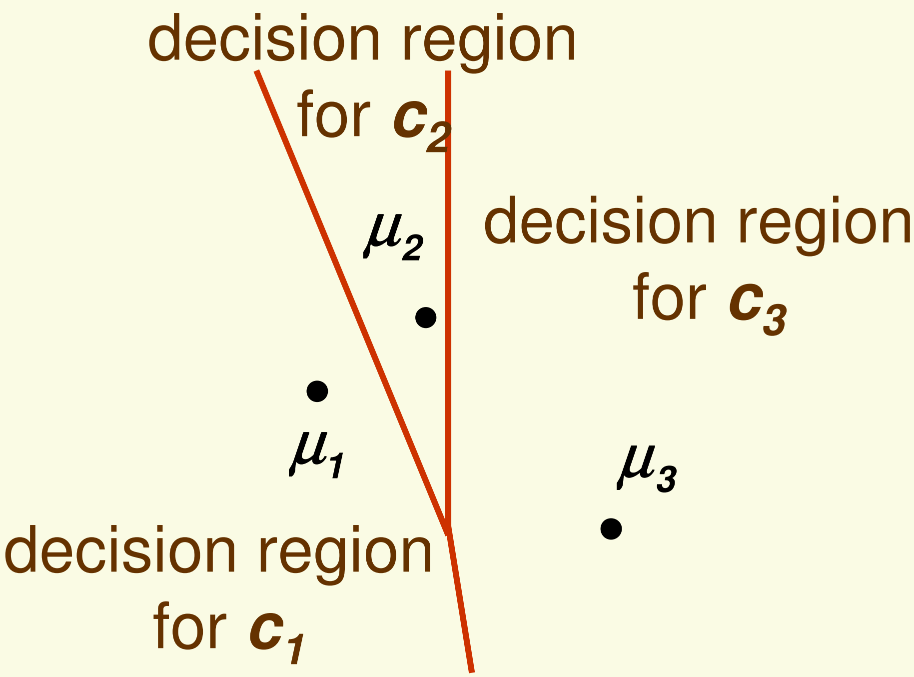


points $\boldsymbol{x}$ at equal Mahalanobis distance from μ lie on an ellipse: Σ stretches cirles to ellipses

Case $\Sigma_i = \Sigma$ Geometric Interpretation

If $\ln P(c_i) = \ln P(c_j)$, **then**

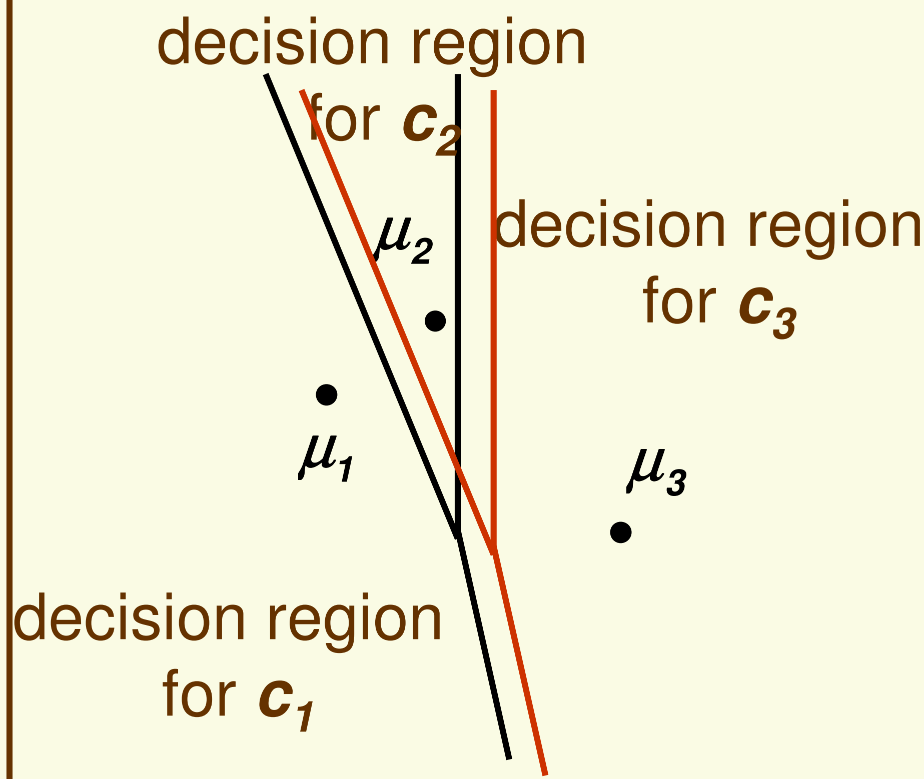
$$g_i(x) = -\|x - \mu_i\|_{\Sigma^{-1}}$$



points in each cell are closer to the mean in that cell than to any other mean under Mahalanobis distance

If $\ln P(c_i) \neq \ln P(c_j)$, **then**

$$g_i(x) = -\frac{1}{2}\|x - \mu_i\|_{\Sigma^{-1}} + \ln P(c_i)$$



Case $\Sigma_i = \Sigma$

- Can simplify discriminant function:

$$g_i(x) = -\frac{1}{2}(x-\mu_i)^t \sum{}^{-1}(x-\mu_i) + \ln P(c_i) =$$

$$= -\frac{1}{2}\left(x^t\Sigma^{-1}x - \mu_i^t\Sigma^{-1}x - x^t\Sigma^{-1}\mu_i + \mu_i^t\Sigma^{-1}\mu_i\right) + \ln P(c_i) =$$

$$= -\frac{1}{2}\left(\cancel{x^t\Sigma^{-1}x} - 2\mu_i^t\Sigma^{-1}x + \mu_i^t\Sigma^{-1}\mu_i\right) + \ln P(c_i) =$$

constant for all classes

$$= -\frac{1}{2}\left(-2\mu_i^t\Sigma^{-1}x + \mu_i^t\Sigma^{-1}\mu_i\right) + \ln P(c_i)$$

$$= \mu_i^t\Sigma^{-1}x + \left(\ln P(c_i) - \frac{1}{2}\mu_i^t\Sigma^{-1}\mu_i\right) = w_i^t x + w_{i0}$$

- Thus in this case discriminant is also linear

Case $\Sigma_i = \Sigma$: Example

- 3 classes, each 2-dimensional Gaussian with

$$\mu_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad \mu_3 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \quad \Sigma_1 = \Sigma_2 = \Sigma_3 = \begin{bmatrix} 1 & -1.5 \\ -1.5 & 4 \end{bmatrix}$$

$$P(c_1) = P(c_2) = \frac{1}{4} \qquad P(c_3) = \frac{1}{2}$$

- Again can be done by solving $\boldsymbol{g_i}(\boldsymbol{x}) = \boldsymbol{g_j}(\boldsymbol{x})$ for i,j=1,2,3

Case $\Sigma_i = \Sigma$: Example

- Let's solve in general first

$$g_j(x) = g_i(x)$$

$$\mu_j^t \Sigma^{-1} x + \underbrace{\left(\ln P(c_j) - \frac{1}{2}\mu_j^t \Sigma^{-1} \mu_j \right)} = \mu_i^t \Sigma^{-1} x + \underbrace{\left(\ln P(c_i) - \frac{1}{2}\mu_i^t \Sigma^{-1} \mu_i \right)}$$

- Let's regroup the terms

$$\left(\mu_j^t \Sigma^{-1} - \mu_i^t \Sigma^{-1}\right) x = -\left(\ln P(c_j) - \frac{1}{2}\mu_j^t \Sigma^{-1} \mu_j \right) + \left(\ln P(c_i) - \frac{1}{2}\mu_i^t \Sigma^{-1} \mu_i \right)$$

- We get the line where $g_j(x) = g_i(x)$

$$\left(\mu_j^t - \mu_i^t\right)\Sigma^{-1} x = \left(\ln \frac{P(c_i)}{P(c_j)} + \frac{1}{2}\mu_j^t \Sigma^{-1} \mu_j - \frac{1}{2}\mu_i^t \Sigma^{-1} \mu_i \right)$$

row vector

scalar

Case $\Sigma_i = \Sigma$: Example

$$\left(\mu_j^t - \mu_i^t\right)\Sigma^{-1}x = \left(\ln\frac{P(c_i)}{P(c_j)} + \frac{1}{2}\mu_j^t\Sigma^{-1}\mu_j - \frac{1}{2}\mu_i^t\Sigma^{-1}\mu_i \right)$$

- Now substitute for i,j=1,2

$$\begin{bmatrix} -2 & 0 \end{bmatrix} x = 0$$
$$x_1 = 0$$

- Now substitute for i,j=2,3

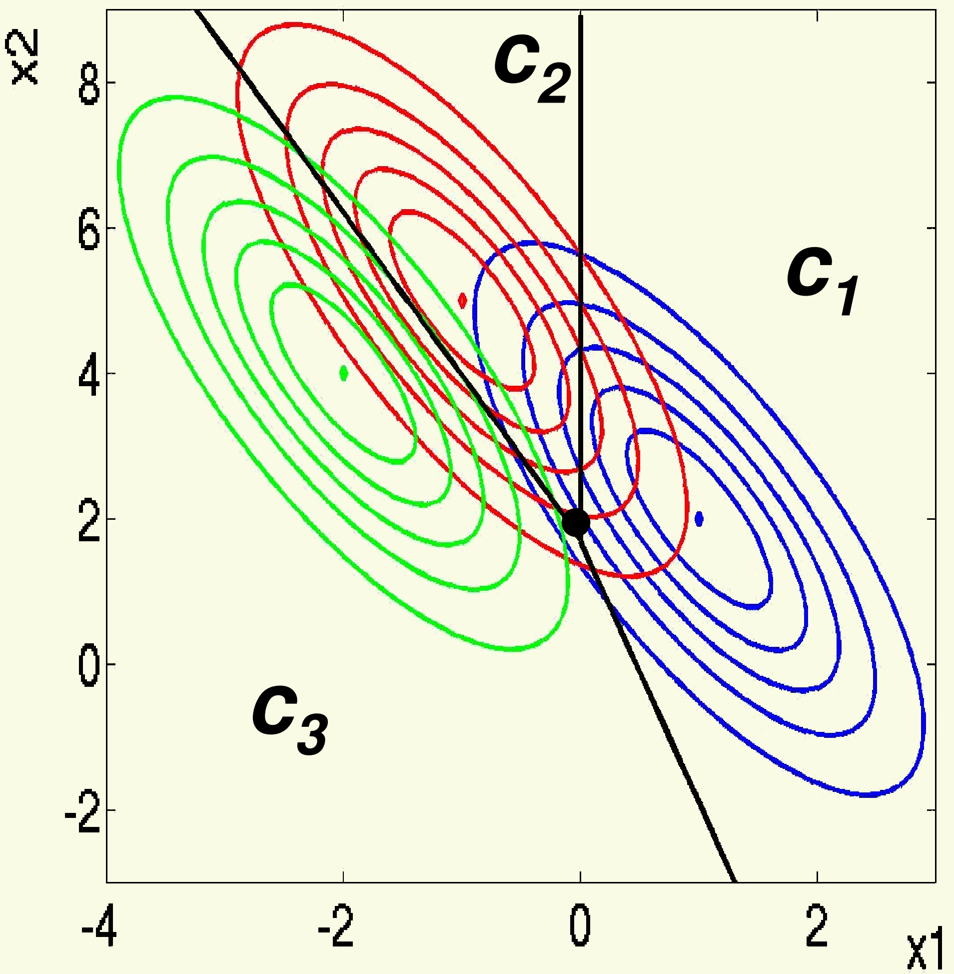
$$\begin{bmatrix} -3.14 & -1.4 \end{bmatrix} x = -2.41$$
$$3.14x_1 + 1.4x_2 = 2.41$$

- Now substitute for i,j=1,3

$$\begin{bmatrix} -5.14 & -1.43 \end{bmatrix} x = -2.41$$
$$5.14x_1 + 1.43x_2 = 2.41$$

Case $\Sigma_i = \Sigma$: Example

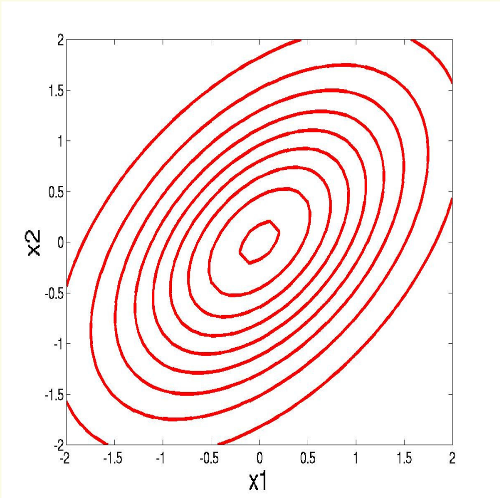
- Priors $P(c_1) = P(c_2) = \frac{1}{4}$ and $P(c_3) = \frac{1}{2}$



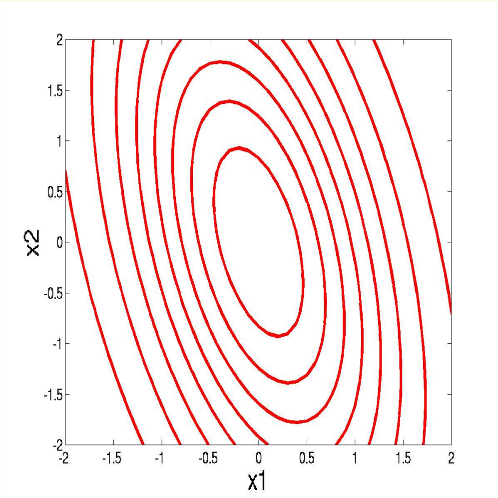
*lines connecting means are **not** in general perpendicular to decision boundaries*

General Case Σ_i are arbitrary

- Covariance matrices for each class are arbitrary
- In this case, features x_1, $x_{2.}$,..., x_d are not necessarily independent



$$\Sigma_i = \begin{bmatrix} \mathbf{1} & \mathbf{0.5} \\ \mathbf{0.5} & \mathbf{1} \end{bmatrix}$$



$$\Sigma_j = \begin{bmatrix} \mathbf{1} & -\mathbf{0.9} \\ -\mathbf{0.9} & \mathbf{4} \end{bmatrix}$$

General Case Σ_i are arbitrary

- From previous discussion,

$$\boxed{g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t \Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{1}{2}\ln|\Sigma_i| + \ln P(c_i)}$$

- This can't be simplified, but we can rearrange it:

$$g_i(\mathbf{x}) = -\frac{1}{2}\left(\mathbf{x}^t\Sigma_i^{-1}\mathbf{x} - 2\mu_i^t\Sigma_i^{-1}\mathbf{x} + \mu_i^t\Sigma_i^{-1}\mu_i\right) - \frac{1}{2}\ln|\Sigma_i| + \ln P(c_i)$$

$$g_i(\mathbf{x}) = \mathbf{x}^t\left(-\frac{1}{2}\Sigma_i^{-1}\right)\mathbf{x} + \mu_i^t\Sigma_i^{-1}\mathbf{x} + \left(-\frac{1}{2}\mu_i^t\Sigma_i^{-1}\mu_i - \frac{1}{2}\ln|\Sigma_i| + \ln P(c_i)\right)$$

$$g_i(\mathbf{x}) = \mathbf{x}^t W \mathbf{x} + \mathbf{w}^t\mathbf{x} + w_{i0}$$

General Case Σ_i are arbitrary

linear in x

constant in x

$$g_i(x) = x^t W x + w^t x + w_{i0}$$

quadratic in x since

$$x^t W x = \sum_{j=1}^{d}\sum_{i=1}^{d} w_{ij} x_i x_j = \sum_{i,j=1}^{d} w_{ij} x_i x_j$$

- Thus the discriminant function is quadratic
- Therefore the decision boundaries are quadratic (ellipses and parabolloids)

General Case Σ_i are arbitrary: Example

- 3 classes, each 2-dimensional Gaussian with

$$\mu_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \quad \mu_3 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 2 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 2 & -2 \\ -2 & 7 \end{bmatrix} \quad \Sigma_3 = \begin{bmatrix} 1 & 1.5 \\ 1.5 & 3 \end{bmatrix}$$

- Priors: $P(c_1) = P(c_2) = \frac{1}{4}$ and $P(c_3) = \frac{1}{2}$

- Again can be done by solving $g_i(x) = g_j(x)$ for i,j=1,2,3

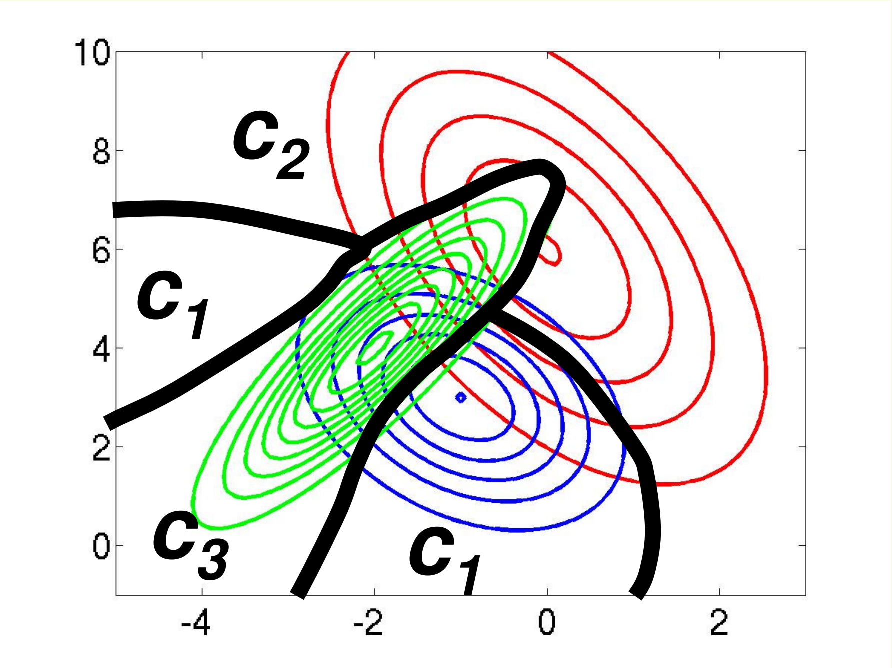
$$g_i(x) = x^t\left(-\frac{1}{2}\Sigma_i^{-1}\right)x + \mu_i^t\Sigma_i^{-1}x + \left(-\frac{1}{2}\mu_i^t\Sigma_i^{-1}\mu_i - \frac{1}{2}\ln\left|\Sigma_i\right| + \ln P(c_i)\right)$$

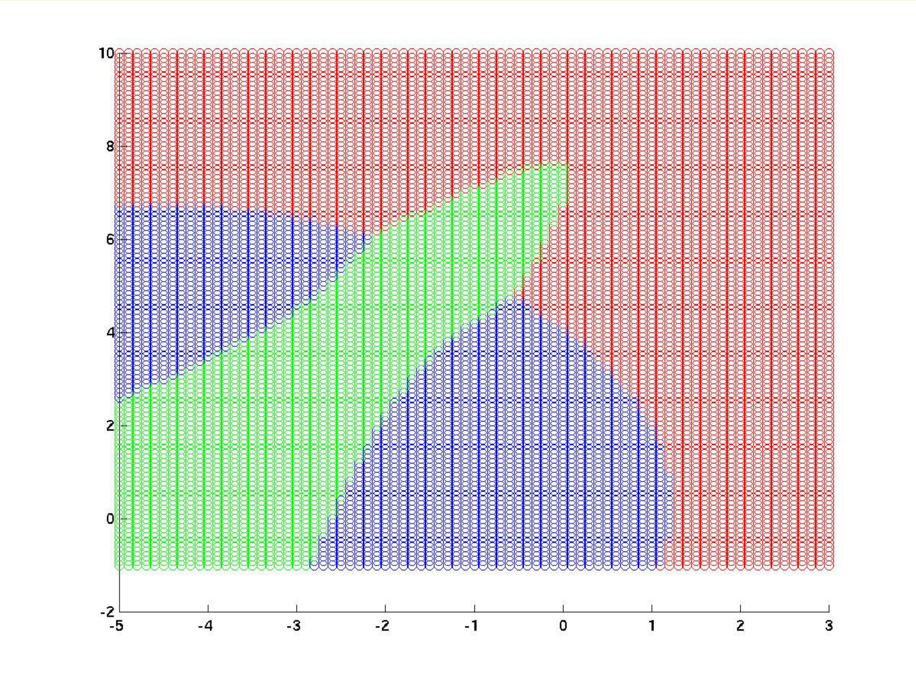
- Need to solve a bunch of quadratic inequalities of 2 variables

General Case Σ_i are arbitrary: Example

$$\mu_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \quad \mu_3 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \qquad \Sigma_1 = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 2 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 2 & -2 \\ -2 & 7 \end{bmatrix} \quad \Sigma_3 = \begin{bmatrix} 1 & 1.5 \\ 1.5 & 3 \end{bmatrix}$$

$$P(c_1) = P(c_2) = \frac{1}{4} \qquad P(c_3) = \frac{1}{2}$$





Important Points

- The Bayes classifier when classes are normally distributed is in general quadratic
  - If covariance matrices are equal and proportional to identity matrix, the Bayes classifier is linear
    - If, in addition the priors on classes are equal, the Bayes classifier is the minimum Eucledian distance classifier
  - If covariance matrices are equal, the Bayes classifier is linear
    - If, in addition the priors on classes are equal, the Bayes classifier is the minimum Mahalanobis distance classifier
- Popular classifiers (Euclidean and Mahalanobis distance) are optimal only if distribution of data is appropriate (normal)