CS 2210 Data Structures and Algorithms

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Introduction

In this and next lectures we introduce the two core components of every computer program:

• Data structures
• Algorithms
A Fundamental Problem

Given a set $S$ of elements and a particular element $x$ the search problem is to decide whether $x$ is in $S$. 
The Search Problem

This problem has a large number of applications:

• $S = \text{Names in a phone book}$
  
  $x = \text{name of a person}$
The Search Problem

This problem has a large number of applications:

- $S = \text{Student records}$
- $x = \text{student ID}$
The Search Problem

This problem has a large number of applications:

• \( S = \) Variables in a program

\( x = \) name of a variable

```java
/* When the user has selected a play, this method is invoked to
process the selected play */
public void actionPerformed(ActionEvent event) {
    if(event.getSource() instanceof JButton) { /* Some position of the
        board was selected */

        int row = -1, col = -1;
        PosPlay pos;

        if (game_ended) System.exit(0);
        /* Find out which position was selected by the player */
        for (int i = 0; i < board_size; i++) {
            for (int j = 0; j < board_size; j++)
                if(event.getSource() == board[i][j]) {
                    row = i;
                    col = j;
                    break;
                }
        }
        if (row != -1) break;
    }
```
The Search Problem

This problem has a large number of applications:

- $S = \text{Web host names}$
- $x = \text{URL}$
Solving a Problem

The solution of a problem has 2 parts:

• How to organize data

• How to solve the problem
Solving a Problem

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  **Data structure:**
  ➢ a systematic way of organizing and accessing data

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  Data structure:
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  Algorithm:
  ➢ a step-by-step procedure for performing some task in finite time
A First Solution

For simplicity, let us assume that $S$ is a set of \textit{n different} integers stored in non-decreasing order in an array $L$.

\begin{center}
\begin{tabular}{cccccccc}
3 & 9 & 11 & 17 & 18 & 26 & 29 & 43 & 48 & 55 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{tabular}
\end{center}
**Algorithm** LinearSearch \((L, n, x)\)

**Input:** Array \(L\) of size \(n\) and value \(x\)

**Output:** Position \(i\), \(0 \leq i < n\), such that \(L[i] = x\), if \(x\) in \(L\), or \(-1\), if \(x\) not in \(L\)

\[
i \leftarrow 0\\
\text{while } (i < n) \text{ and } (L[i] \neq x) \text{ do}\\
\quad i \leftarrow i + 1\\
\text{if } i = n \text{ then return } -1\\
\text{else return } i
\]
Proving the Correctness of an Algorithm

To prove that an algorithm is correct we need to show 2 things:

• The algorithm terminates
• The algorithm produces the correct output
Correctness of Linear Search

Termination

• $i$ takes values 0, 1, 2, 3, ...
• The while loop cannot perform more than $n$ iterations because of the condition ($i < n$)
Correctness of Linear Search

Correct Output

• The algorithm compares $x$ with $L[0]$, $L[1]$, $L[2]$, ...

• Hence, if $x$ is in $L$ then $x = L[i]$ in some iteration of the **while** loop; this ends the loop and then the algorithm correctly returns the value $i$

• If $x$ is not in $L$ then in some iteration $i = n$; this ends the loop and the algorithm returns -1.
Algorithm BinarySearch (L, x, first, last)

Input: Array L of size n and value x
Output: Position i, 0 ≤ i < n, such that L[i] = x, if x in L, or -1, if x not in L

if first > last then return -1
else mid ← \(\lceil (\text{first} + \text{last}) / 2 \rceil\)

if x = L[mid] then return mid
else if x < L[mid] then
    return BinarySearch (L, x, first, mid - 1)
else return BinarySearch (L, x, mid + 1, last)
Correctness of Binary Search

Termination

• If \( x = L[mid] \) the algorithm terminates
• If \( x < L[mid] \) or \( x > L[mid] \), the value \( L[mid] \) is discarded from the next recursive call. Hence, in each recursive call the size of L decreases by at least 1.
• After a finite # recursive calls the size of L is zero and the algorithm ends
Correctness of Binary Search

Correct Output

• If \( x = L[mid] \) the algorithm correctly returns \( mid \)

• The algorithm only discards values different from \( x \) so if all values of \( L \) are discarded (so \( L \) is empty) it is because \( x \) is not in \( L \) and the algorithm correctly returns \(-1\).
Comparing Algorithms

We have several algorithms for solving the same problem. Which one is better?
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Criteria that we can use to compare algorithms:

• Conceptual simplicity
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- Running time
Comparing Algorithms

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Criteria that we can use to compare algorithms:

• Conceptual simplicity
• Difficulty to implement
• Difficulty to modify
• Running time
• Space (memory) usage
We define the **complexity** of an algorithm as the amount of **computer resources** that it uses.
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We define the complexity of an algorithm as the amount of computer resources that it uses. We are particularly interested in two computer resources: memory and time. Consequently, we define two types of complexity functions:

- **Space complexity**: amount of memory that the algorithm needs.
- **Time complexity**: amount of time needed by the algorithm to complete.
Complexity Function

The complexity of an algorithm is a non-decreasing function on the size of the input.
Types of Complexity Functions

For both kinds of complexity functions we can define 3 cases:

• **Best case**: Least amount of resources needed by the algorithm to solve an instance of the problem of size \( n \).
Types of Complexity Functions

For both kinds of complexity functions we can define 3 cases:

- **Worst case**: Largest amount of resources needed by the algorithm to solve an instance of the problem of size $n$. 
Types of Complexity Functions

For both kinds of complexity functions we can define 3 cases:

• **Average case:**
  amount of resources to solve instance 1 of size n +
  amount of resources to solve instance 2 of size n +
  . . .
  amount of resources to solve last instance of size n

  number of instances of size n
Types of Complexity Functions

In this course we will study worst case complexity.
How do we compute the time complexity of an algorithm?
How do we compute the time complexity of an algorithm?

We need a clock to measure time.
Experimental way of measuring the time complexity
Experimental way of measuring the time complexity

We need:

• a computer
Experimental way of measuring the time complexity

We need:

• a computer

• a compiler for the programming language in which the algorithm will be implemented
Experimental way of measuring the time complexity

We need:

• a computer
• a compiler for the programming language in which the algorithm will be implemented
• an operating system
Experimental way of measuring the time complexity

Drawbacks

• Expensive
Experimental way of measuring the time complexity

Drawbacks

• Expensive
• Time consuming
Experimental way of measuring the time complexity

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• Expensive
• Time consuming
• Results depend on the input selected
Experimental way of measuring the time complexity

Drawbacks

• Expensive
• Time consuming
• Results depend on the input selected
• Results depend on the particular implementation
Computing the time complexity

• We wish to compute the time complexity of an algorithm without having to implement it.
Computing the time complexity

• We wish to compute the time complexity of an algorithm *without having to implement it.*
• We want the time complexity to characterize the performance of an algorithm on *ALL* inputs and all implementations (i.e. all computers and all programming languages).
Algorithm LinearSearch \((L,n,x)\)

**Input:** Array \(L\) of size \(n\) and value \(x\)

**Output:** Position \(i\), \(0 \leq i < n\), such that \(L[i] = x\), if \(x\) in \(L\), or \(-1\), if \(x\) not in \(L\)

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i \leftarrow 0 \\
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\quad i \leftarrow i+1 \\
\textbf{if} \ i=n \ \textbf{then return} \ -1 \\
\textbf{else return} \ i
\]
Primitive Operations

A basic or primitive operation is an operation that requires a constant amount of time in any implementation.

Examples:

\(\leftarrow, +, -, \times, \div, <, >, =, \leq, \geq, \neq\)
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Examples:

\( \rightarrow, +, -, \times, /, <, >, =, \leq, \geq, \neq \)

Constant, means independent from the size of the input.
When do we need to compute the time complexity function?

Assume a computer with speed $10^8$ operations per second.

<table>
<thead>
<tr>
<th>Time Complexity</th>
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</tr>
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<tbody>
<tr>
<td>$f(n) = n$</td>
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<td>$f(n) = n$</td>
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<td>$2 \times 10^{-7}$ s</td>
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<td>$10^{-6}$ s</td>
<td>$4 \times 10^{-6}$ s</td>
</tr>
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<td>$10^{-5}$ s</td>
<td>$8 \times 10^{-5}$ s</td>
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<td>$n = 10$</td>
<td>$n = 20$</td>
<td>$n = 1000$</td>
<td>$n = 10^6$</td>
<td></td>
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<tr>
<td>$f(n) = n$</td>
<td>$10^{-7}$ s</td>
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<td>$10^{-5}$ s</td>
<td>$10^{-2}$ s</td>
<td></td>
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<td>$f(n) = n^2$</td>
<td>$10^{-6}$ s</td>
<td>$4 \times 10^{-6}$ s</td>
<td>$10^{-2}$ s</td>
<td>2.4 hrs</td>
<td></td>
</tr>
<tr>
<td>$f(n) = n^3$</td>
<td>$10^{-5}$ s</td>
<td>$8 \times 10^{-5}$ s</td>
<td>10 s</td>
<td>360 yrs</td>
<td></td>
</tr>
<tr>
<td>$f(n) = 2^n$</td>
<td>$10^{-5}$ s</td>
<td>$10^{-1}$ s</td>
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Asymptotic Notation

We want to characterize the time complexity of an algorithm for large inputs irrespective of the value of implementation dependent constants.
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The mathematical notation used to express time complexities is the asymptotic notation:
Asymptotic or Order Notation

Let $f(n)$ and $g(n)$ be functions from $\mathbb{I}$ to $\mathbb{R}$. We say that $f(n)$ is $O(g(n))$ (read ``$f(n)$ is big-Oh of $g(n)$'' or "$f(n)$ is of order $g(n)$") if there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that

$$f(n) \leq c \times g(n) \text{ for all } n \geq n_0$$
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Constant = independent from $n$
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Constant = independent from $n$

We sometimes write $f(n) = O(g(n))$ or $f(n) \in O(g(n))$. 