Sorting

Using ADTs to Implement Sorting Algorithms

Objectives

- Examine several sorting algorithms that can be implemented using collections and in-place:

  Insertion Sort

  Selection Sort

  Quick Sort

- Analyse the time complexity of these algorithms

Sorting Problem

- Suppose we have an unordered list of objects that we wish to have sorted into ascending order
- We will discuss the implementation of several sort methods with a header of the form:

```
public void someSort( UnorderedList list)
// precondition: list holds a sequence of objects in
//                           some random order
// postcondition: list contains the same objects,
//                           now sorted into ascending order
```

Comparing Sorts

- We will compare the following sorts:
  - ***Insertion Sort*** using stacks and in-place
  - ***Selection Sort*** using queues and in-place
  - ***Quick Sort***

- Assume that there are **n** items to be sorted into ascending order

Insertion Sort

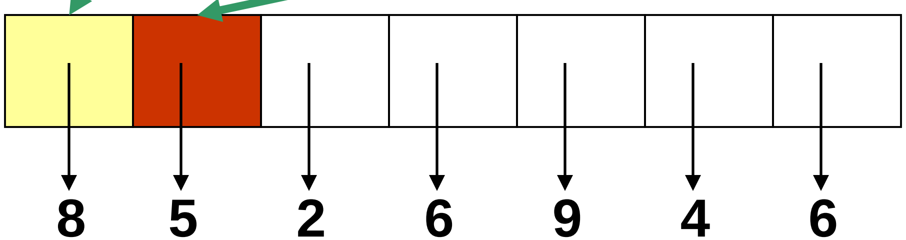
- ***Insertion Sort*** orders a sequence of values by repetitively inserting the next value into a ***sorted subset*** of the sequence
- More specifically:
  - Consider the first item to be a ***sorted subsequence*** of length **1**
  - Insert the second item into the ***sorted subsequence***, now of length **2**
  - Repeat the process, always inserting the ***first*** item from the ***unsorted portion*** into the ***sorted subsequence*, until the entire sequence is in order**

Insertion Sort Algorithm

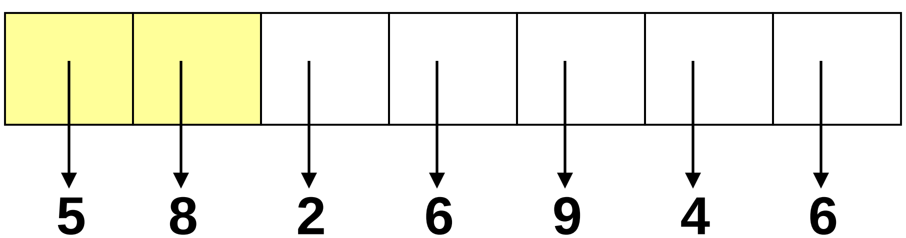
Example: **sorting a sequence of Integer objects**

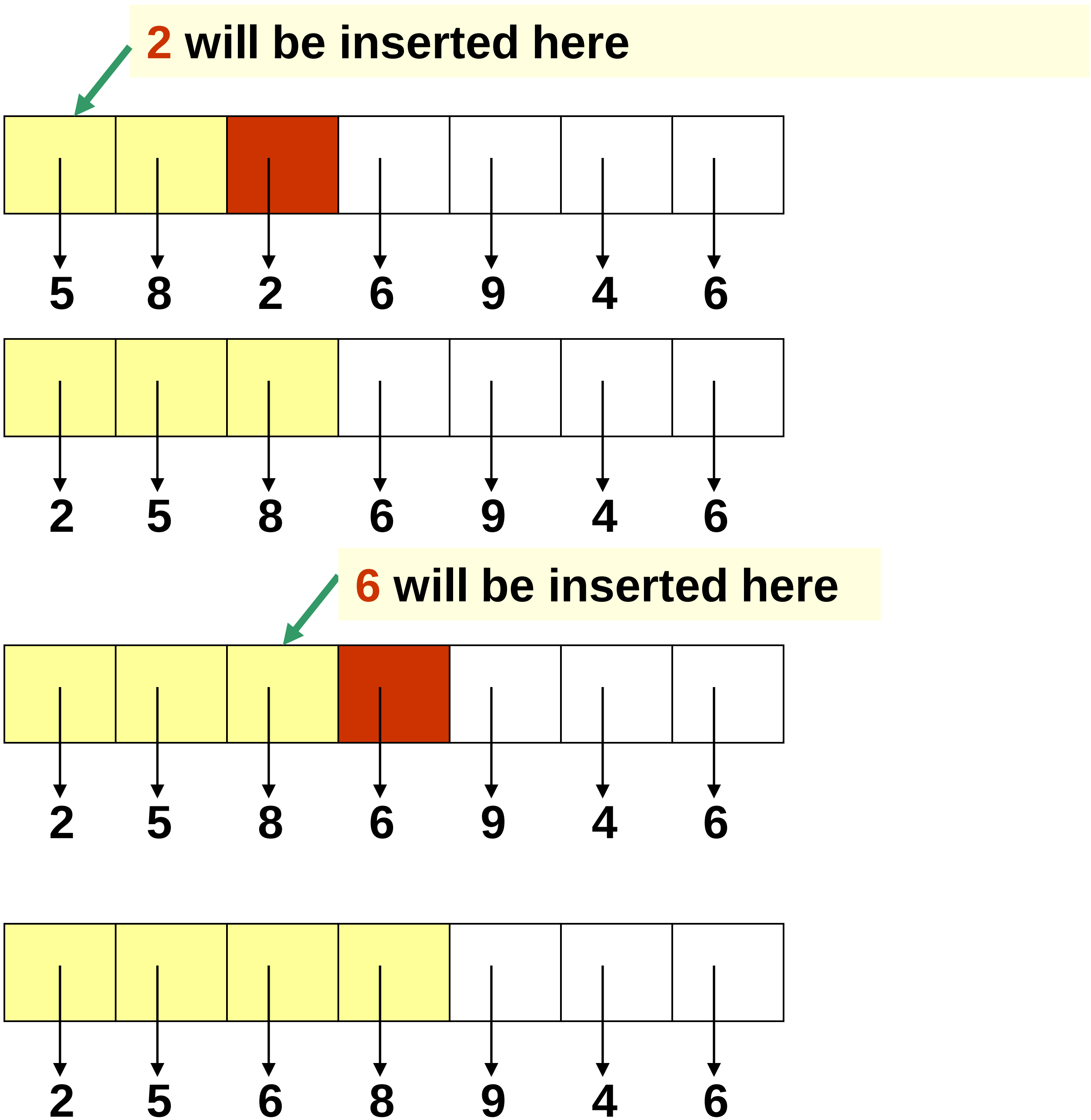
Sorted subsequence

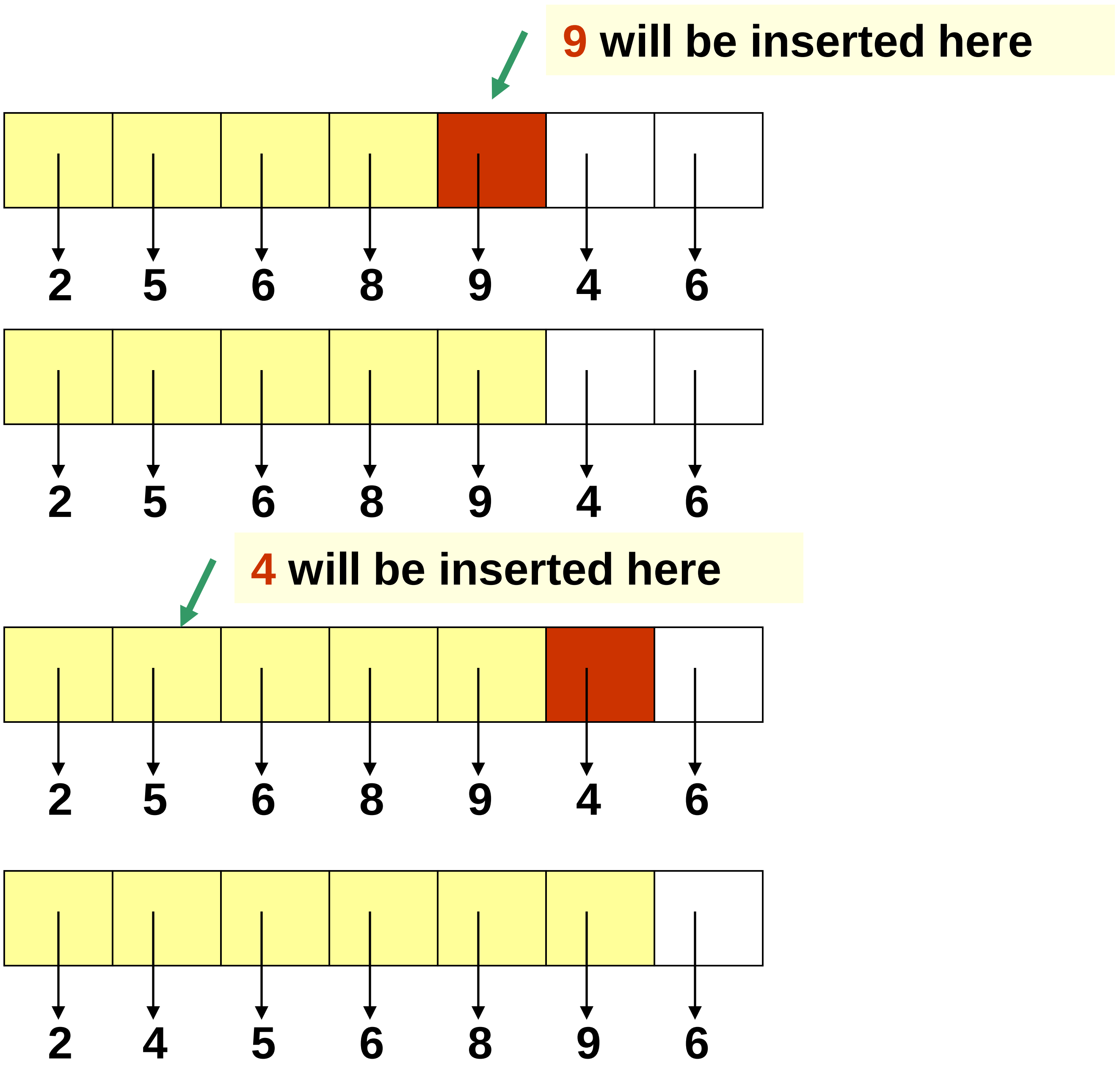
Value to be "inserted"



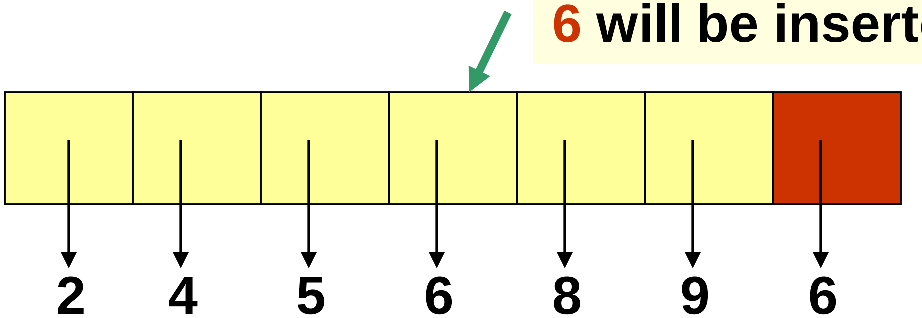
Value 5 is to be inserted where the 8 is: reference to 8 will be copied to where the 5 is, the 5 will be put in the vacated position, and the sorted subsequence now has length 2



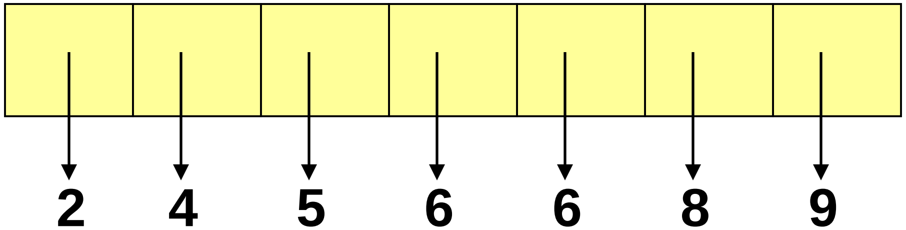

2 will be inserted here
5 8 2 6 9 4 6
2 5 8 6 9 4 6
6 will be inserted here
2 5 8 6 9 4 6
2 5 6 8 9 4 6


9 will be inserted here
2 5 6 8 9 4 6
2 5 6 8 9 4 6
4 will be inserted here
2 5 6 8 9 4 6
2 4 5 6 8 9 6

6 will be inserted here



And we're done!



Insertion Sort using Stacks

***Approach to the problem*:**

- Use two temporary stacks **sorted** and **temp**, both of which are originally empty
- The contents of **sorted** will always be in order, with the smallest item on the top of the stack
  - This will be the "sorted subsequence"
- **temp** will temporarily hold items that need to be "shifted" out in order to insert the new item in the proper place in **sorted**

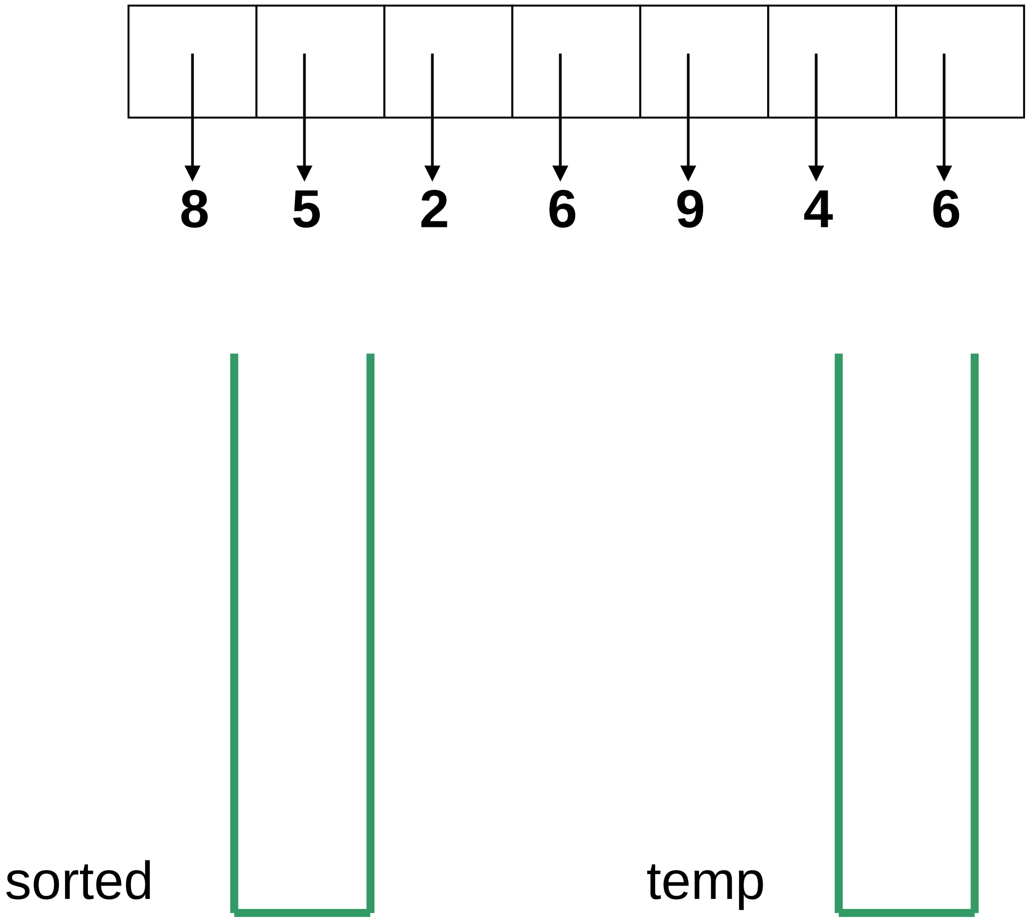
Algorithm insertionSort (A,n)

In: Array A storing n elements

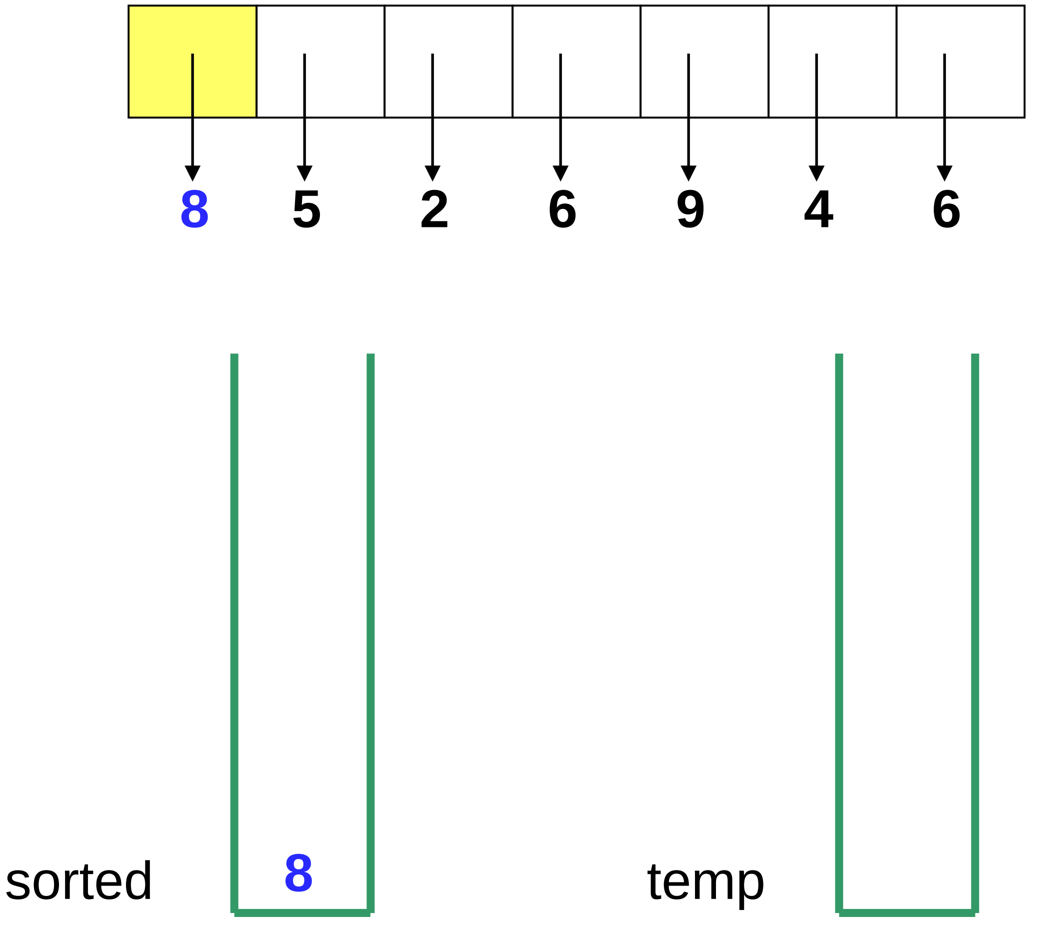
Out: Sorted array

```
sorted = empty stack
temp = empty stack
for i = 0 to n-1 do {
    while (sorted is not empty) and (sorted.peek() < A[i]) do
        temp.push (sorted.pop())
    sorted.push (A[i])
    while temp is not empty do
        sorted.push (temp.pop())
}
for i = 0 to n-1 do
    A[i] = sorted.pop()
```

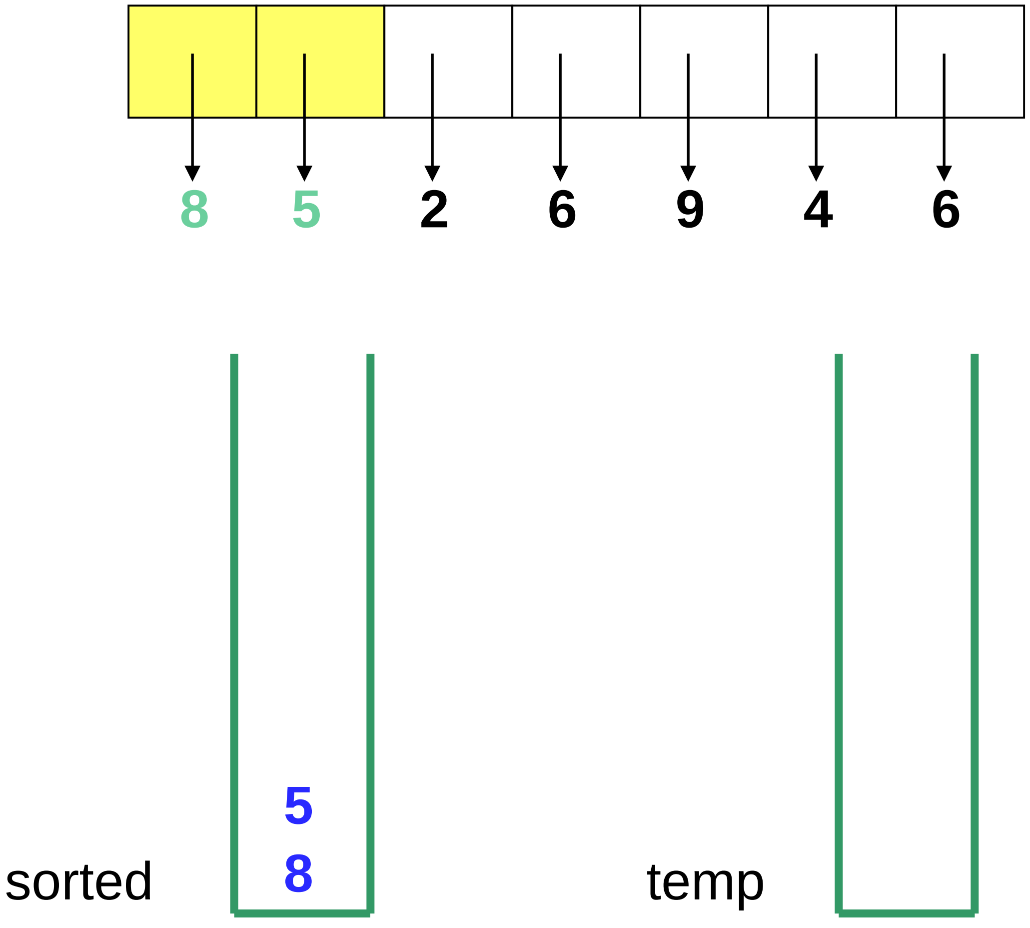
Insertion Sort



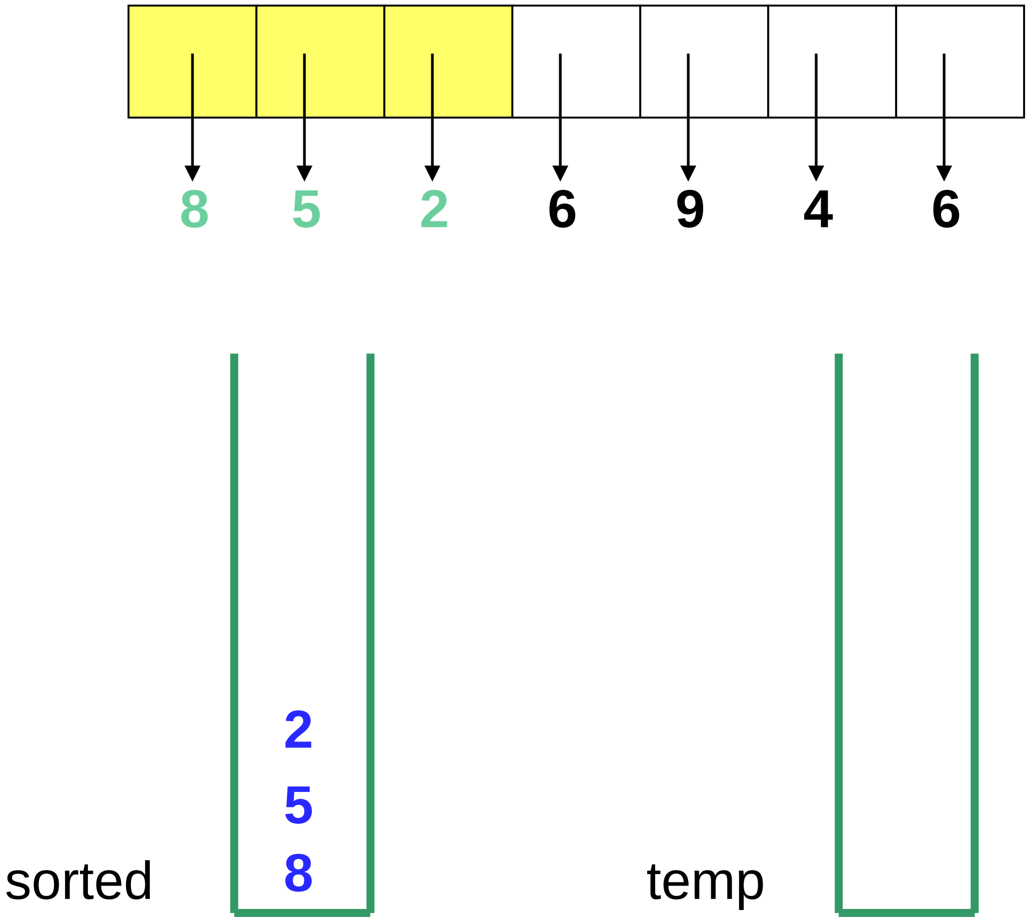
Insertion Sort



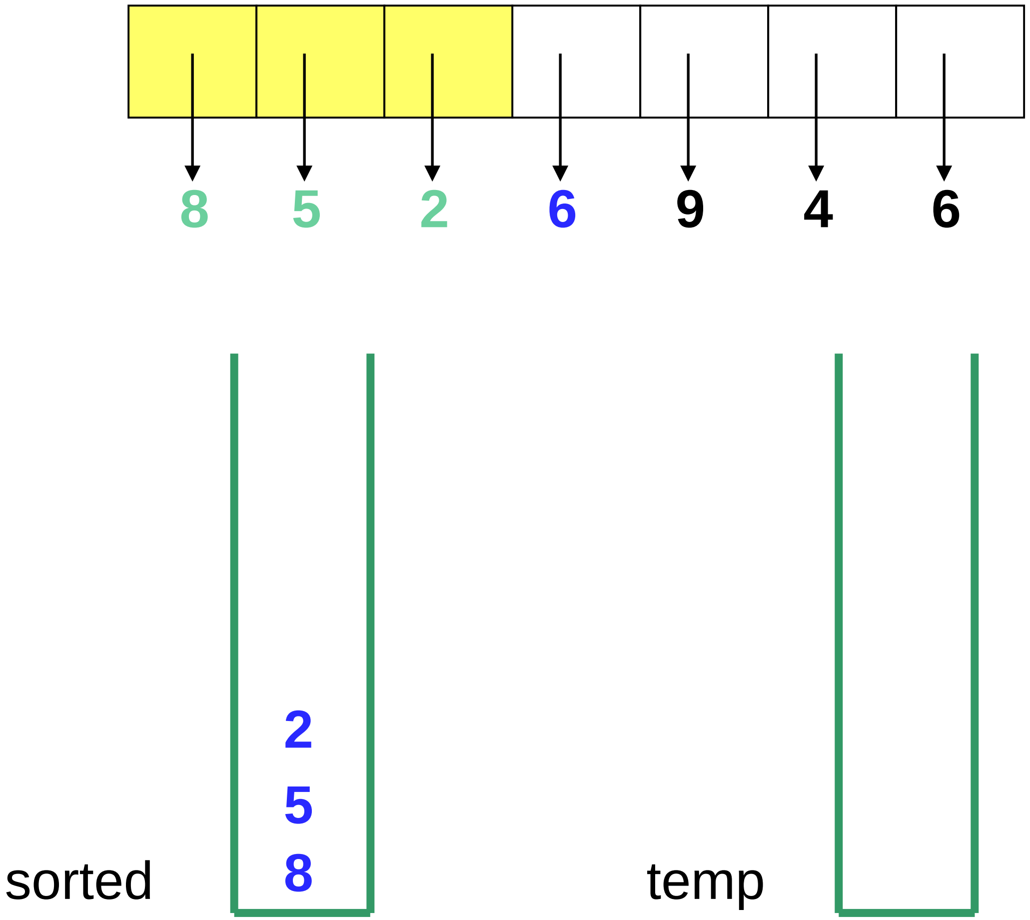
Insertion Sort



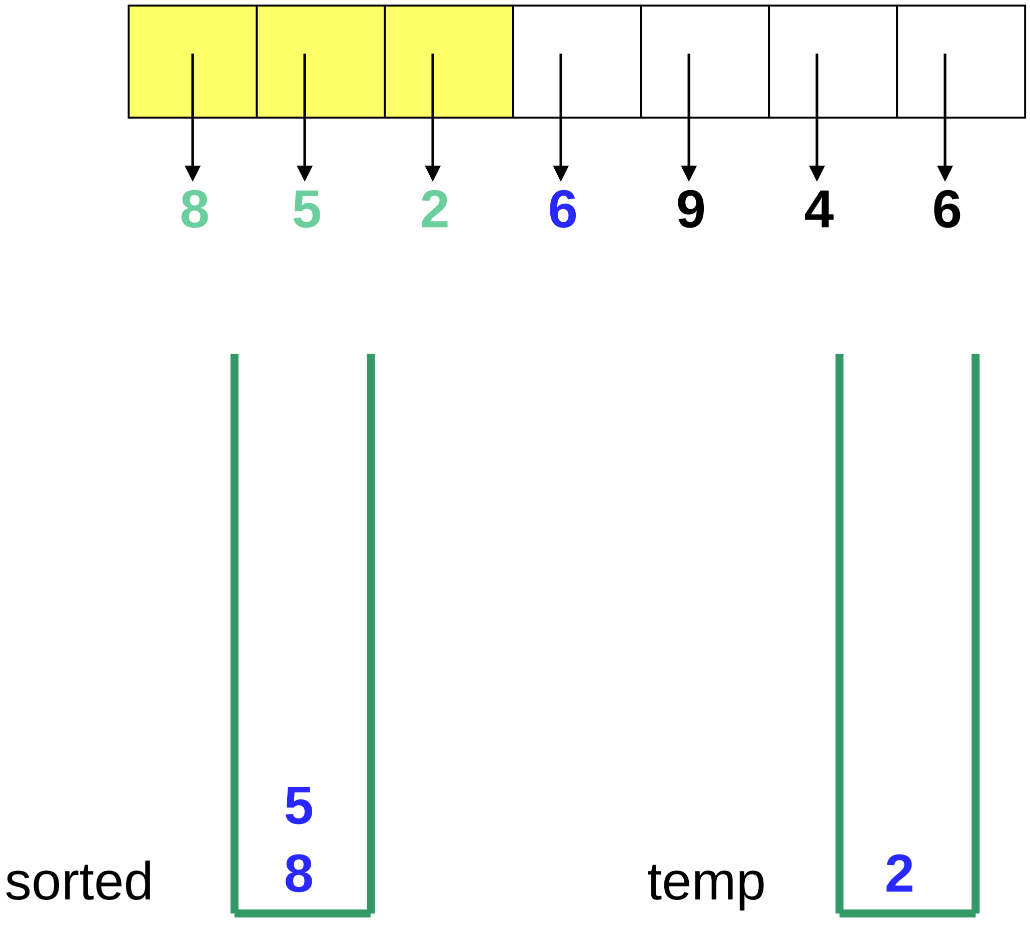
Insertion Sort



Insertion Sort



Insertion Sort



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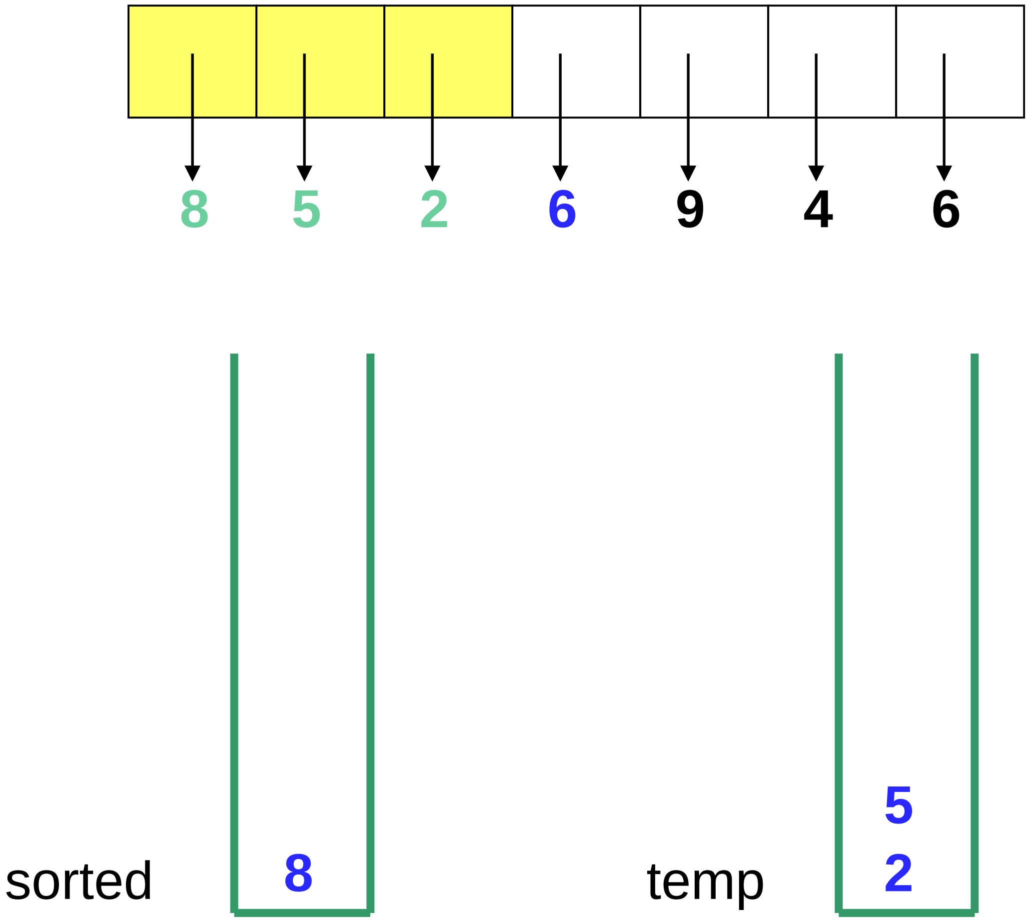
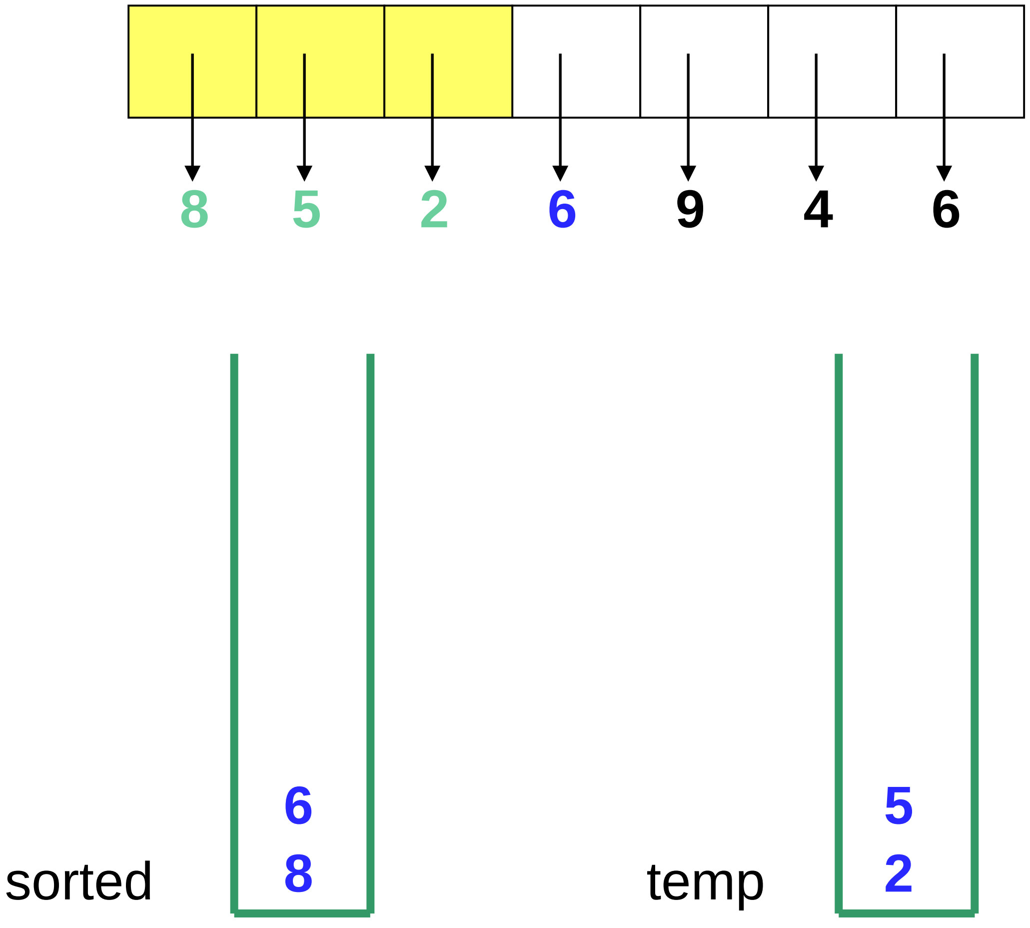
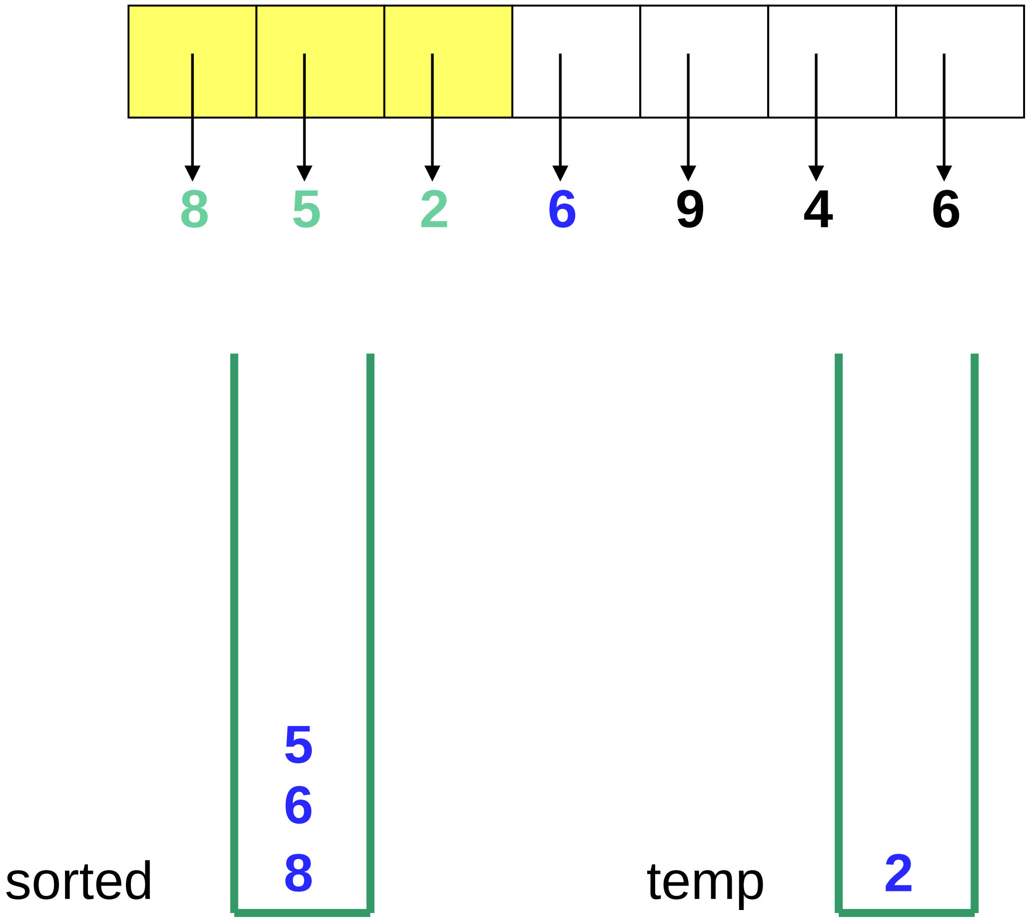


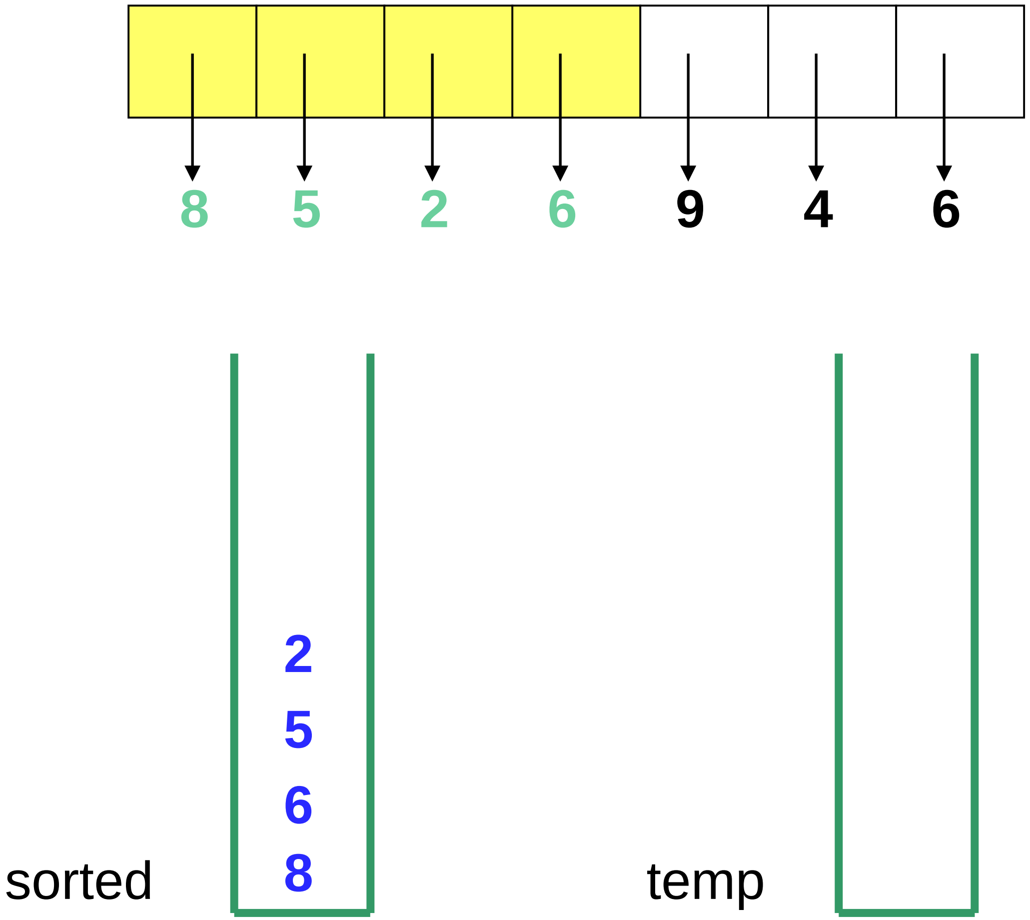
Insertion Sort



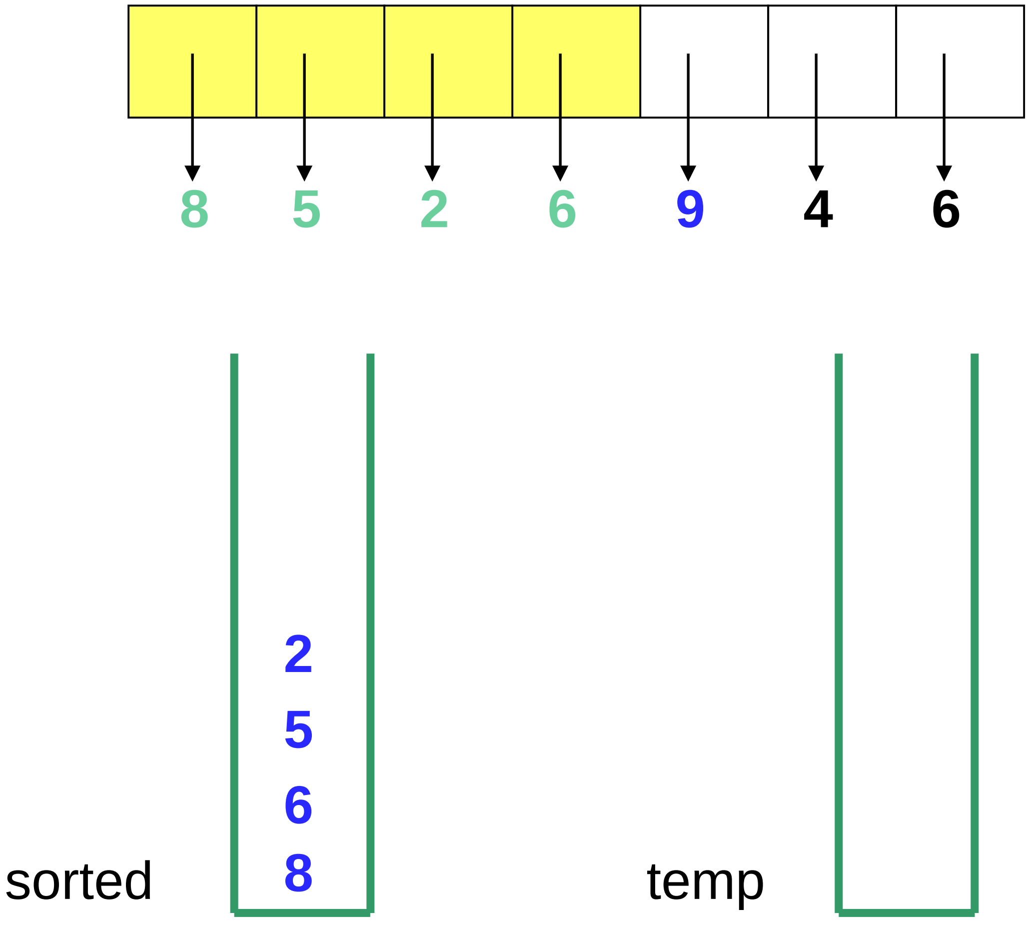
Insertion Sort



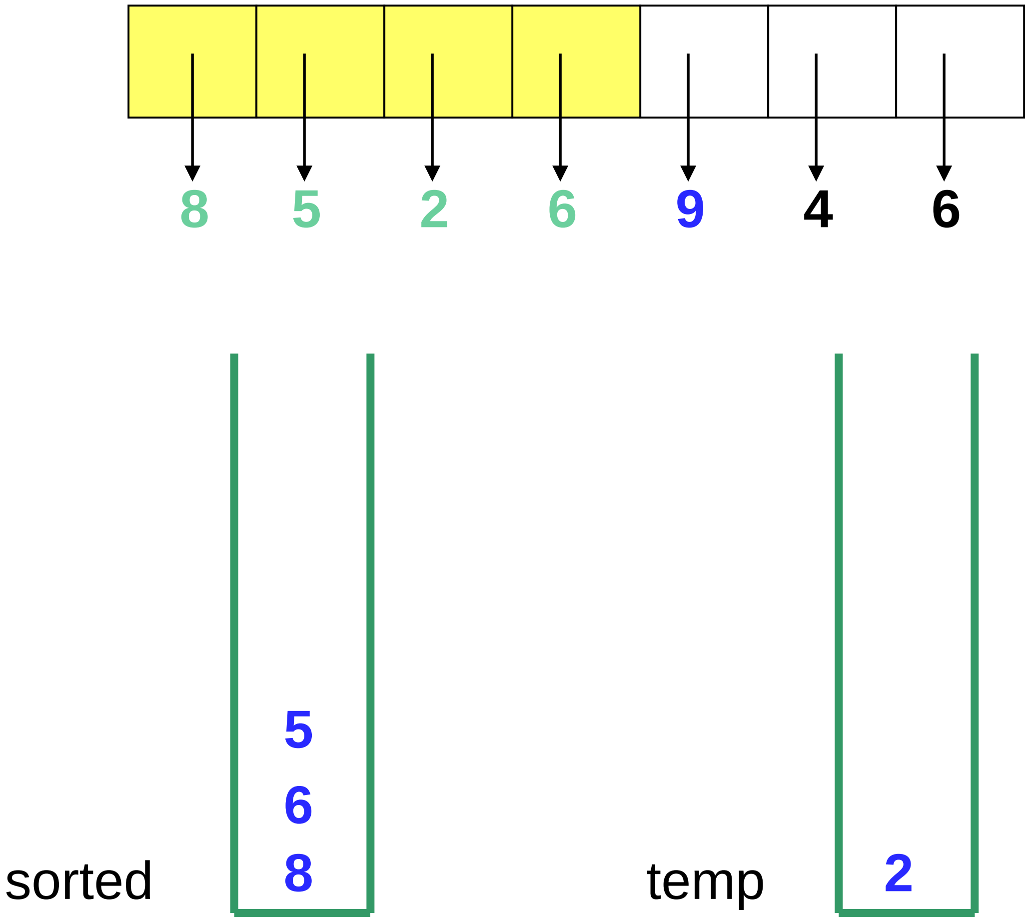
Insertion Sort



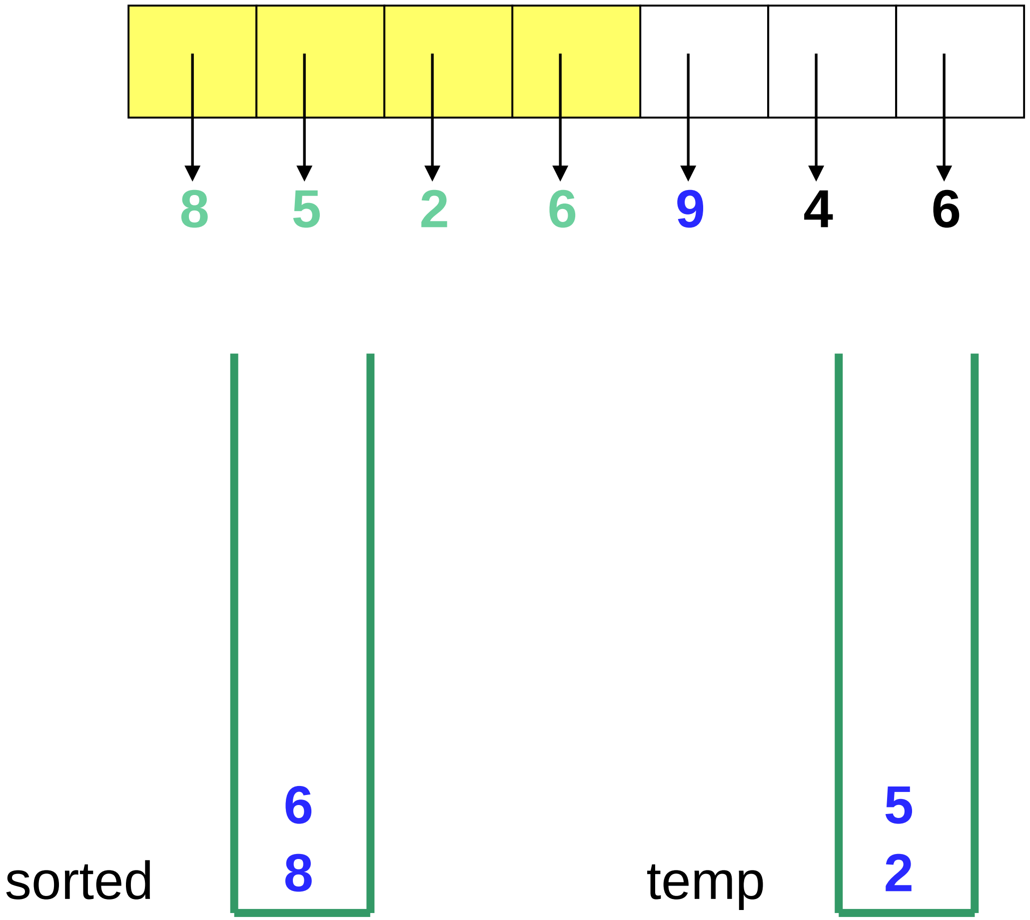
Insertion Sort



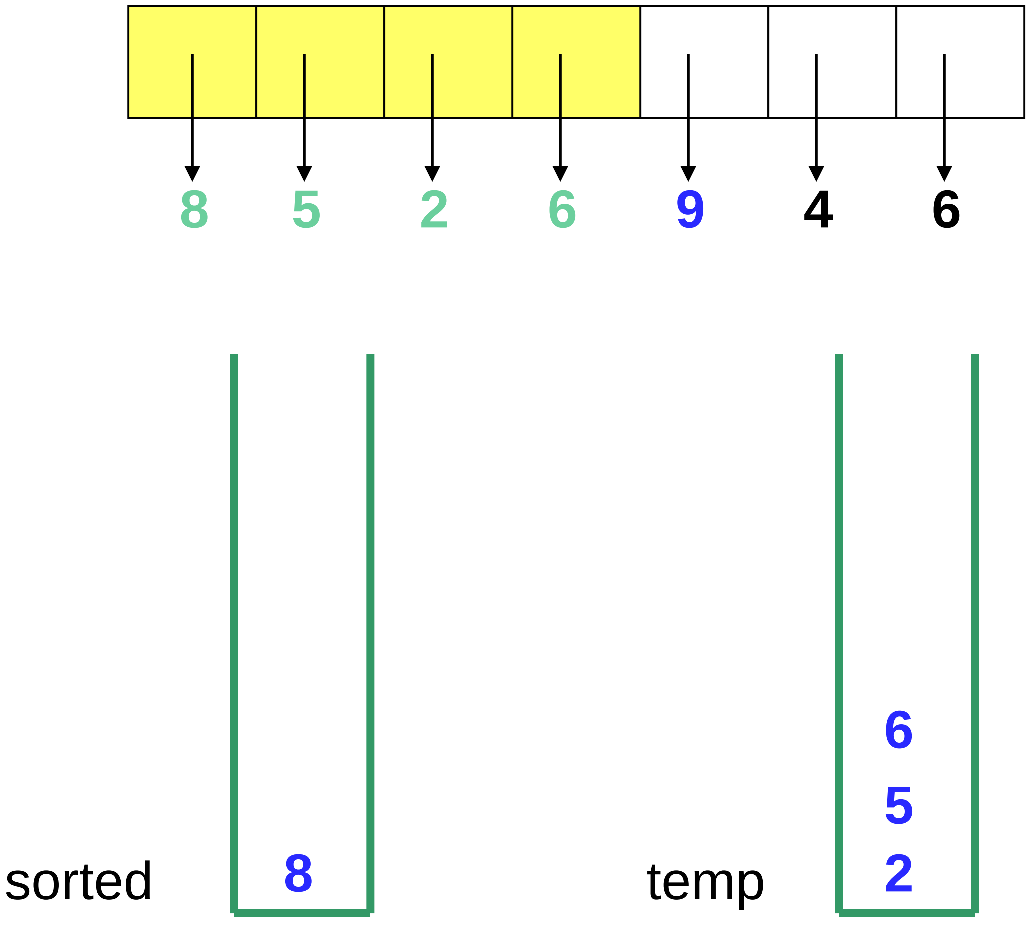
Insertion Sort



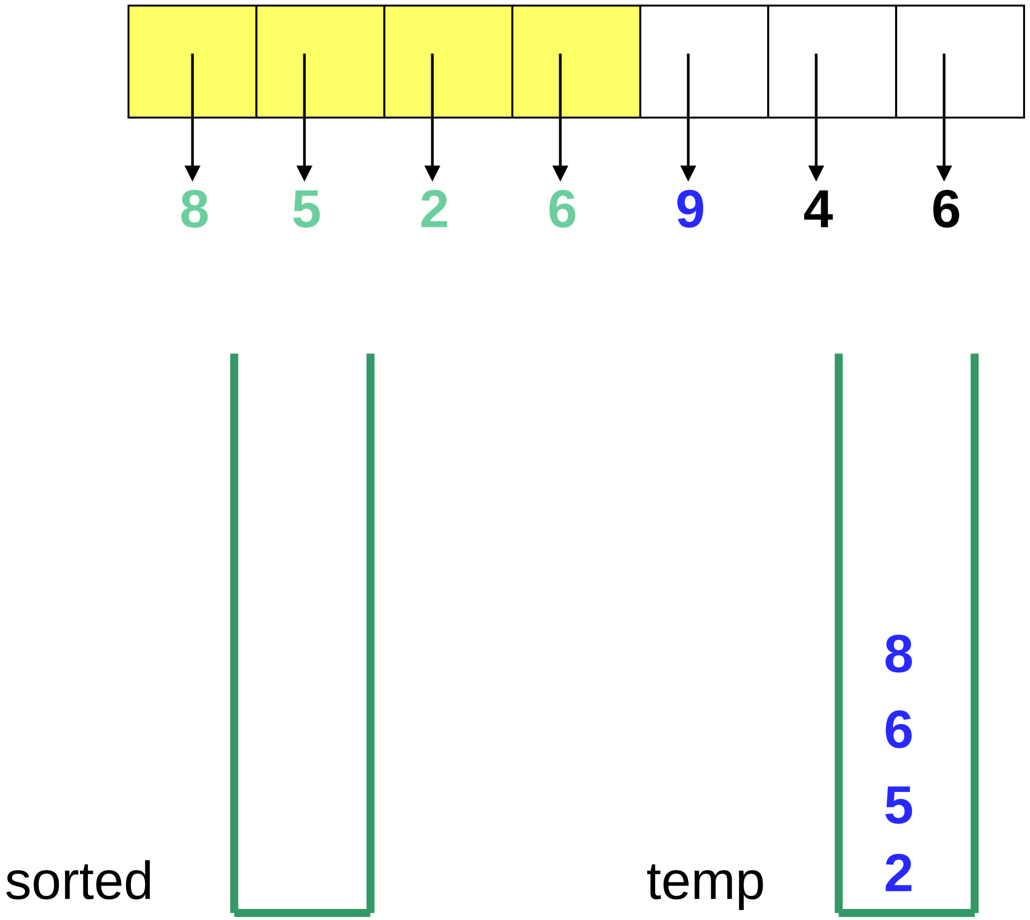
Insertion Sort



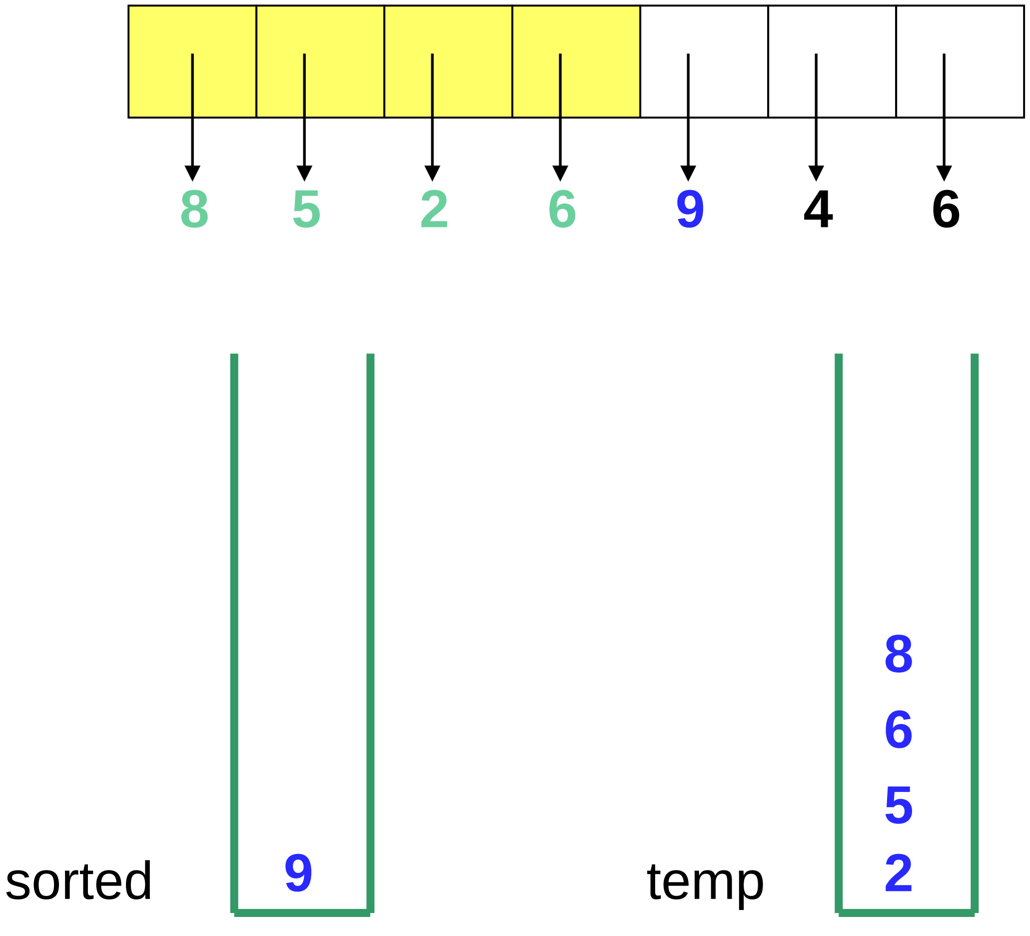
Insertion Sort



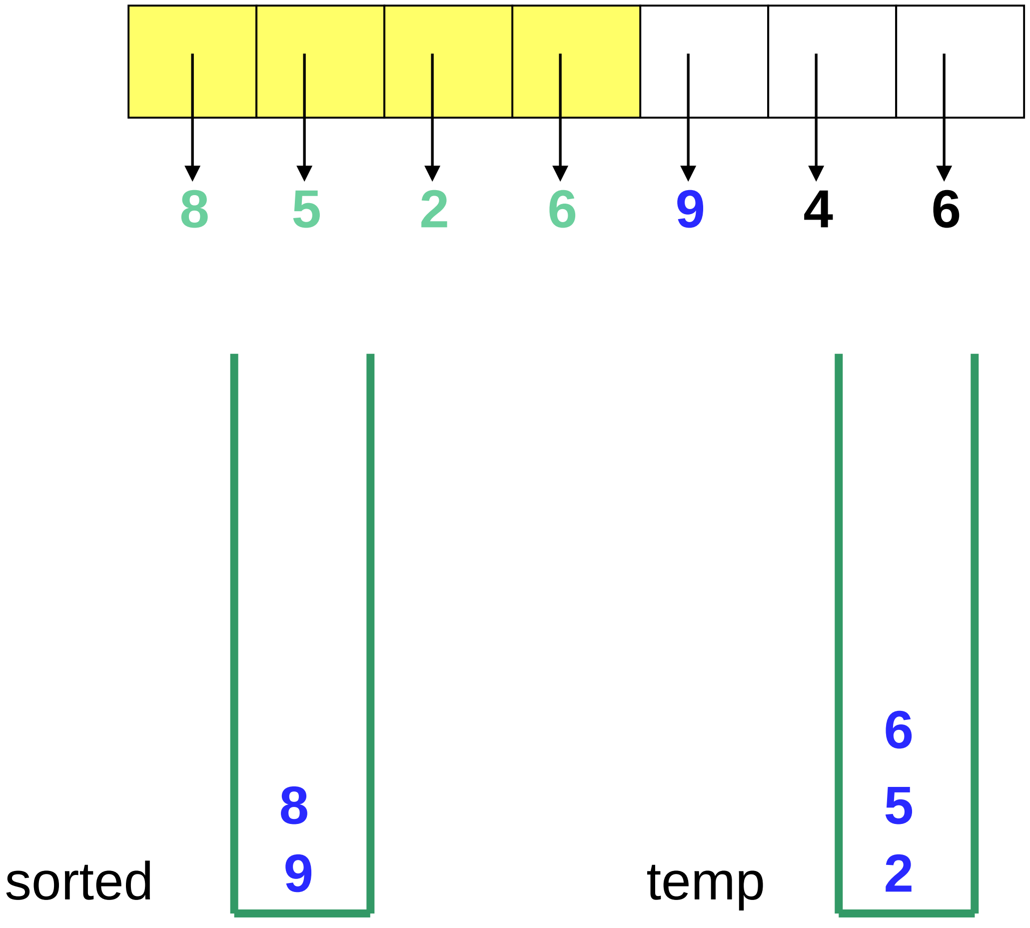
Insertion Sort



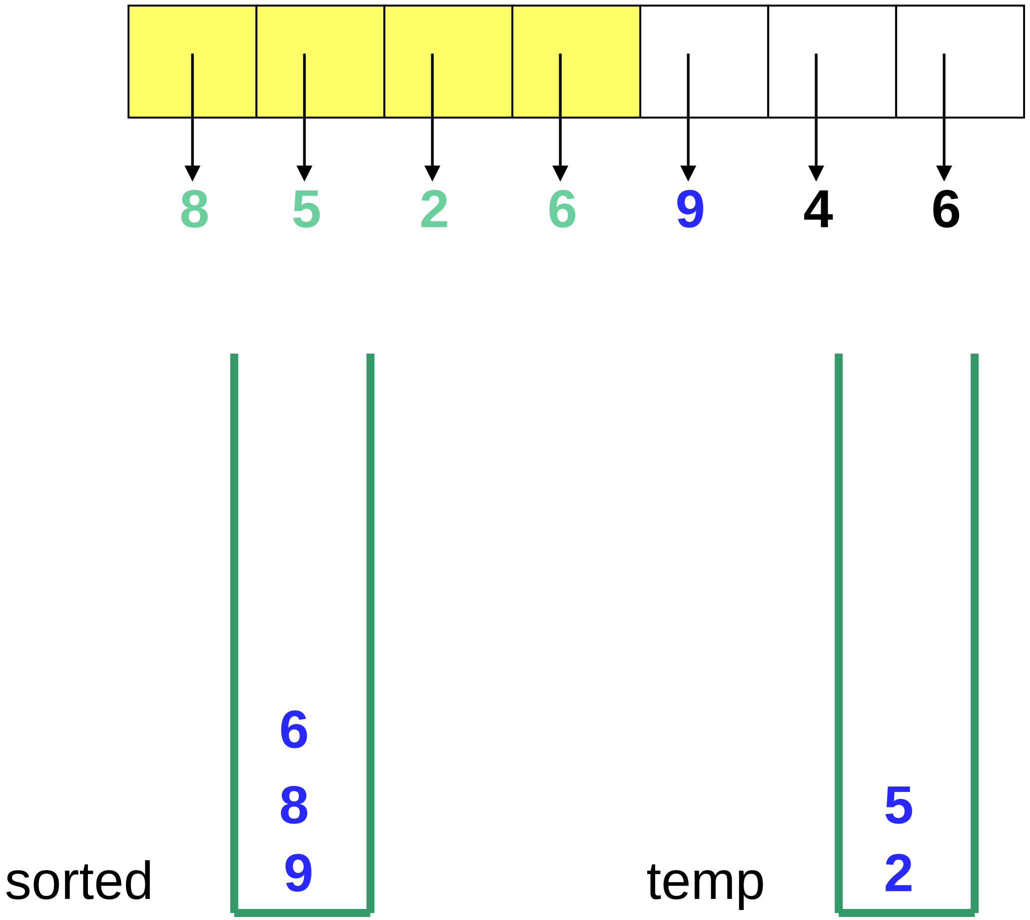
Insertion Sort



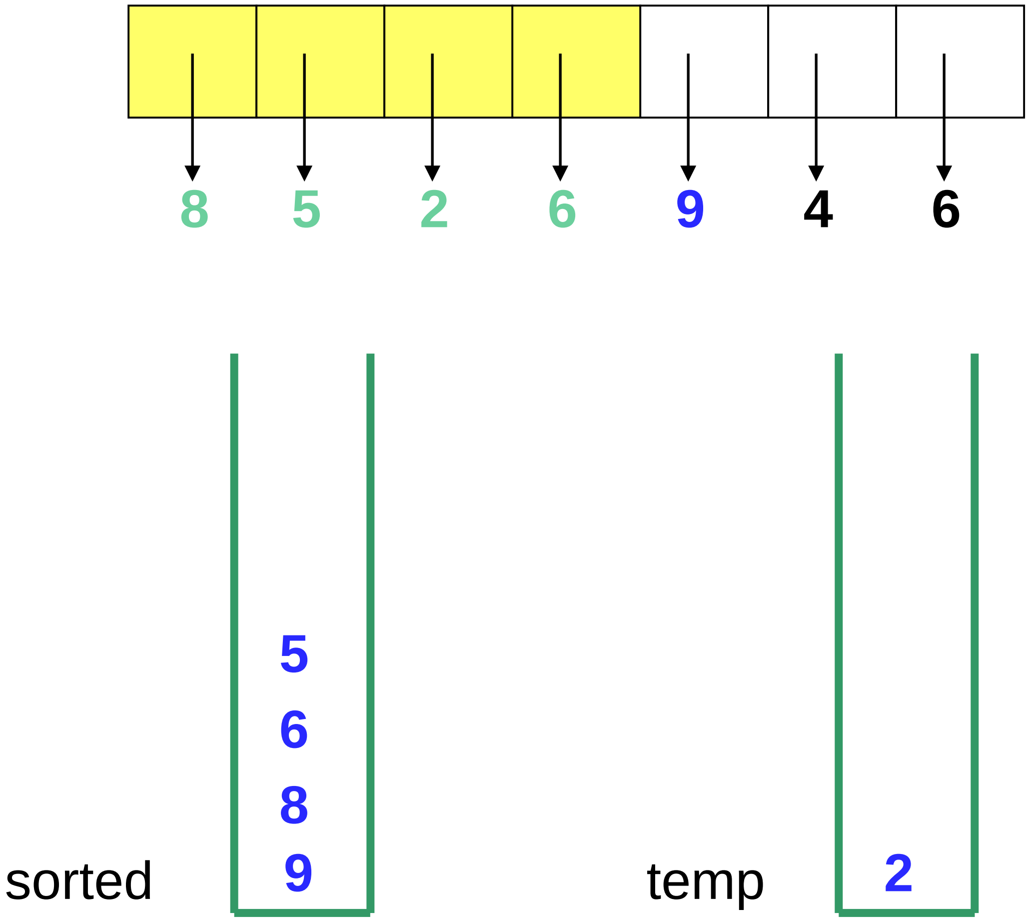
Insertion Sort



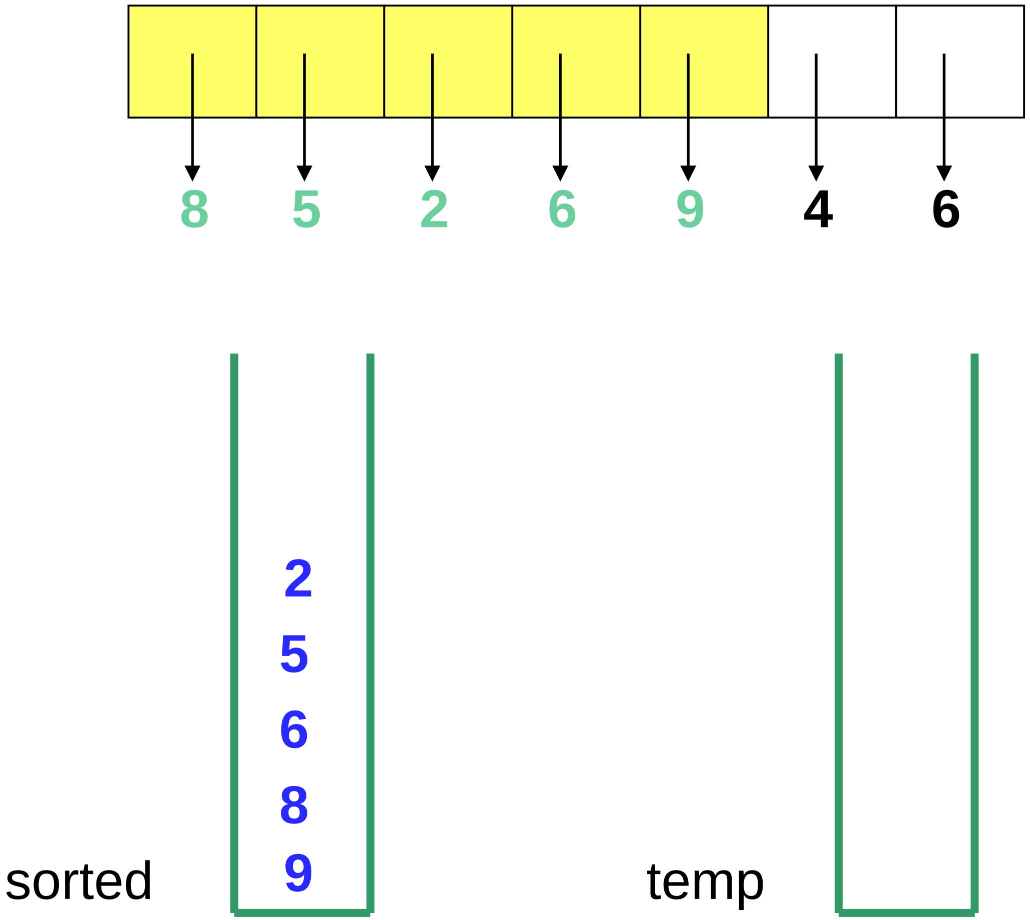
Insertion Sort



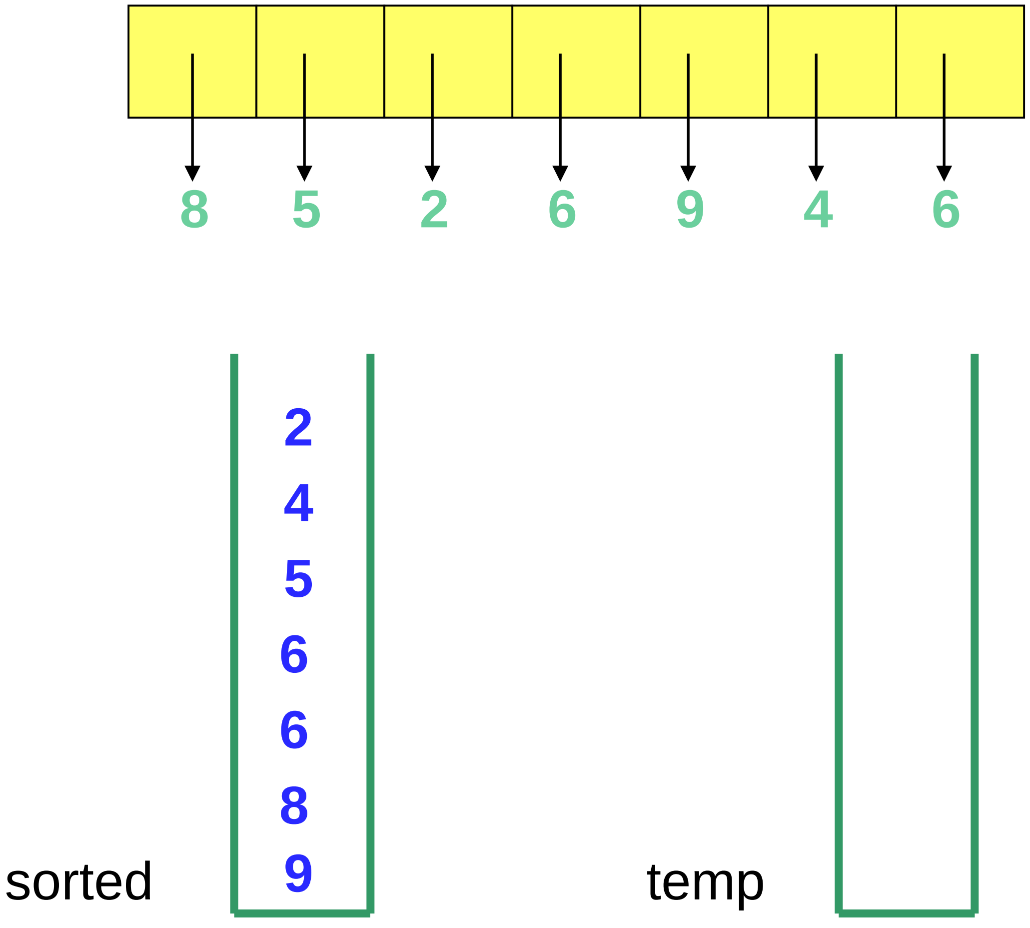
Insertion Sort



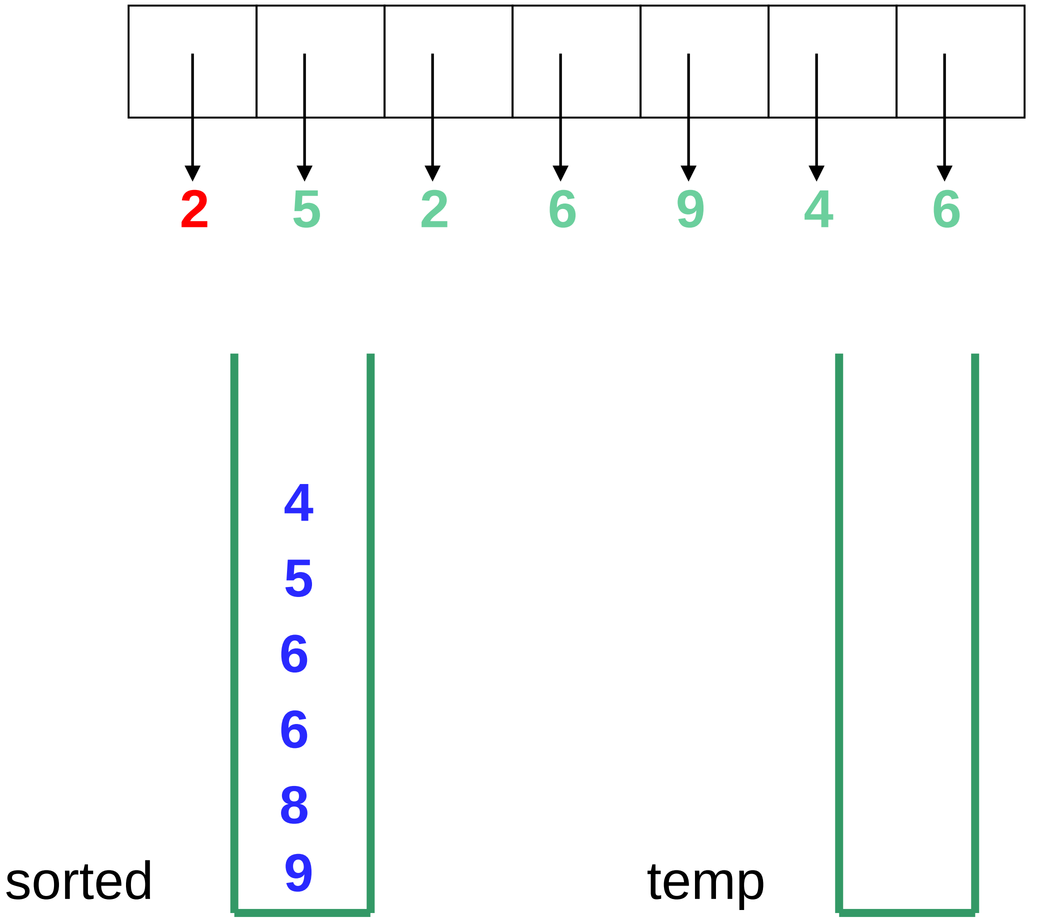
Insertion Sort



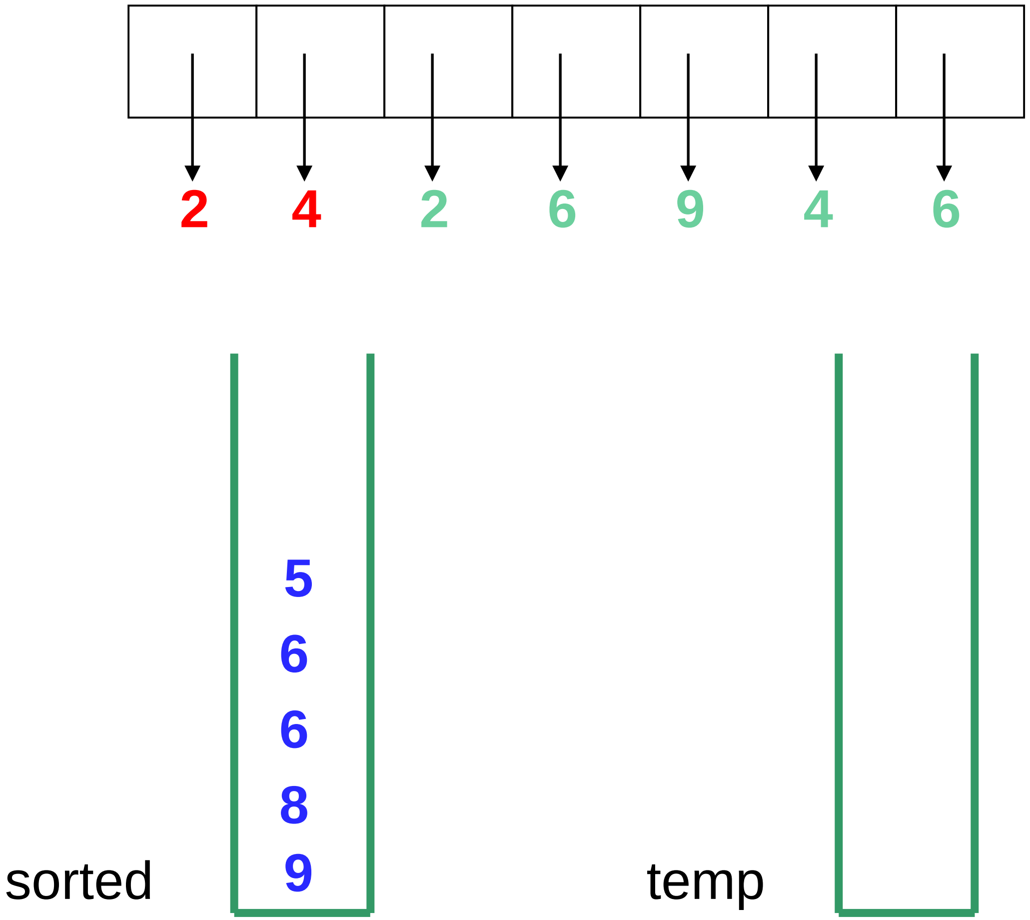
Insertion Sort



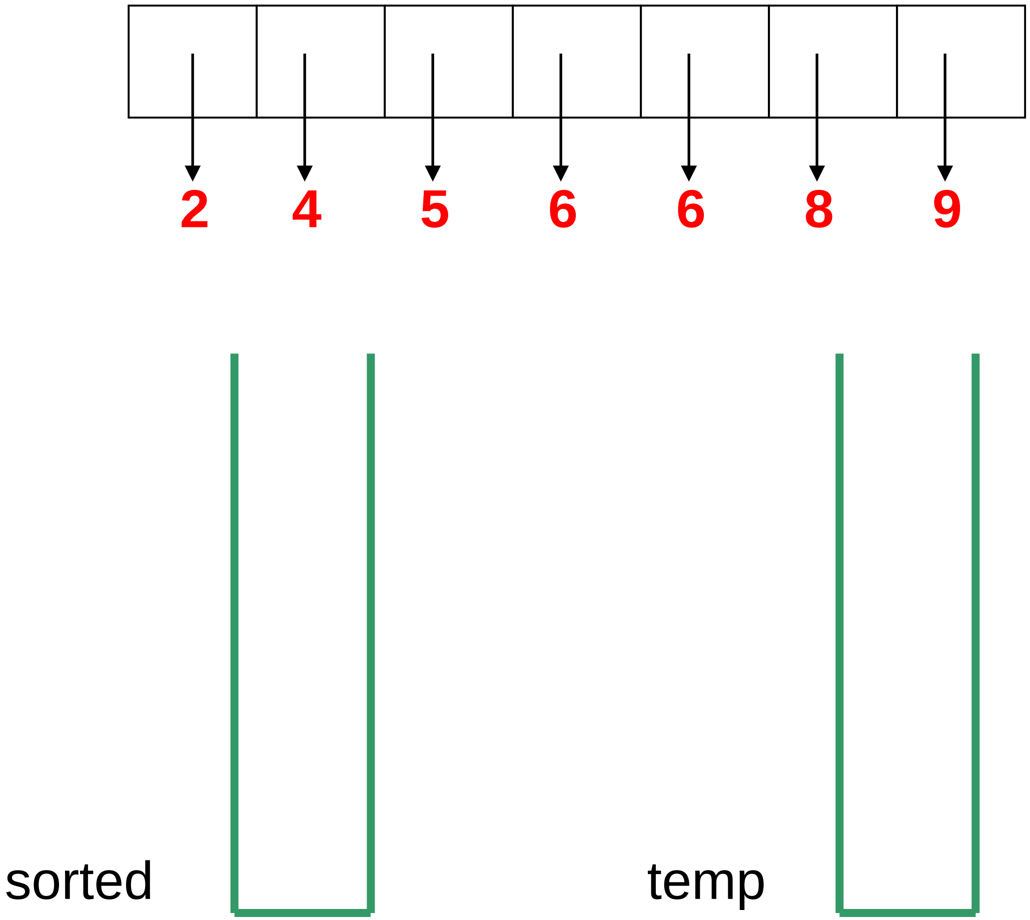
Insertion Sort



Insertion Sort



Insertion Sort



Analysis of Insertion Sort Using Stacks

- Each time through the outer for loop, one more item is taken from the array and put into place on **sorted**. So the outer loop is repeated n times. Consider one iteration of the for loop:
  - Assume that there are **i** items in **sorted.**
    Worst case: every item has to be popped from **sorted** and pushed onto **temp, so**
    i pops and **i** pushes
  - New item A[i] is pushed onto **sorted**
  - Items in **temp** are popped and pushed onto **sorted, so i** pops and **i** pushes
  - If we implement the stacks using a singly linked list, each stack operation performs a constant number of primitive operations.

Analysis of Insertion Sort Using Stacks

Hence, assuming that sorted has i items, one iteration of the first while loop performs a constant number c_1 of primitive operations and the loop is repeated i times in the worst case, so the number of operations that it performs is ic_1.

The second while loop also performs a constant number c_2 of operations per iteration and the loop is repeated i times in the worst case, so it performs ic_2 operations.

Pushing A[i] into the stack performs a constant number c_3 of operations.

Therefore one iteration of the for loop performs

$$ic_1 + ic_2 + c_3$$

operations.

Analysis of Insertion Sort Using Stacks

The outer for loop is executed **n** times, *but* each time the number of elements in **sorted** increases by **1**, from **0** to **(n-1)**

- So, the total number of operations performed by the outer for loop, in the worst case, is $(0\times c_1+0\times c_2+c_3)+(1\times c_1+1\times c_2+c_3)+(2\times c_1+2\times c_2+c_3)+ \ldots$ $(n-1)\times c_1+(n-1)\times c_2+c_3 = n(n-1)(c_1+c_2)/2+n\times c_3$
- Then there are $\mathbf{n}\times c_4$ additional operations to move the sorted items back onto the array, where c_4 is a constant. Finally, creating the empty stacks requires a constant number c_5 of operations.

- So, the total number of operations performed by the algorithm is $n(n-1)(c_1+c_2)/2+n\times c_3+n\times c_4+c_5$, which is $O(n^2)$.

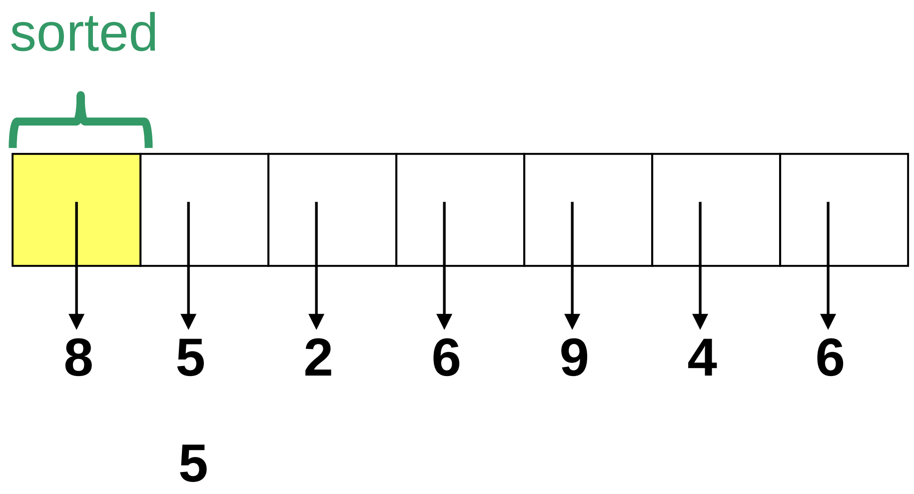
Discussion

- Is there a best case?
  - Yes: the items are already sorted, but in reverse order (largest to smallest)
  - What is the time complexity then?

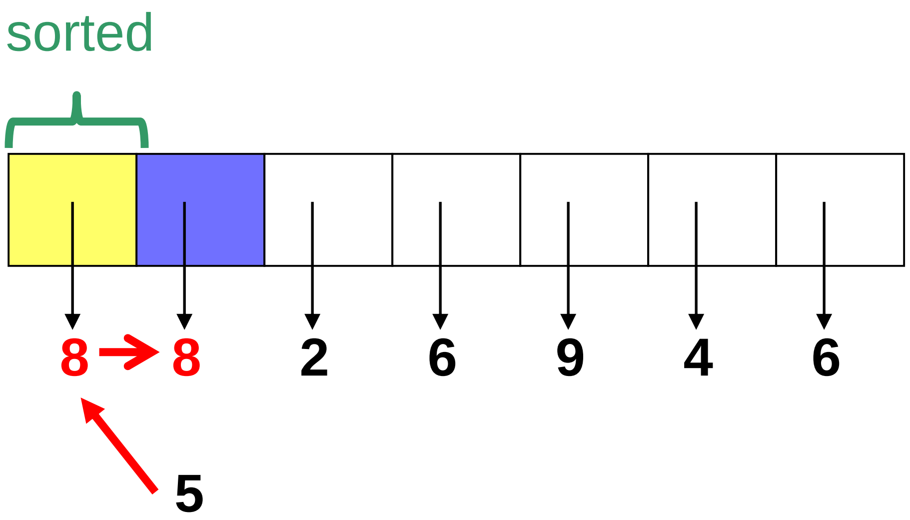
- What is the worst case?
  - The items are already sorted, in the correct order!!
  - Why is this the worst case?

In-Place Insertion Sort

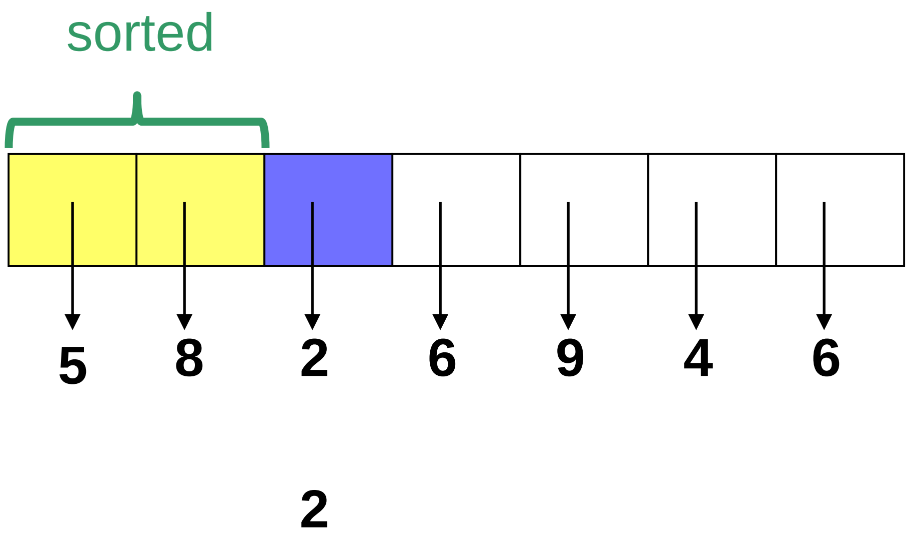
In-Place: the algorithm does not use auxiliary data structures.



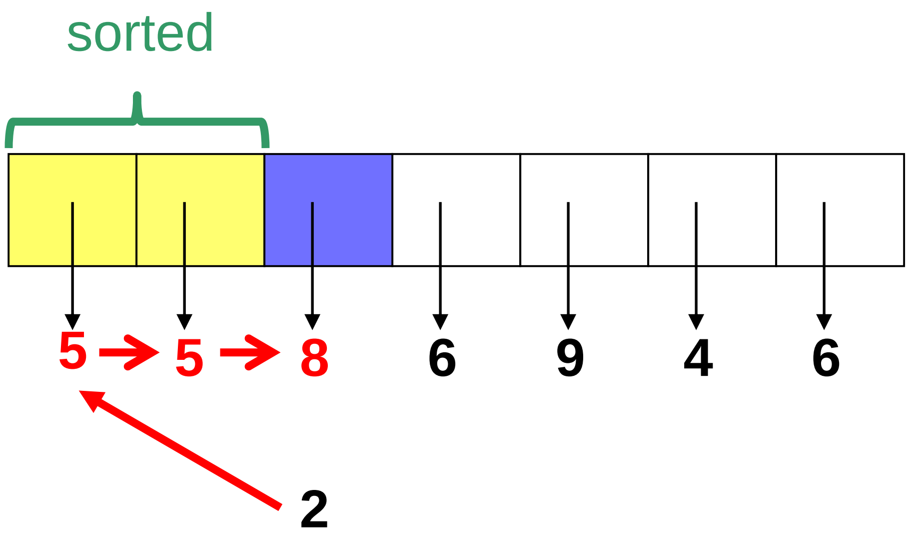
In-Place Insertion Sort



In-Place Insertion Sort



In-Place Insertion Sort



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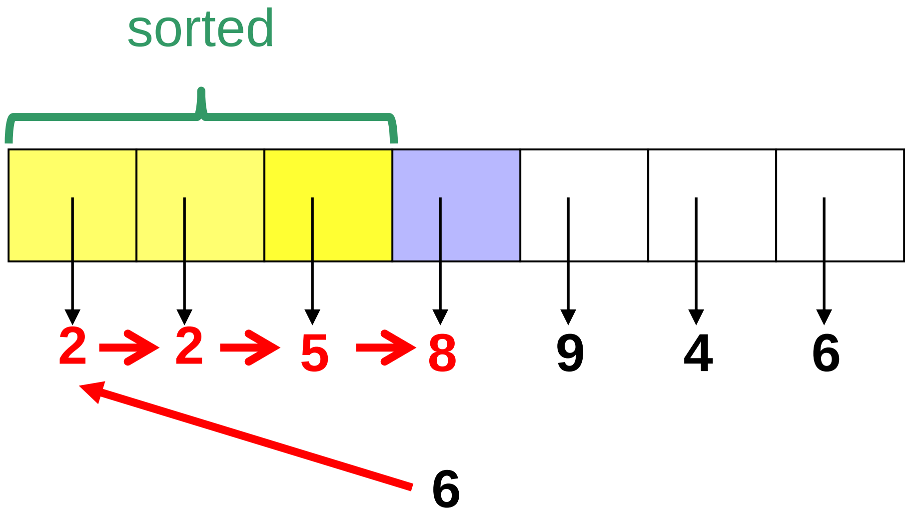
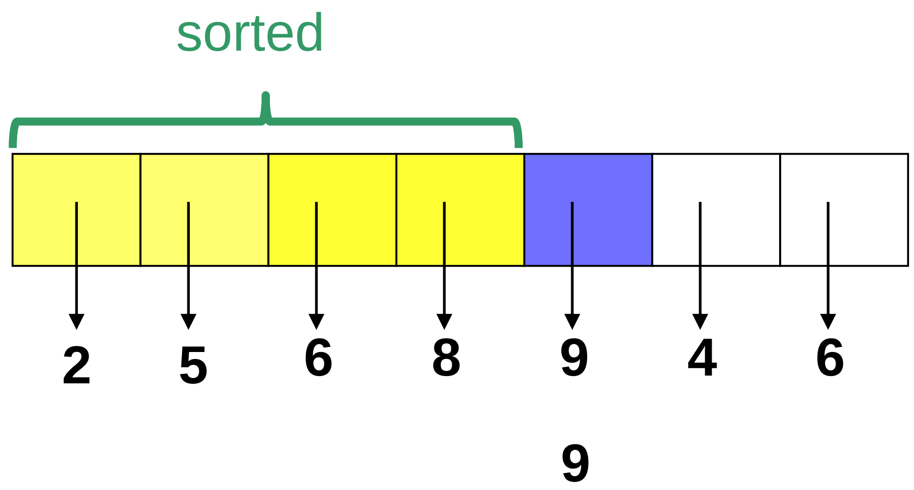
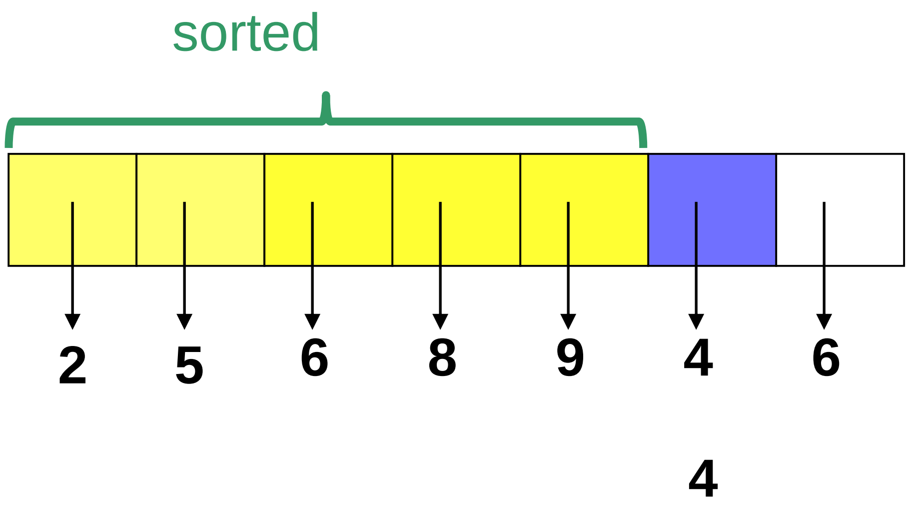


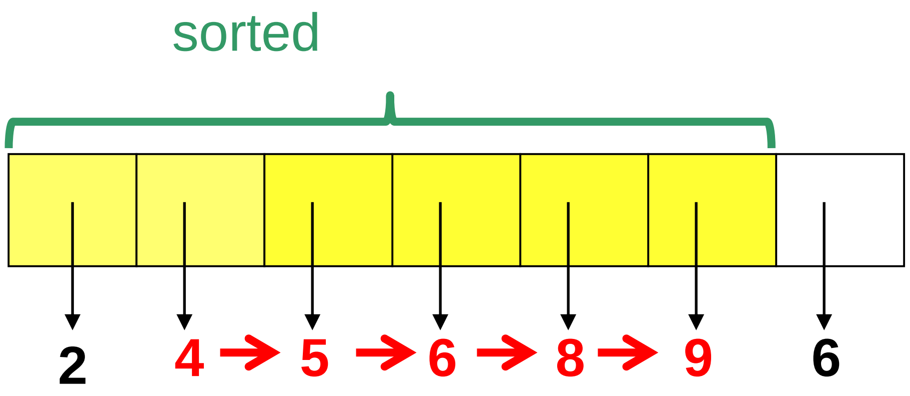
In-Place Insertion Sort



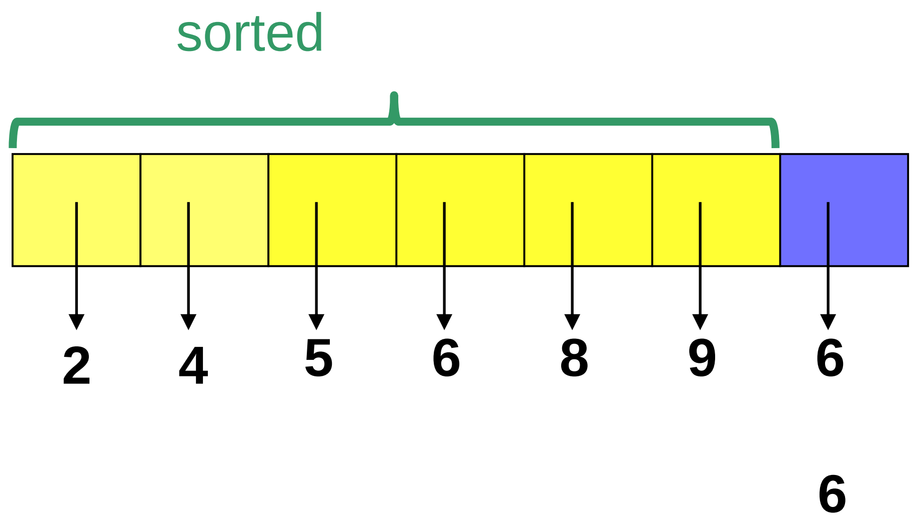
In-Place Insertion Sort



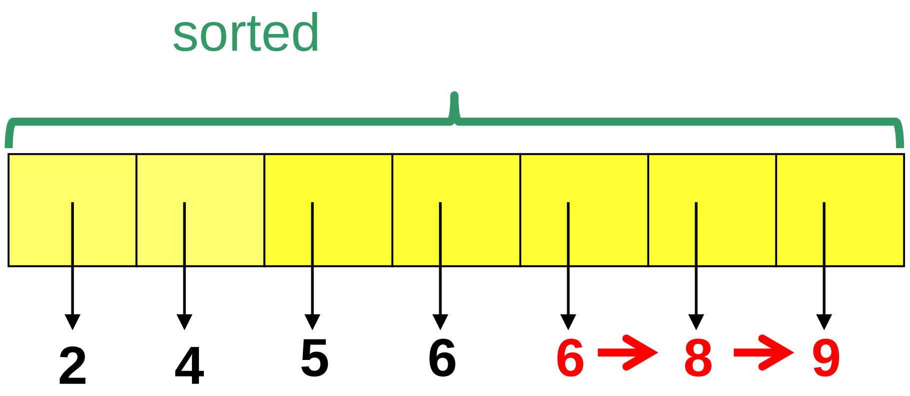
In-Place Insertion Sort



In-Place Insertion Sort



In-Place Insertion Sort



```
Algorithm insertionSort (A,n)
In: Array A storing n values
Out: {Sort A in increasing order}
for i = 1 to n-1 do {
  // Insert A[i] in the sorted sub-array A[0..i-1]
   temp = A[i]
   j = i – 1
   while (j >= 0) and (A[j] > temp) do {
        A[j+1] = A[j]
        j = j – 1
   }
   A[j+1] = temp
}
```

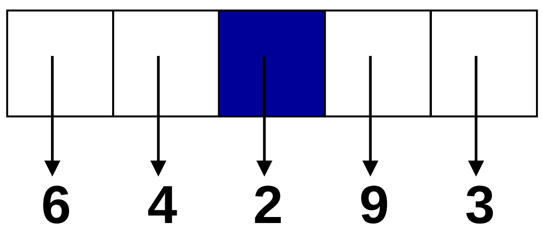
Selection Sort

- ***Selection Sort*** orders a sequence of values by repetitively putting a particular value into its ***final*** position
- More specifically:
  - Find the smallest value in the sequence
  - Switch it with the value in the first position
  - Find the next smallest value in the sequence
  - Switch it with the value in the second position
  - Repeat until all values are in their proper places

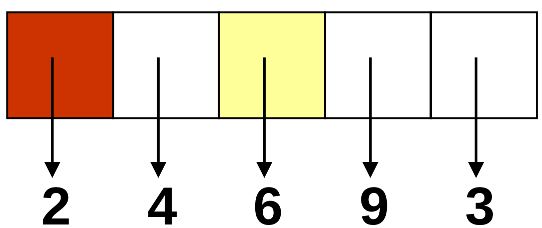
Selection Sort Algorithm

Initially, the *entire* container is the "*unsorted portion*" of the container.

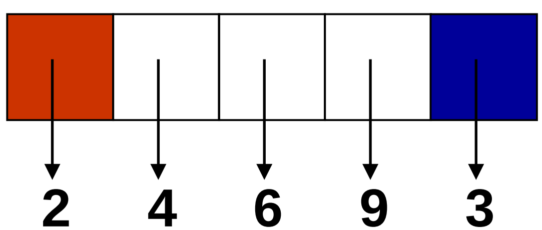
Sorted portion is coloured red.



Find smallest element in unsorted portion of container



Interchange the smallest element with the one at the front of the unsorted portion



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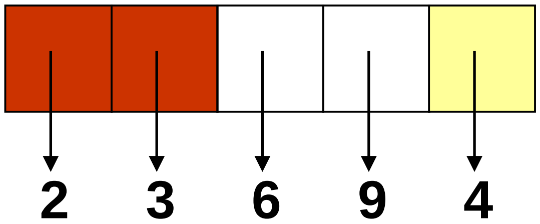

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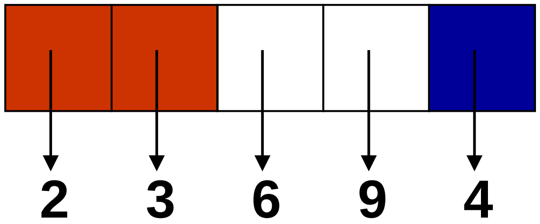
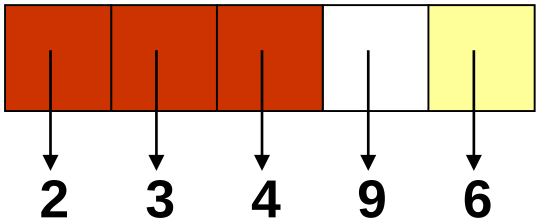
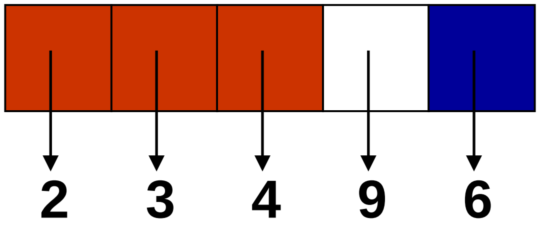


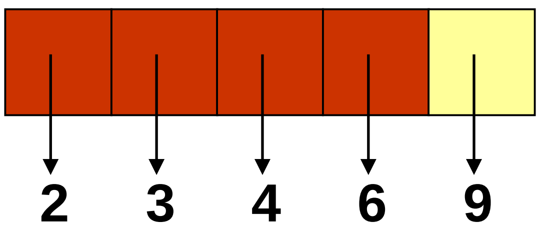
Find smallest element in unsorted portion of container



Interchange the smallest element with the one at the front of the unsorted portion



Find smallest element in unsorted portion of container



Interchange the smallest element with the one at the front of the unsorted portion

After n-1 repetitions of this process, the last item has automatically fallen into place

Selection Sort Using a Queue

***Approach to the problem*:**

- Create a queue **sorted**, originally empty, to hold the items that have been sorted ***so far***
- The contents of **sorted** will always be in order, with new items added at the end of the queue

Selection Sort Using Queue Algorithm

- While the unordered list **list** is not empty:
  - ***remove*** the smallest item from **list** and ***enqueue*** it to the end of **sorted**
- The list is now empty, and **sorted** contains the items in ascending order, from front to rear
- To restore the original list, ***dequeue*** the items one at a time from **sorted**, and ***add them to the rear*** of **list**

```
Algorithm selectionSort(list)
temp = empty queue
sorted = empty queue
while list is not empty do {
   smallestSoFar = remove first item from list
   while list is not empty do {
        item = remove first item from list
         if item < smallestSoFar {
               temp.enqueue(smallestSoFar)
               smallestSoFar = item
          }
         else temp.enqueue(item)
     }
     sorted.enqueue(smallestSoFar)
     while temp is not empty do
          add temp.dequeue() to the end of list
}
while sorted is not empty do
    add sorted.dequeue() to the end of list
```

Selection Sort is an **$O(n^2)$** algorithm

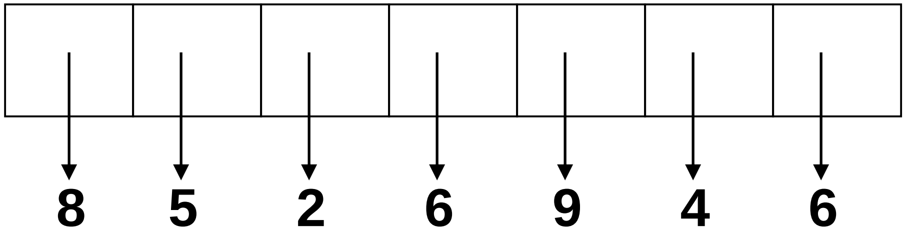
The analysis is similar to that of Insertion Sort. We will leave it as an exercise for you to analyze this algorithm.

Discussion

- Is there a best case?
  - No, we have to step through the entire remainder of the list looking for the next smallest item, no matter what the ordering
- Is there a worst case?
  - No

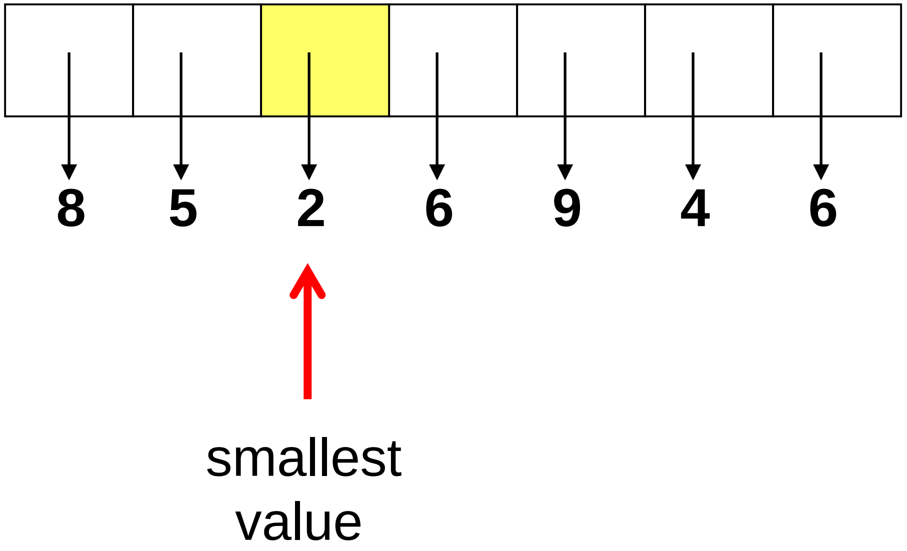
In-Place SelectionSort

Selection sort without using any additional data structures. Assume that the values to sort are stored in an array.



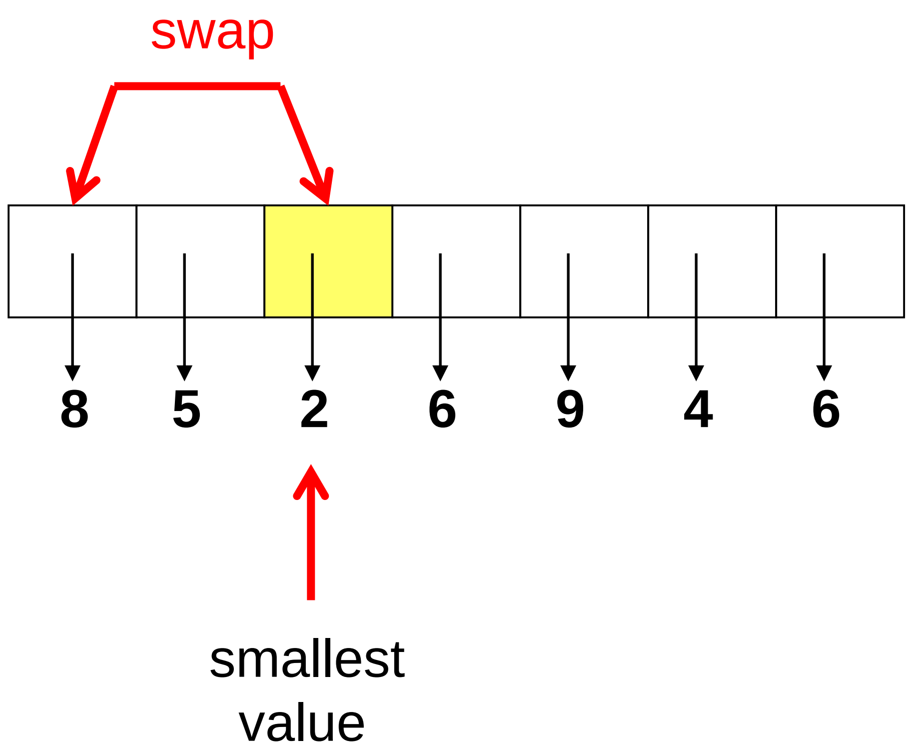
In-Place SelectionSort

First find the smallest value



In-Place SelectionSort

Swap it with the element in the first position of the array.



In-Place SelectionSort

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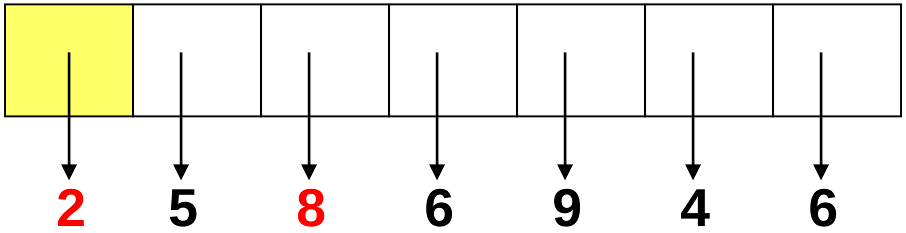
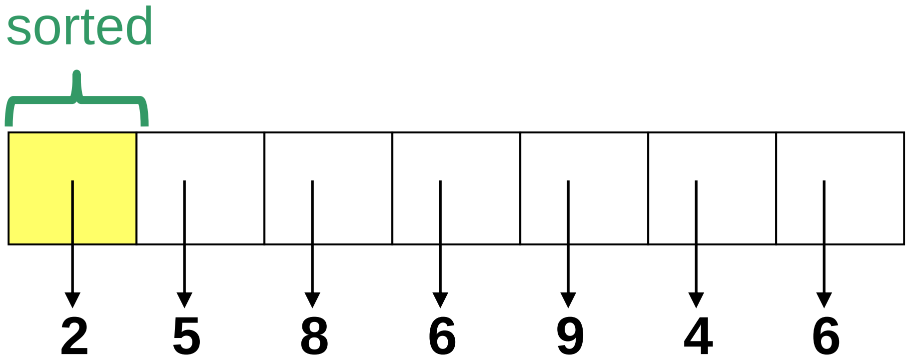
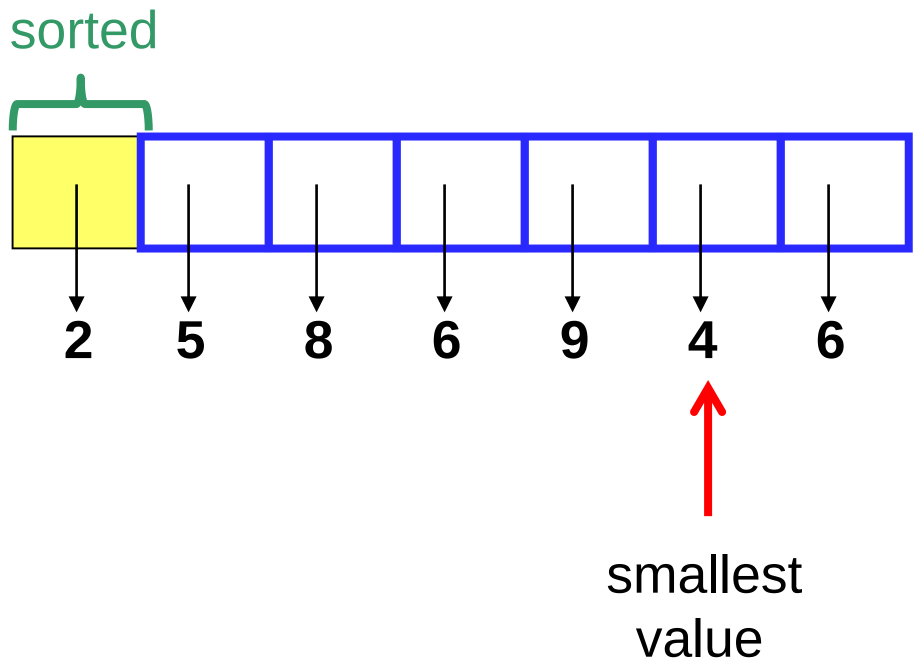


In-Place SelectionSort



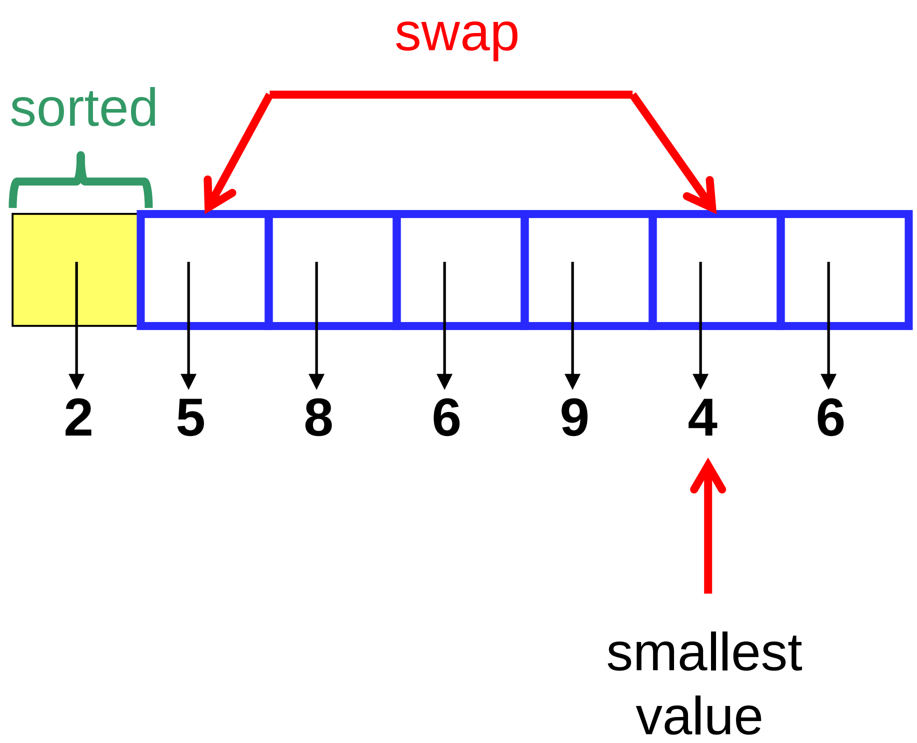
In-Place SelectionSort

Now consider the rest of the array and again find the smallest value.

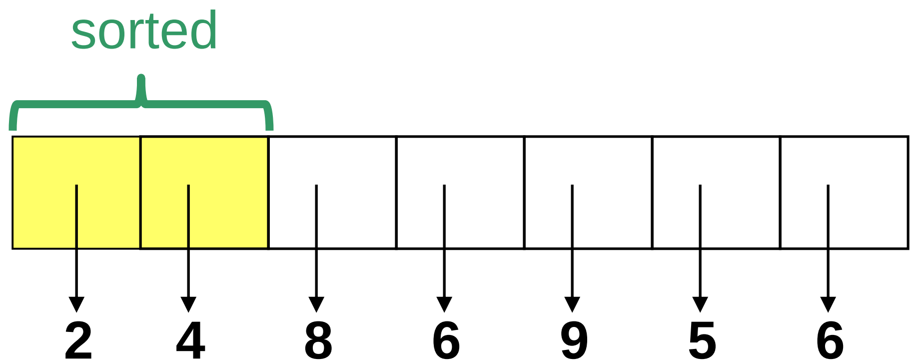


In-Place SelectionSort

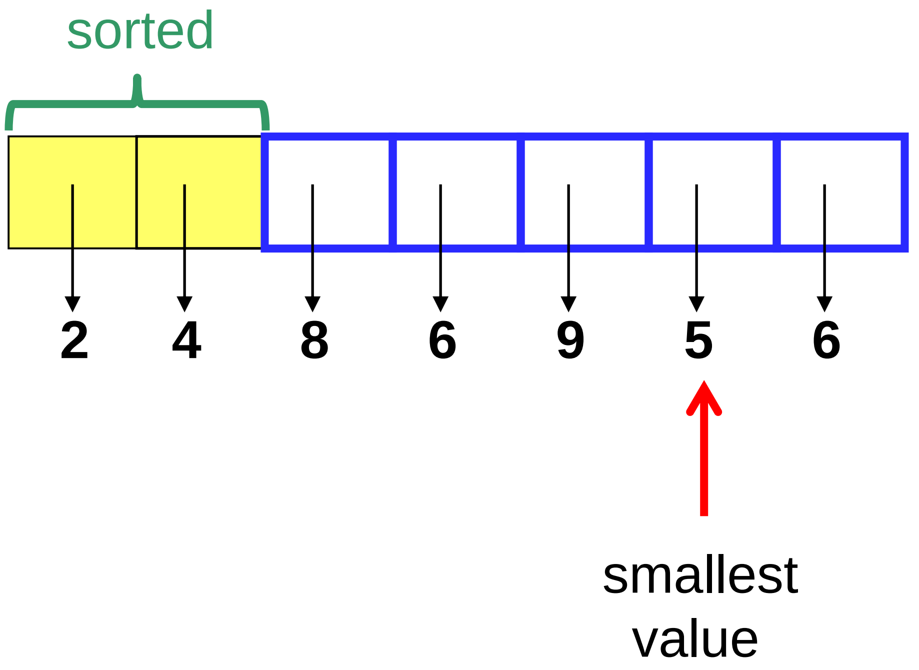
Swap it with the element in the second position of the array, and so on.



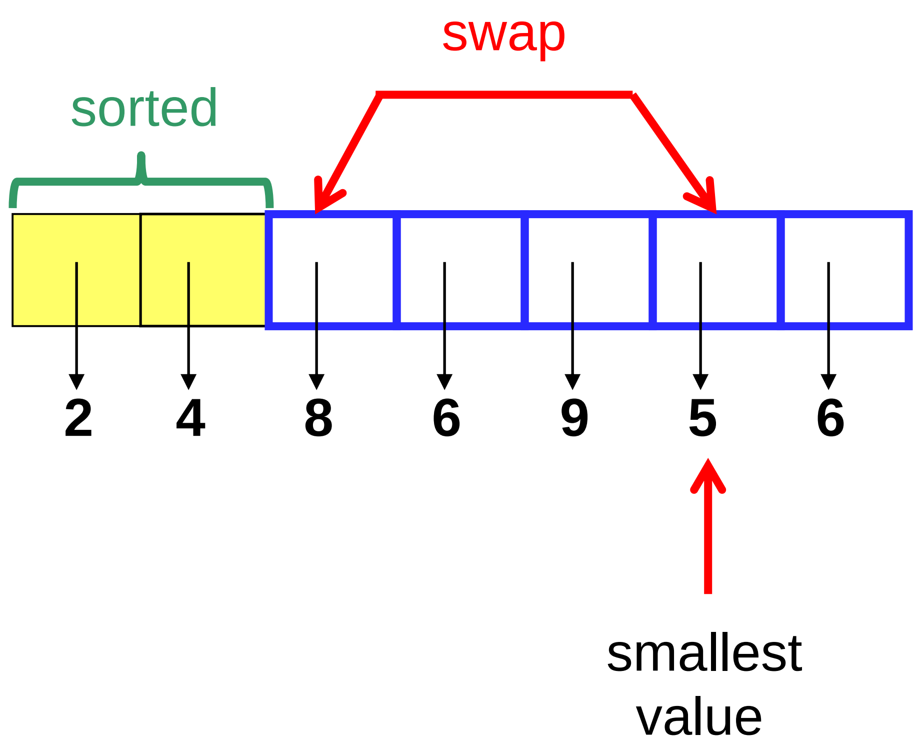
In-Place SelectionSort



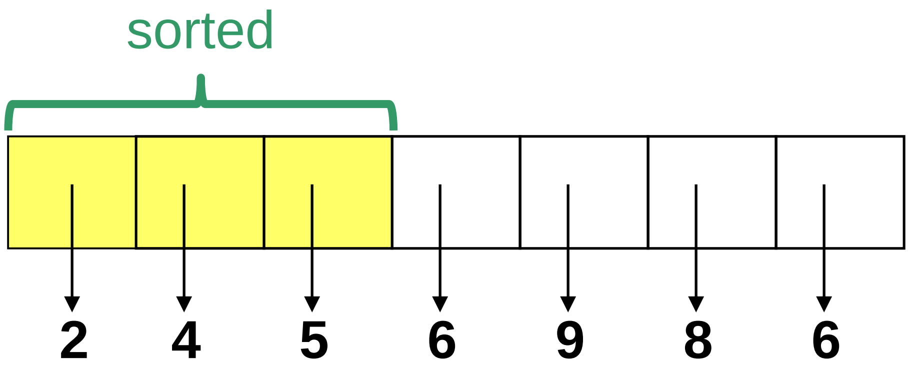
In-Place SelectionSort



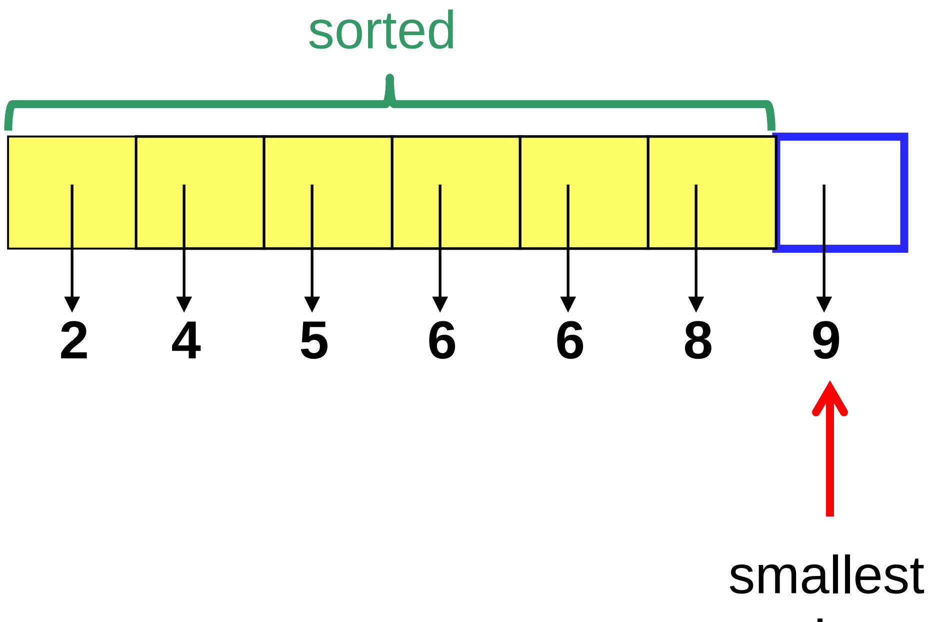
In-Place SelectionSort



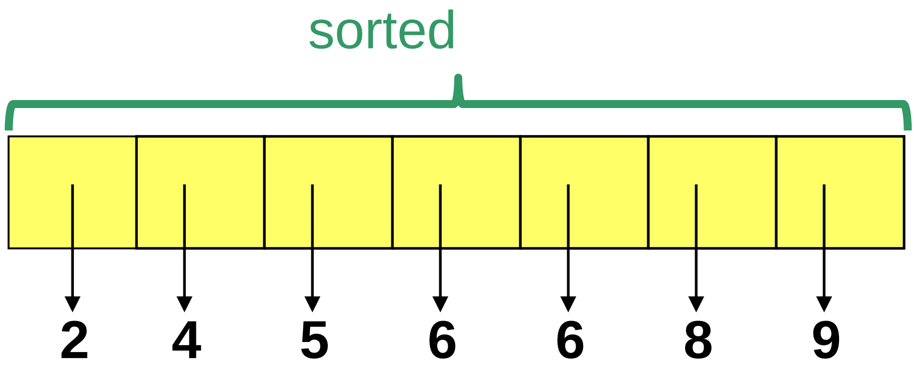
In-Place SelectionSort



In-Place SelectionSort



In-Place SelectionSort



```
Algorithm selectionSort (A,n)
In: Array A storing n values
Out: {Sort A in increasing order}
for i = 0 to n-2 do {
    // Find the smallest value in unsorted subarray A[i..n-1]
    smallest = i
    for j = i + 1 to n - 1do {
        if A[j] < A[smallest] then
            smallest = j
    }
    // Swap A[smallest] and A[i]
    temp = A[smallest]
    A[smallest] = A[i]
    A[i] = temp
}
```

Quick Sort

- ***Quick Sort*** orders a sequence of values by ***partitioning*** the list around one element (called the ***pivot*** or ***partition element***), then sorting each partition
- More specifically:
  - Choose one element in the sequence to be the pivot
  - Organize the remaining elements into three groups (***partitions***): those ***greater than*** the pivot, those ***less than*** the pivot, and those ***equal*** to the pivot
  - Then sort each of the first two partitions (recursively)

Quick Sort

Partition element or ***pivot***:

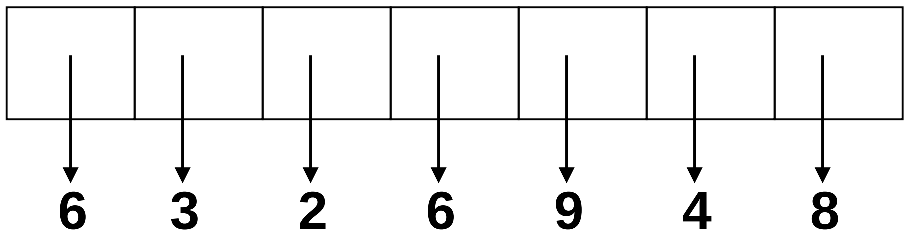
- The choice of the **pivot** is arbitrary
- For efficiency, it would be nice if the pivot divided the sequence roughly in half
  - However, the algorithm will work in any case

Quick Sort

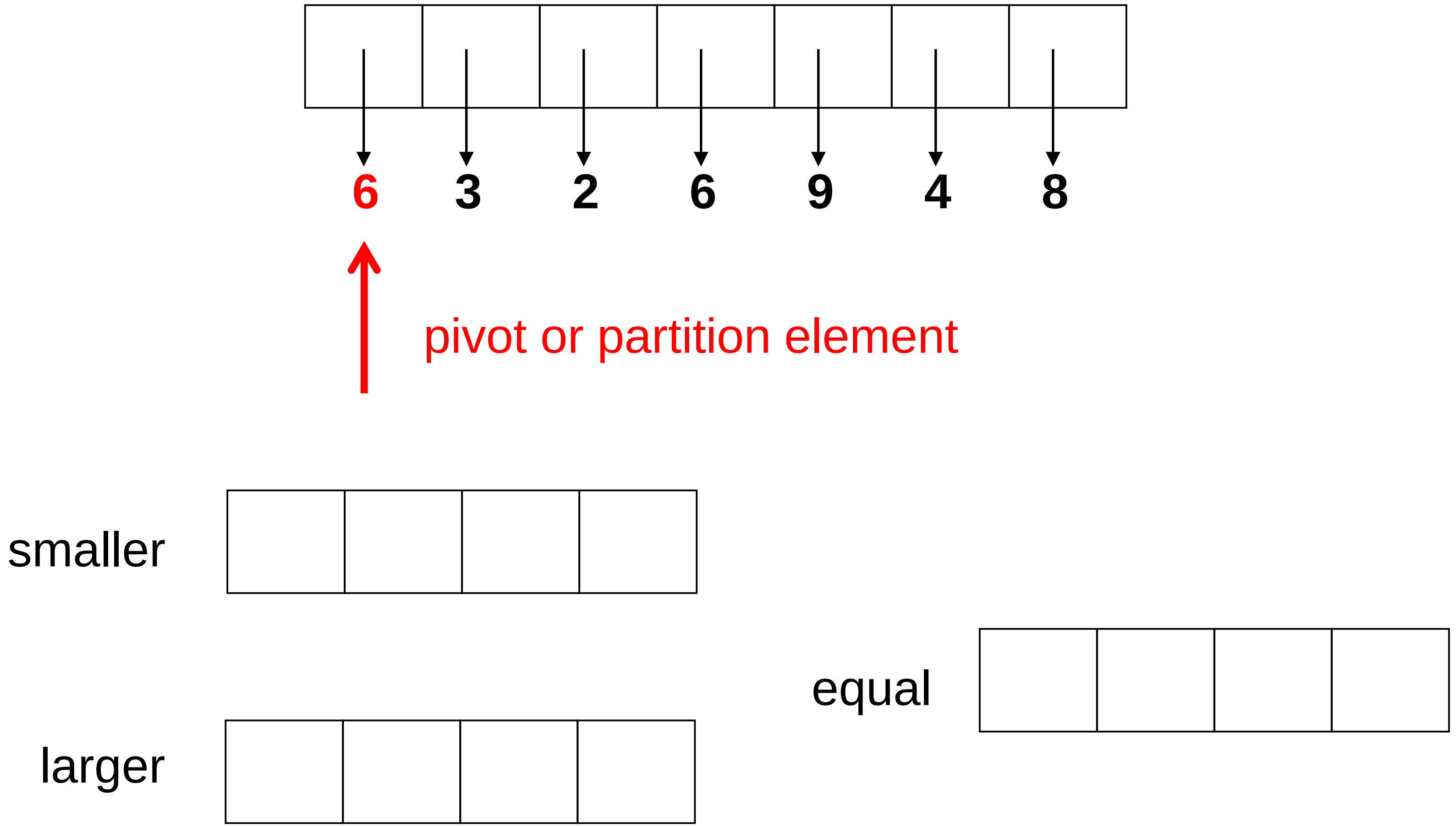
Approach to the problem:

- We put all the items to be sorted into a container (e.g. an array)
- We choose the pivot (partition element) as the first element from the container
- We use a container **smaller** to hold the items that are smaller than the pivot, a container **larger** to hold the items that are larger than the pivot, and a container **equal** to hold the items of the same value as the pivot
- We then ***recursively*** sort the items in the containers **smaller** and **larger**
- Finally, copy the elements from **smaller** back to the original container, followed by the elements from **equal**, and finally the ones from **larger**

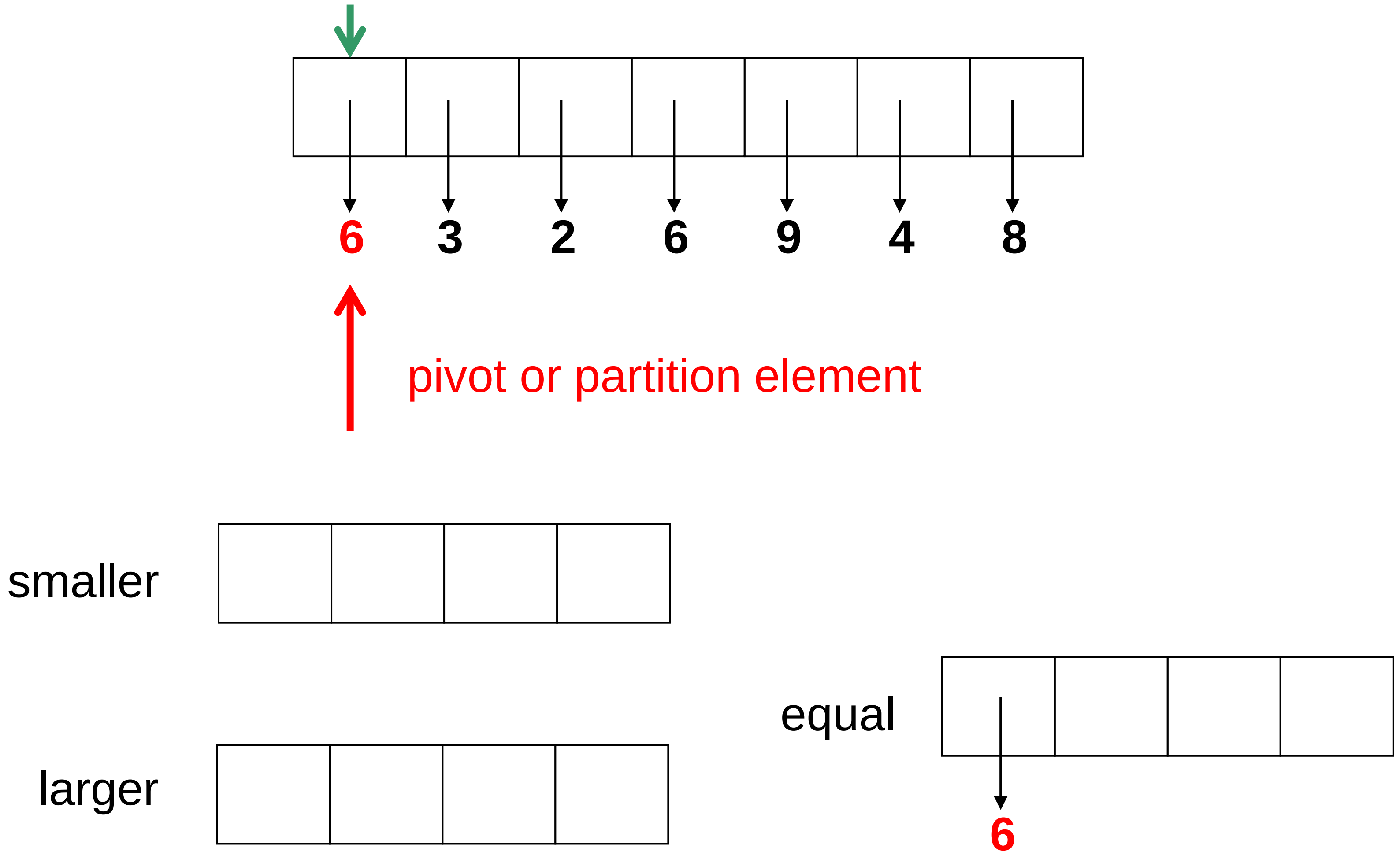
QuickSort



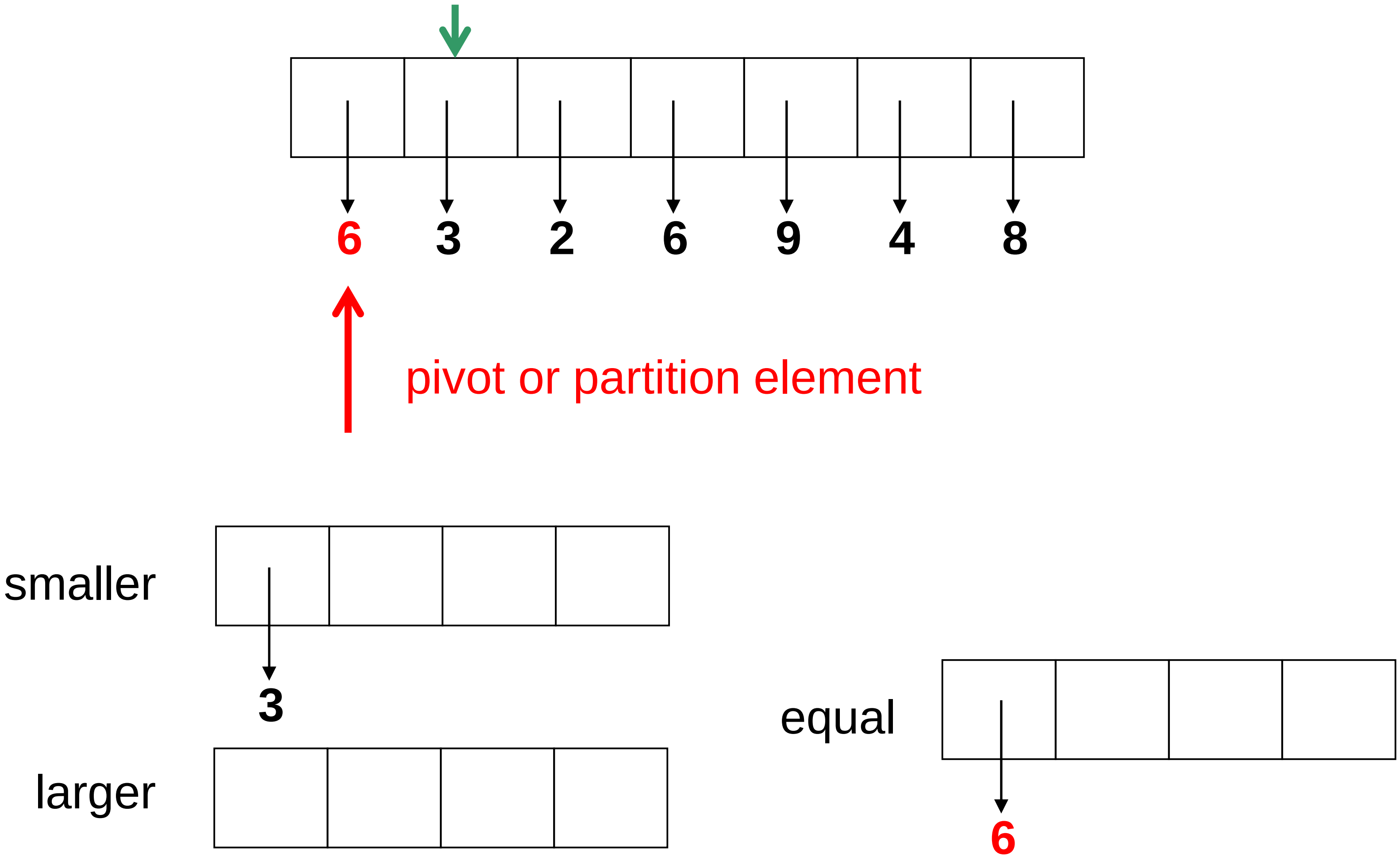
QuickSort



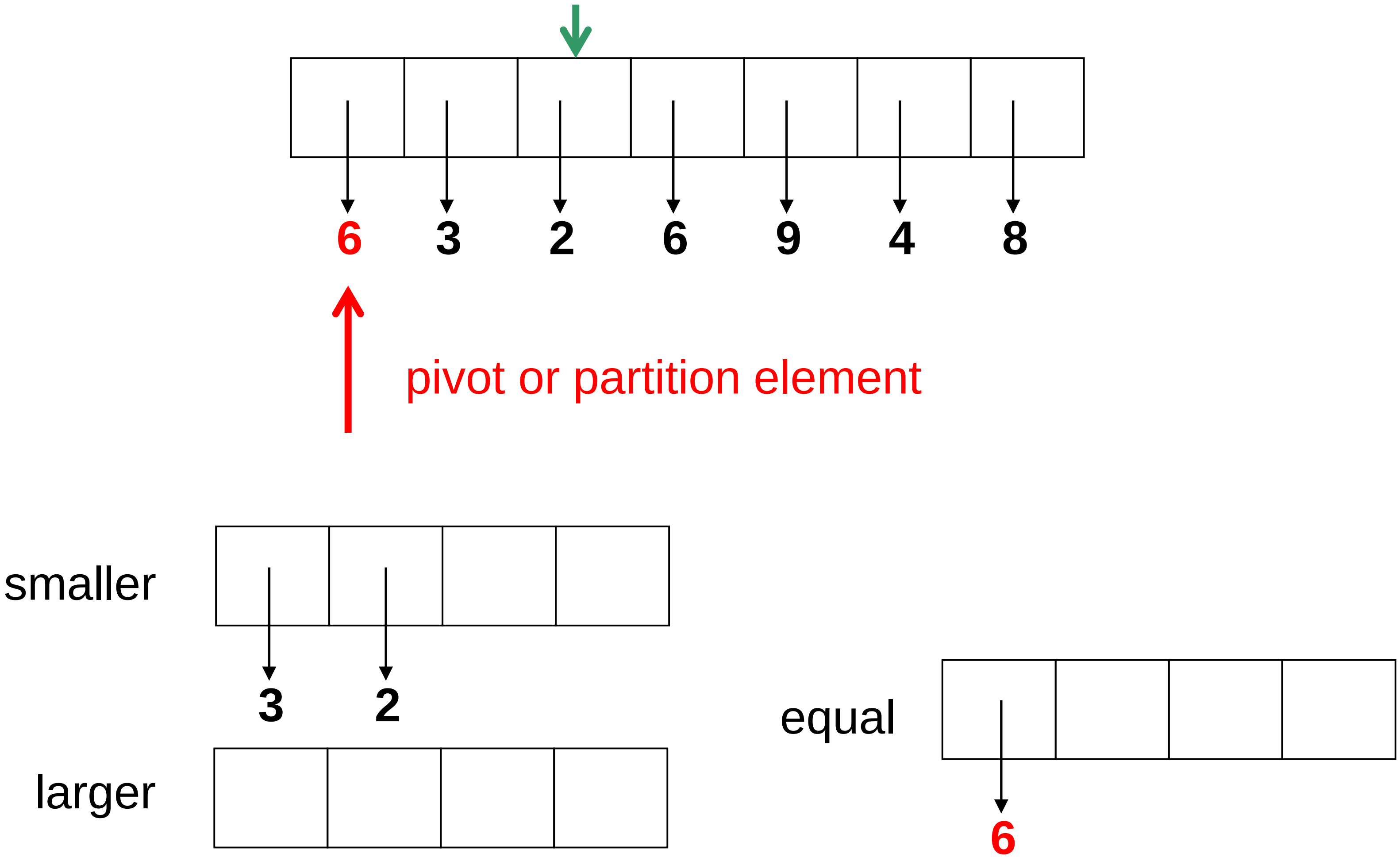
QuickSort



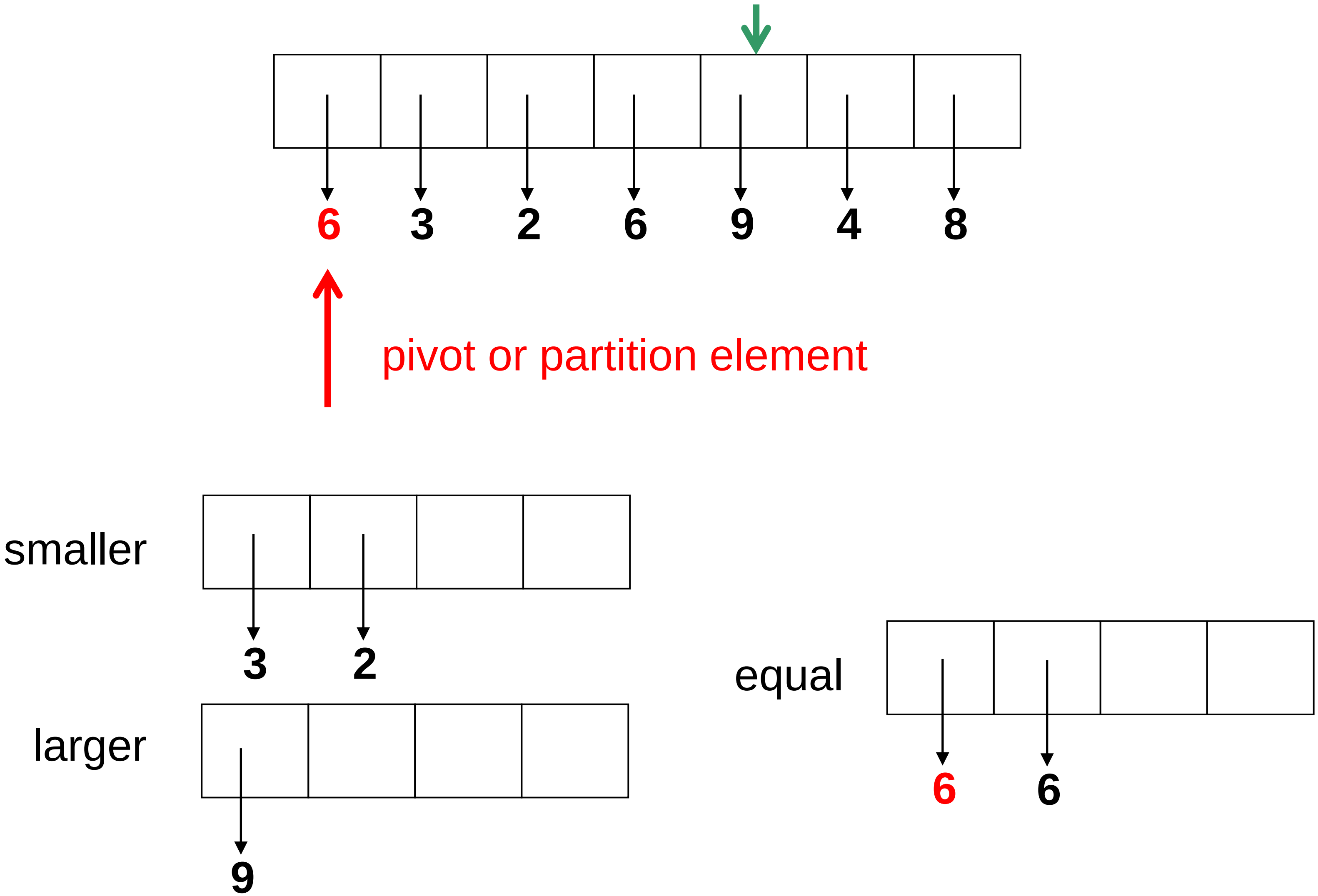
QuickSort


6 3 2 6 9 4 8
pivot or partition element
smaller
3
larger
equal
6

QuickSort

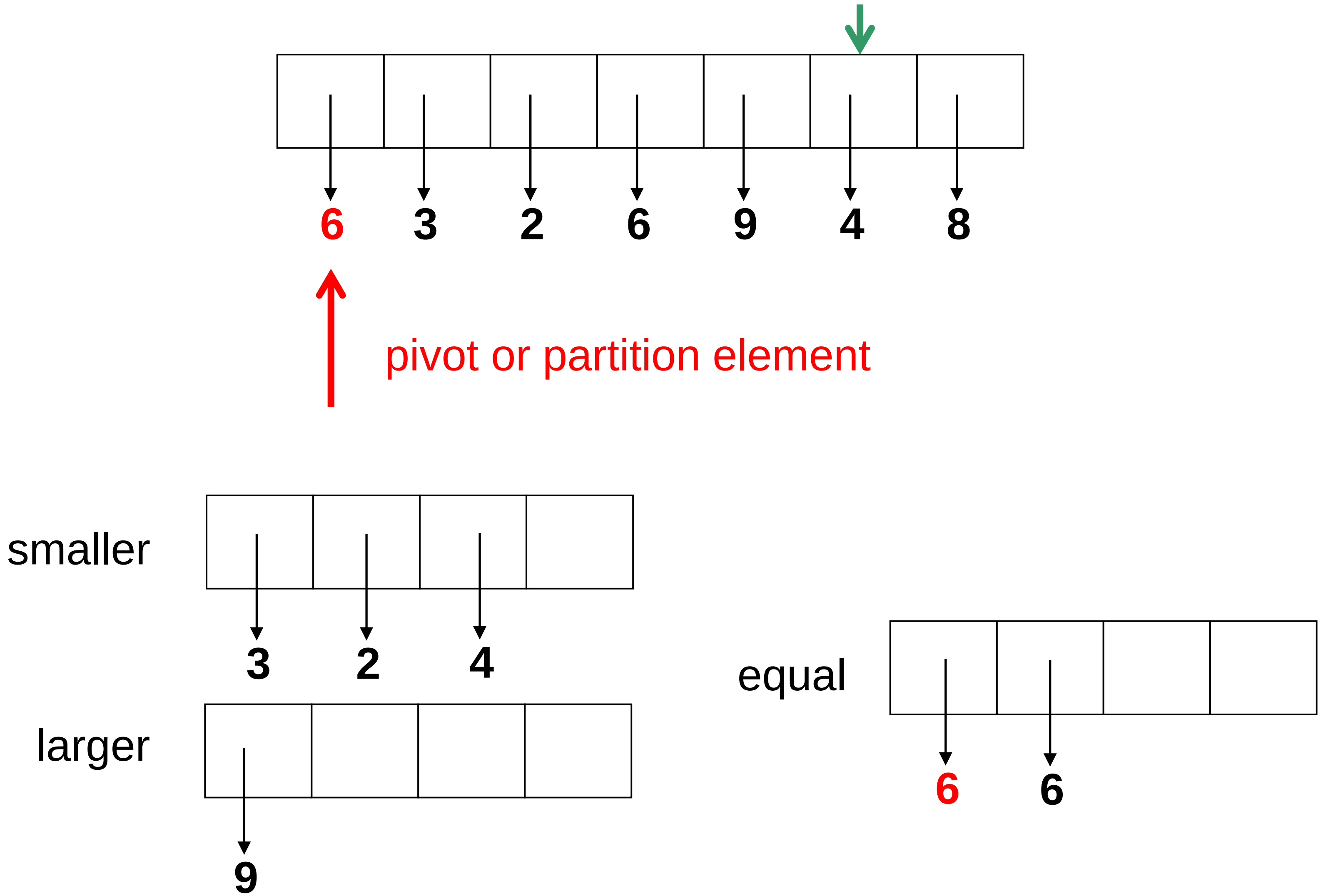

6 3 2 6 9 4 8
pivot or partition element
smaller
3 2
larger
equal
6

QuickSort



6
3
2
6
9
4
8
pivot or partition element
smaller
3
2
larger
9
equal
6
6

QuickSort


6
3
2
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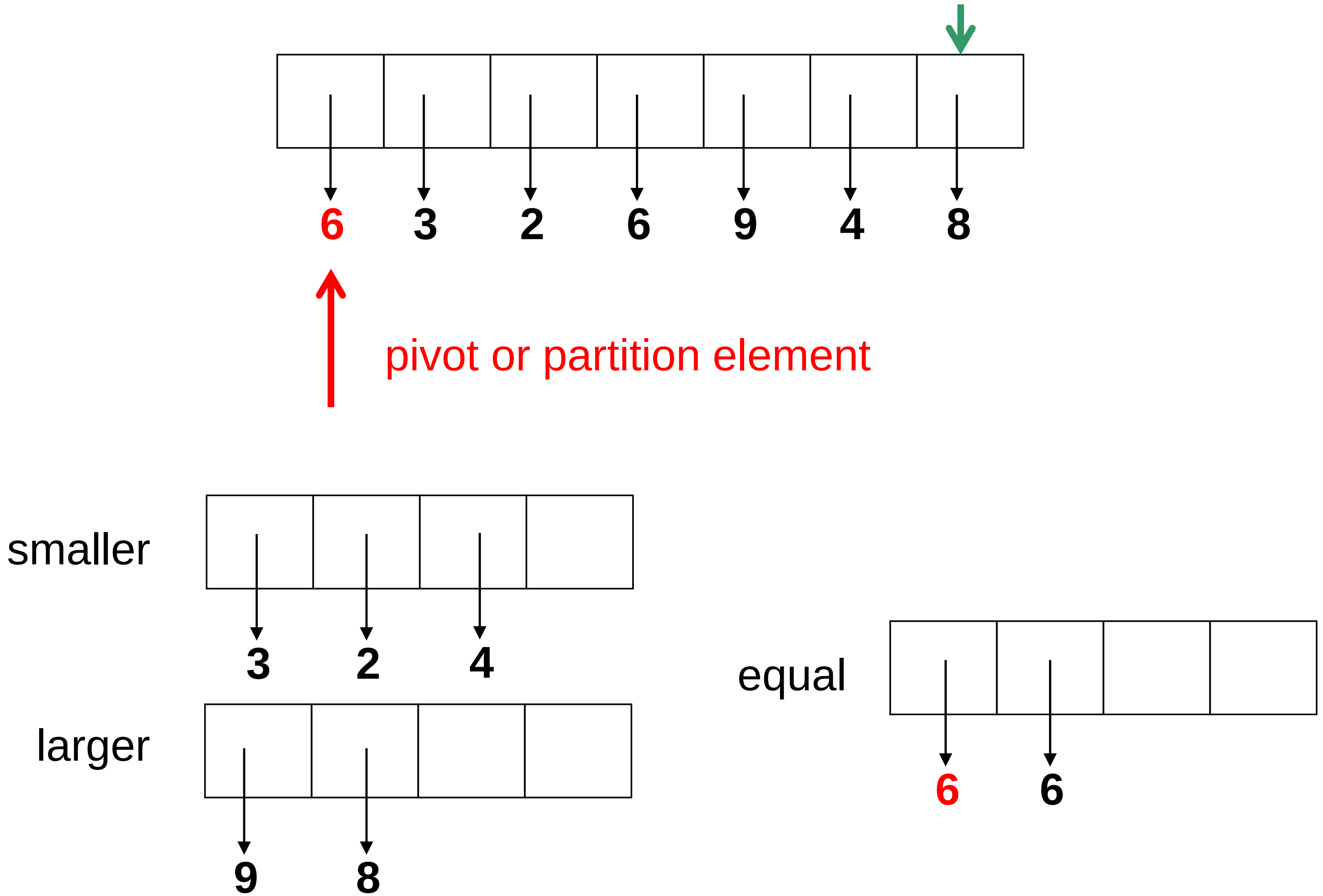
pivot or partition element
smaller
3
2
4
larger
9
equal
6
6

QuickSort


6
3
2
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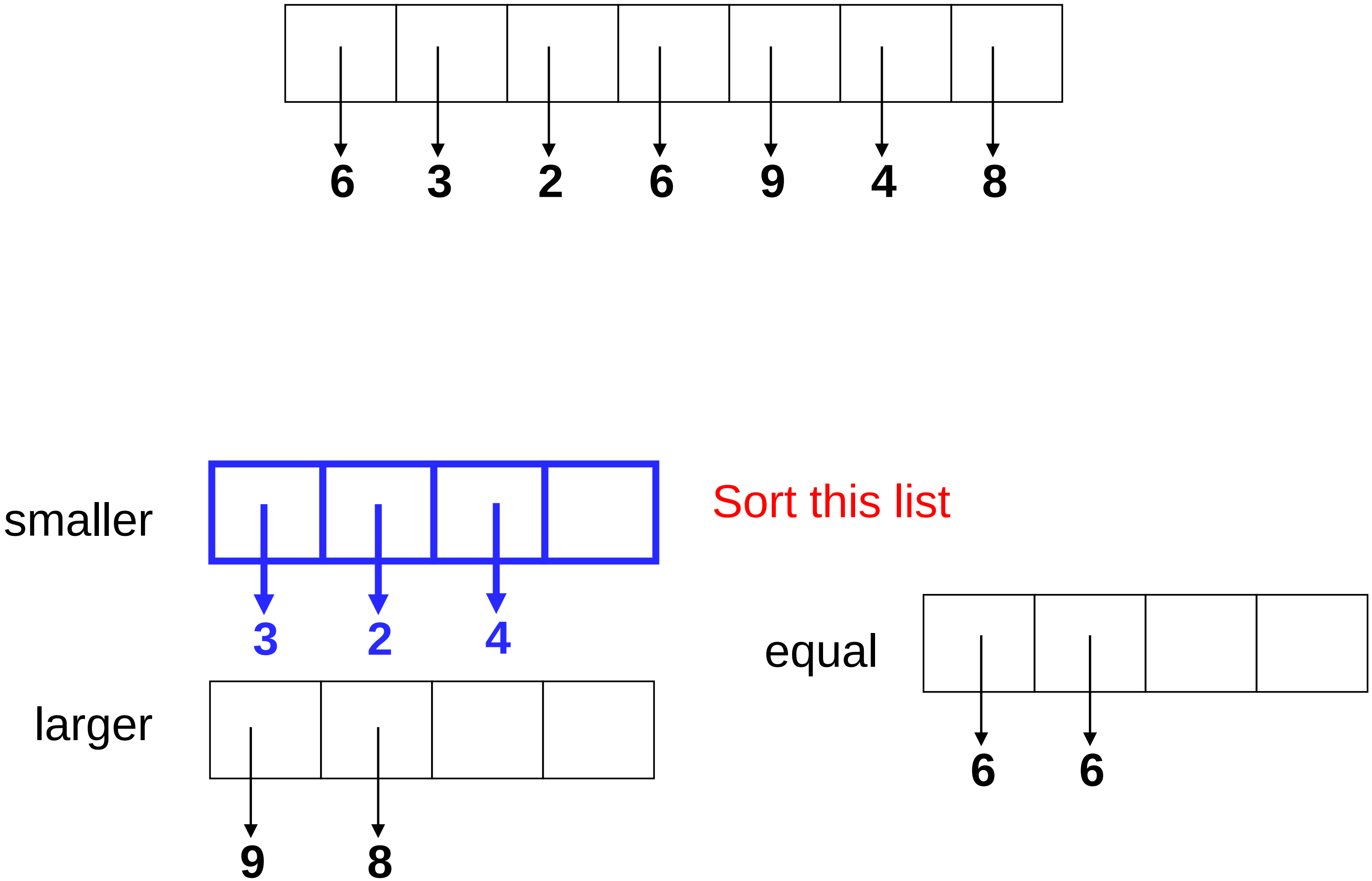
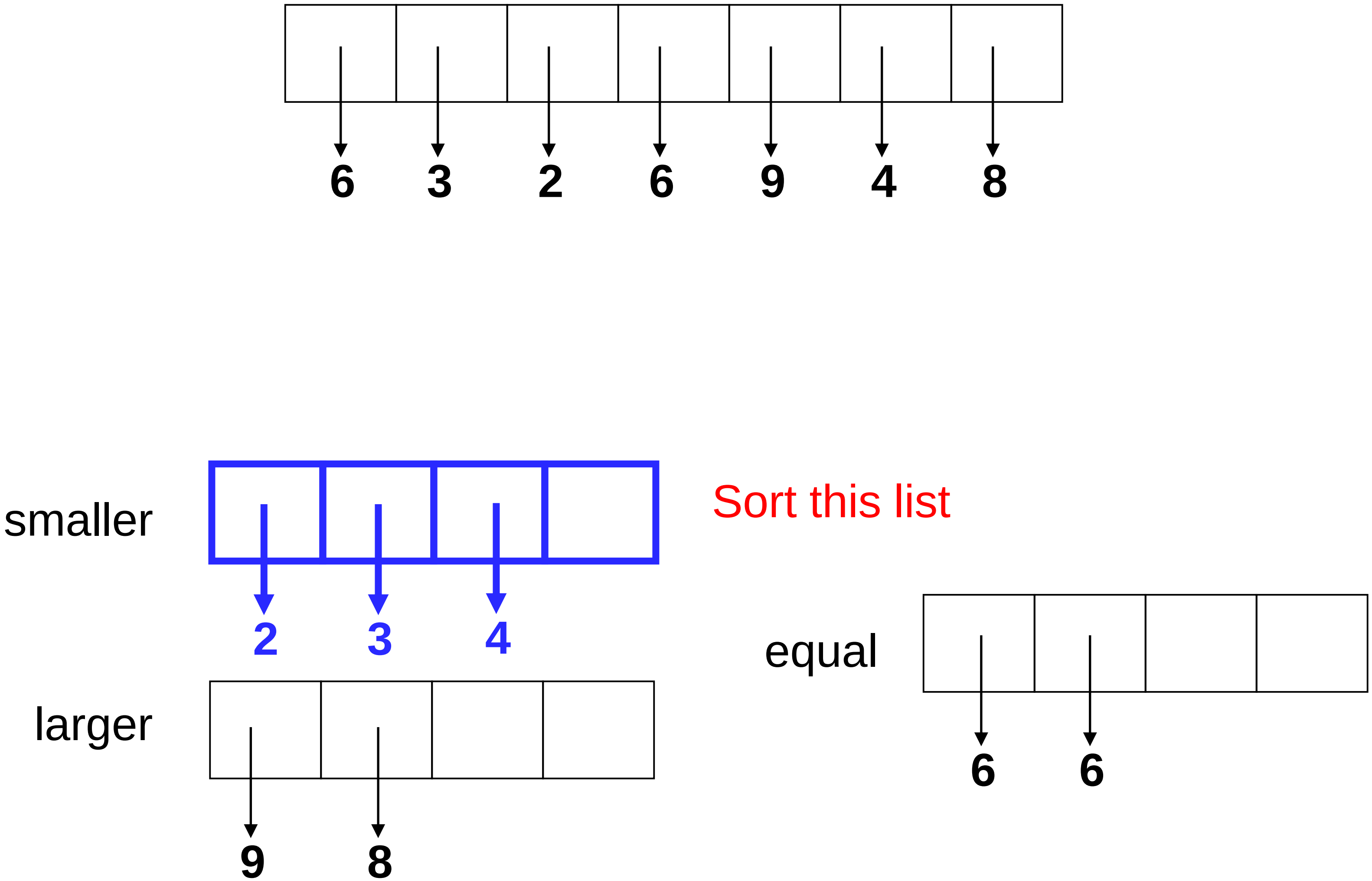
pivot or partition element
smaller
3
2
4
larger
9
8
equal
6
6

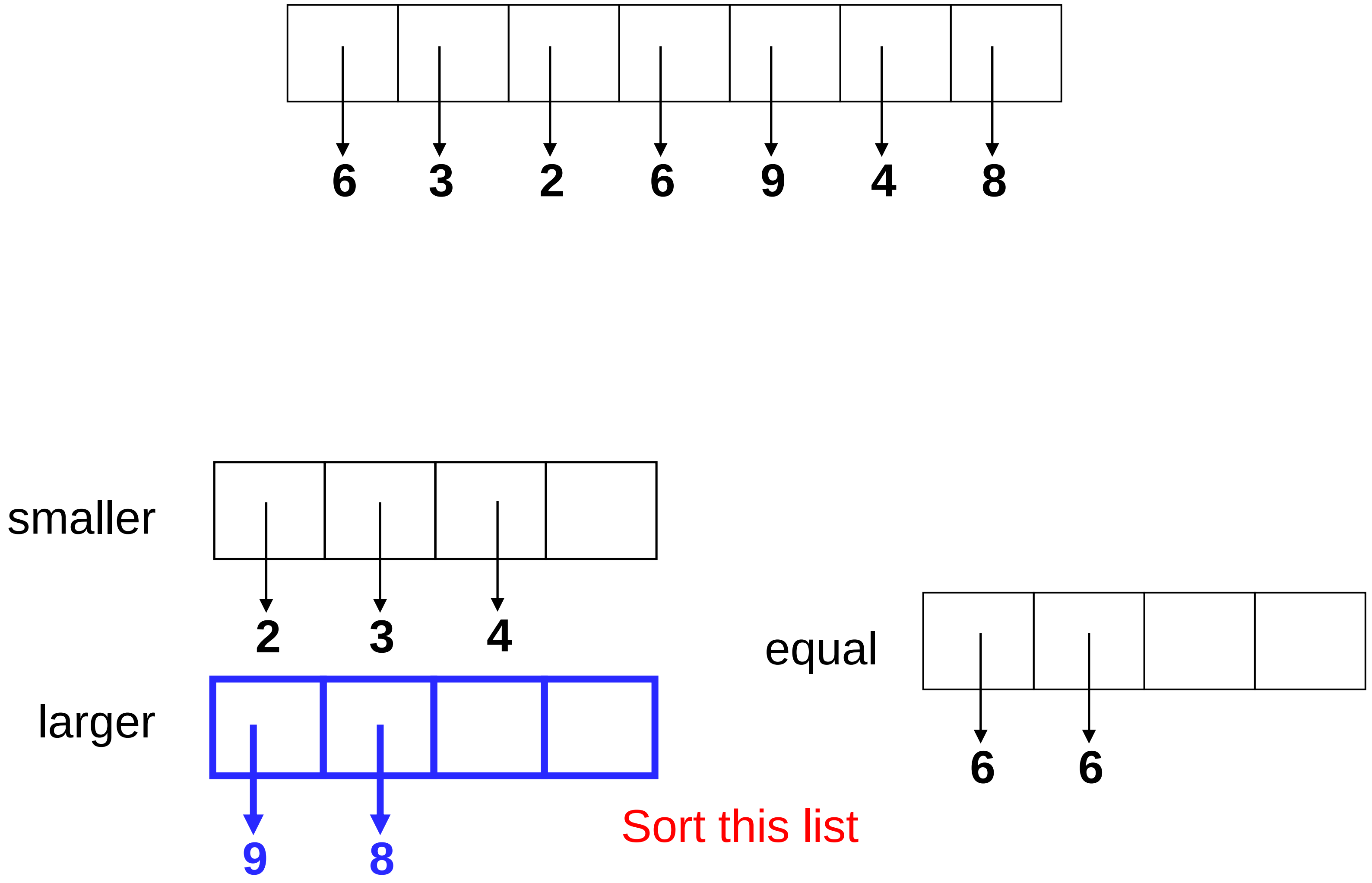
QuickSort



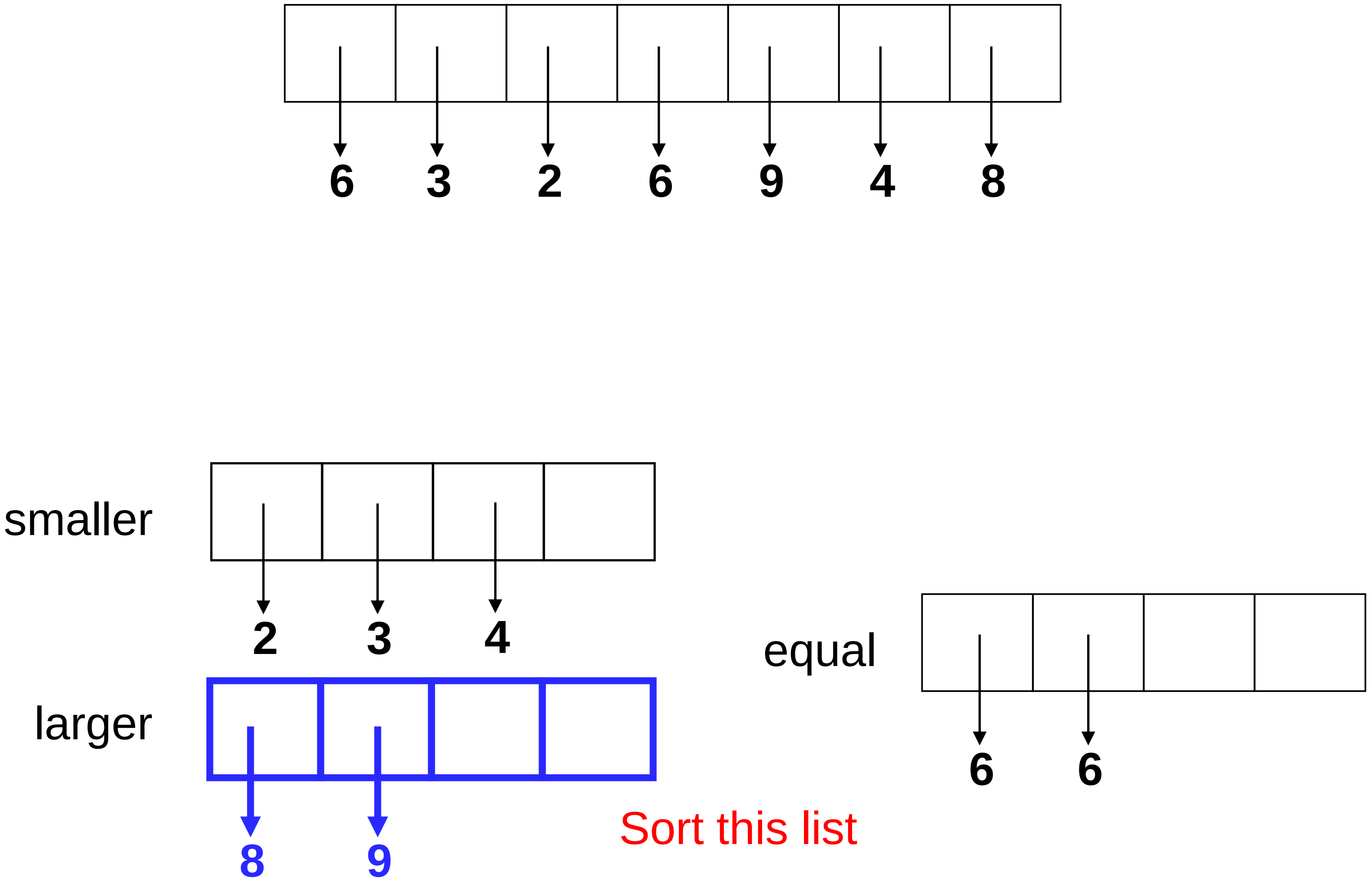
QuickSort



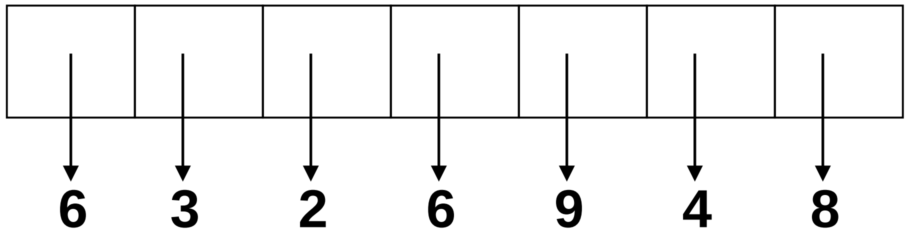
QuickSort



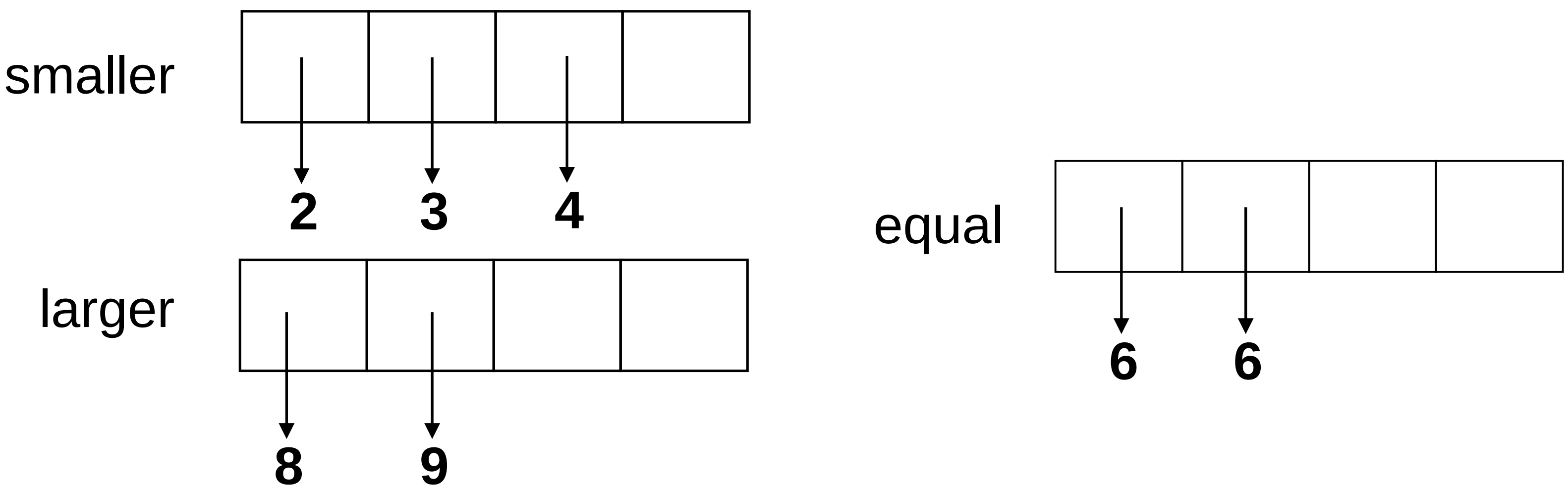
QuickSort



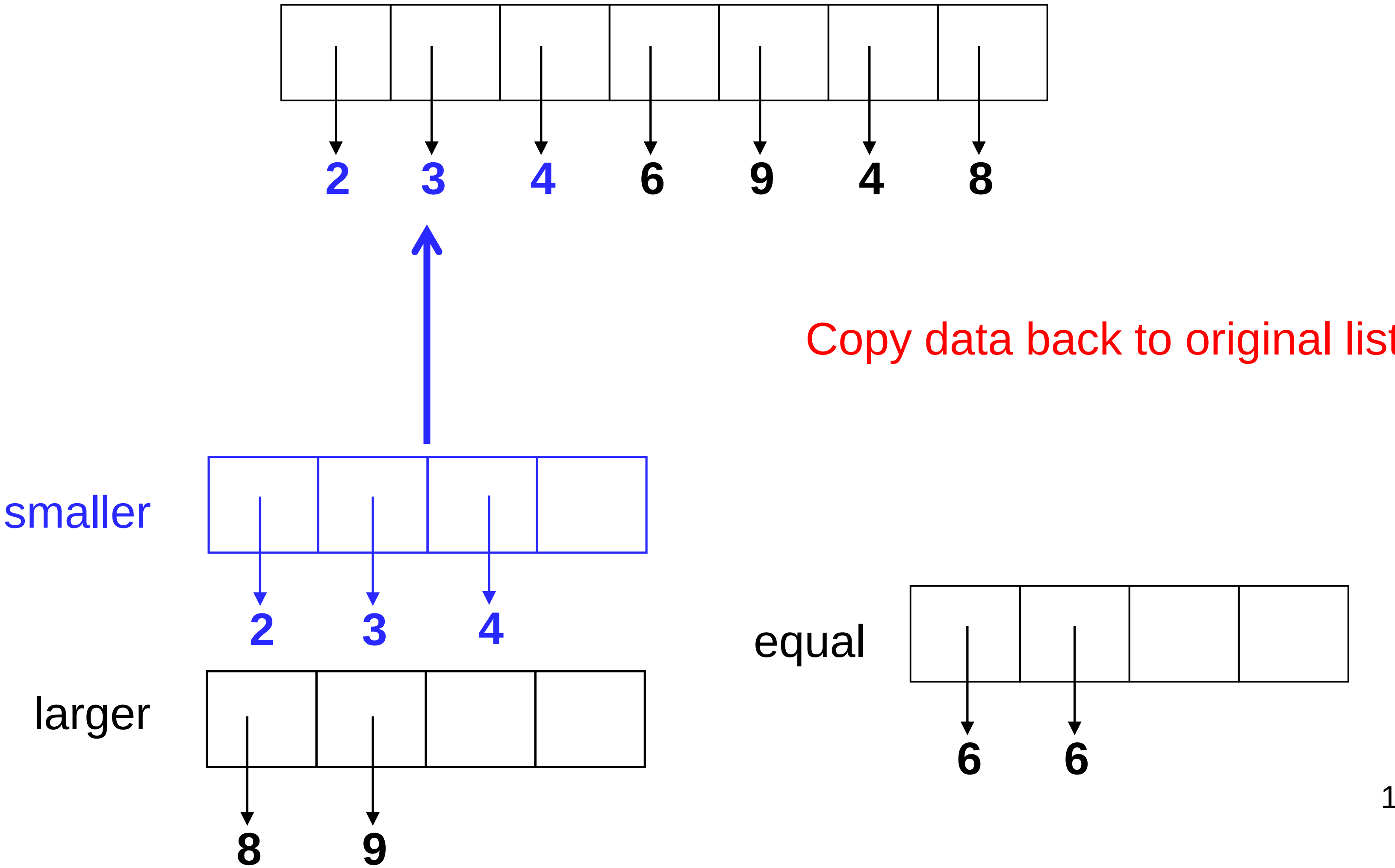
QuickSort



Copy data back to original list

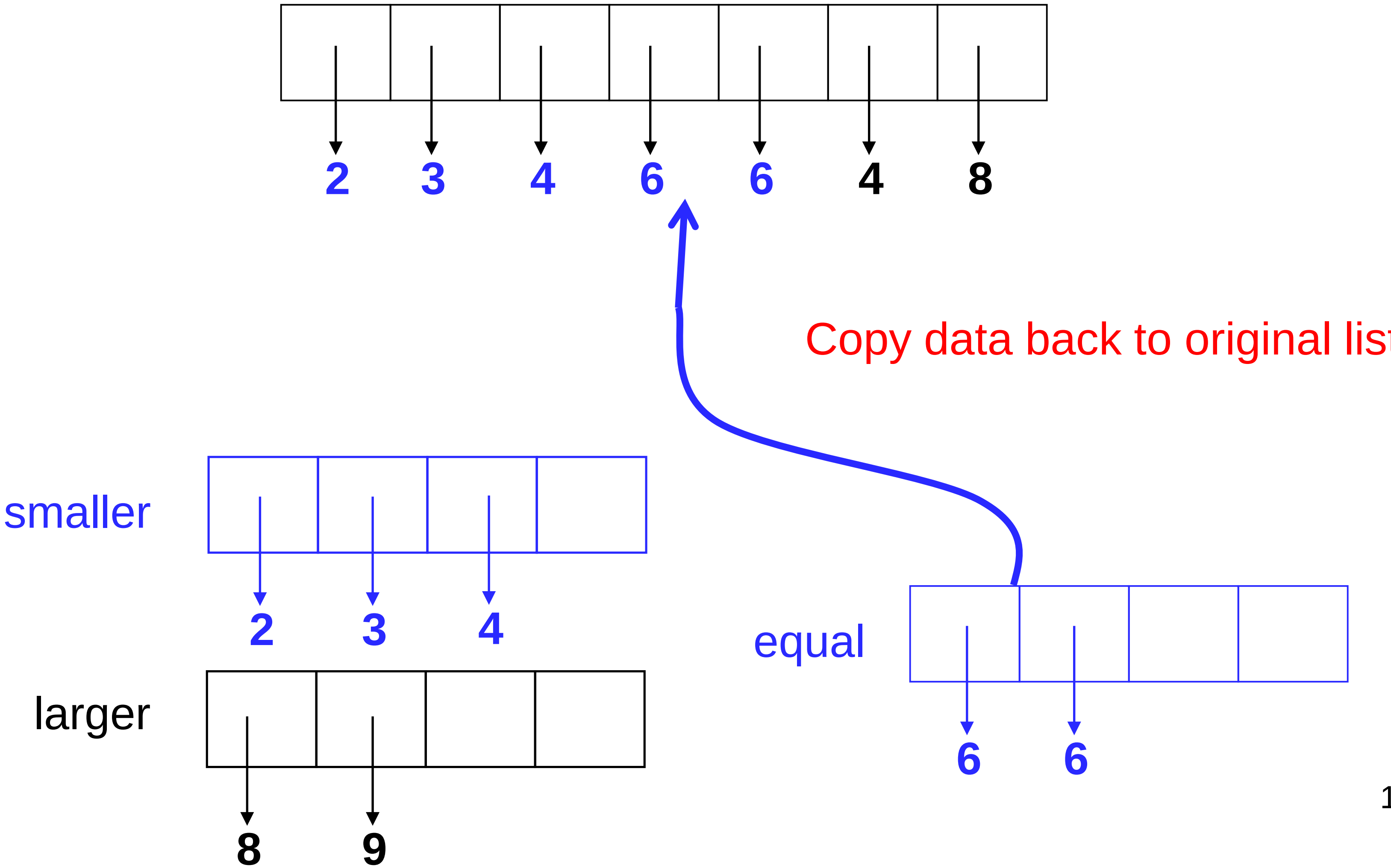


QuickSort



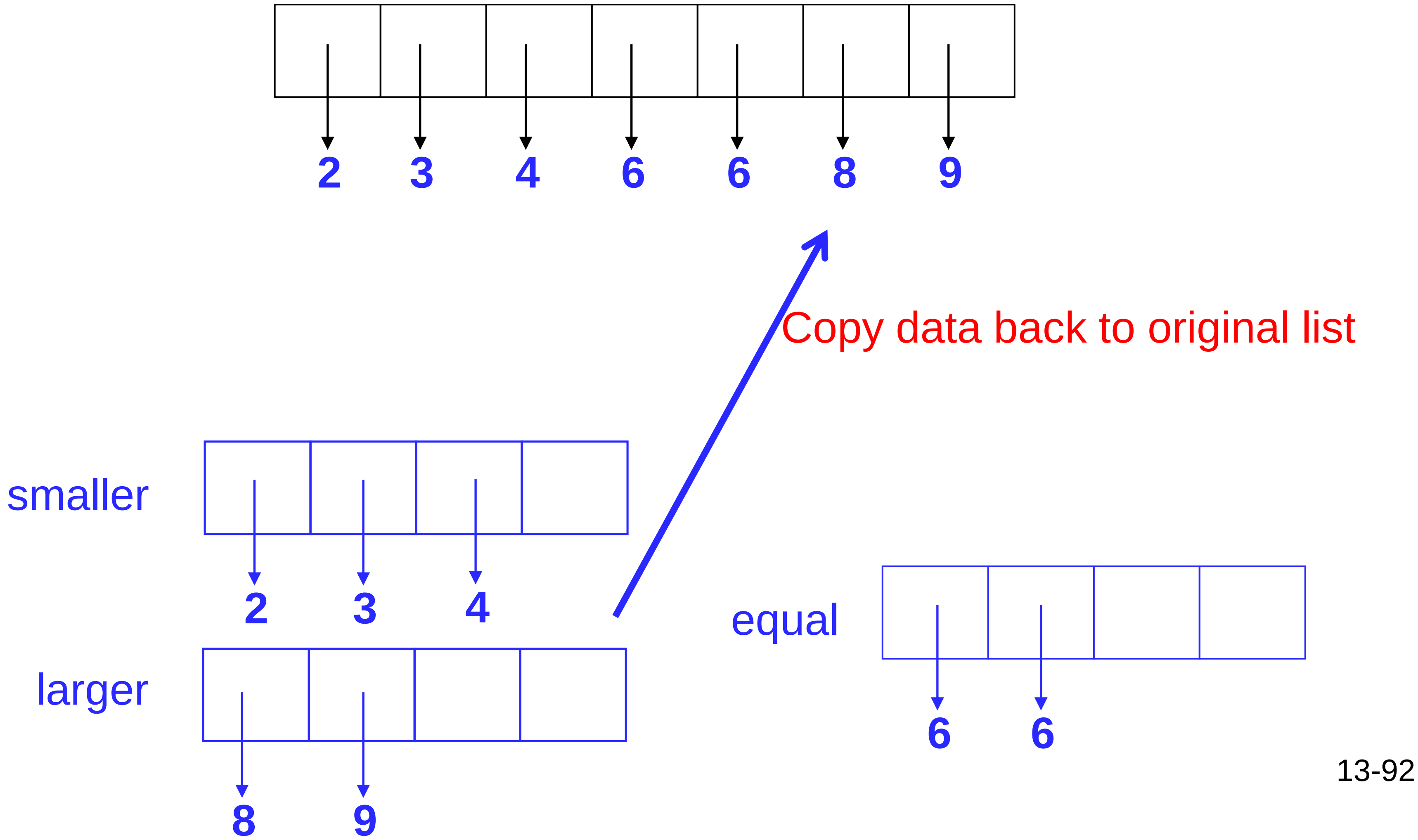
2
3
4
6
9
4
8
Copy data back to original list
smaller
2
3
4
equal
6
6
larger
8
9

QuickSort

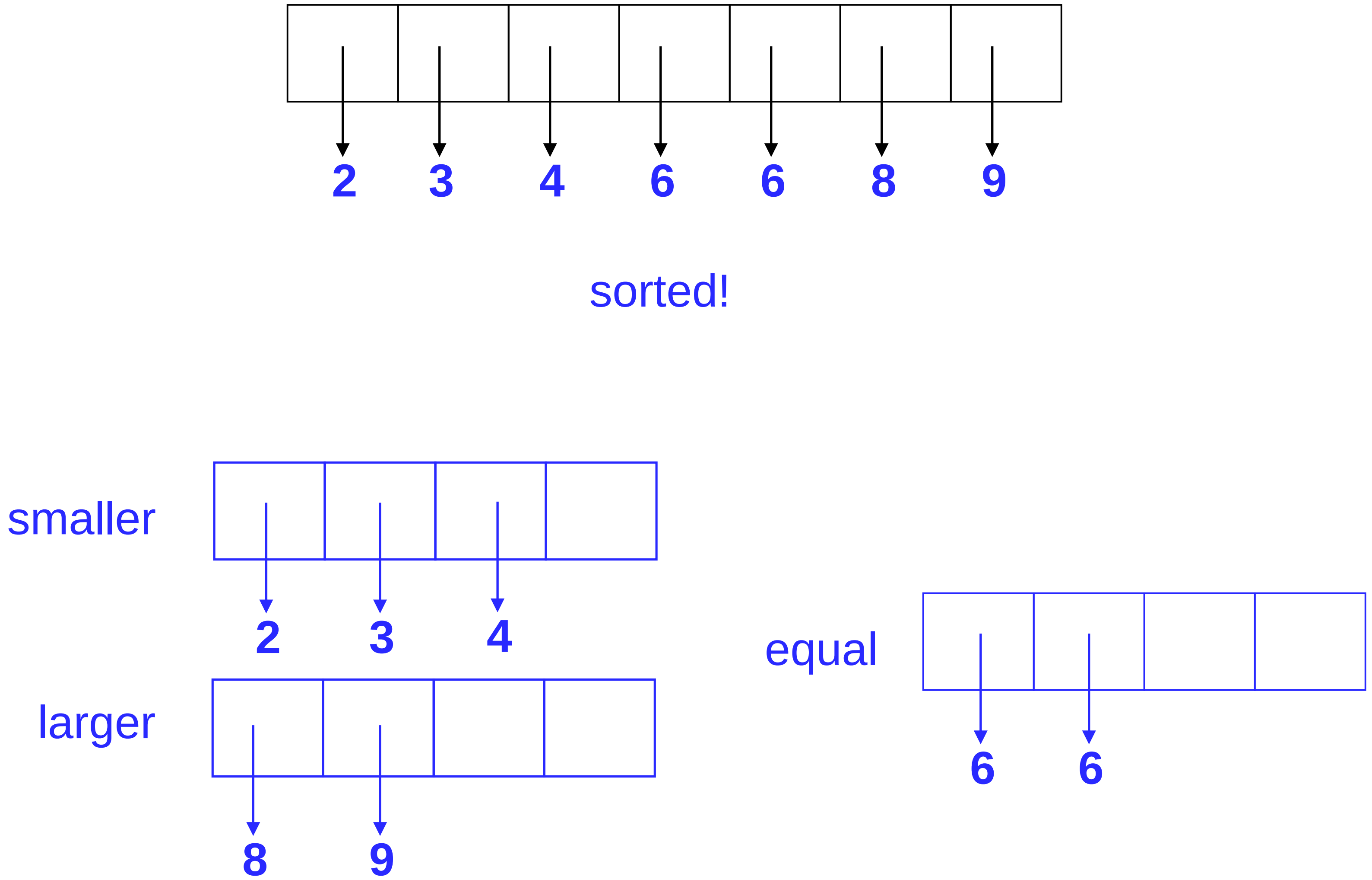


2
3
4
6
6
4
8
Copy data back to original list
smaller
2
3
4
larger
8
9
equal
6
6

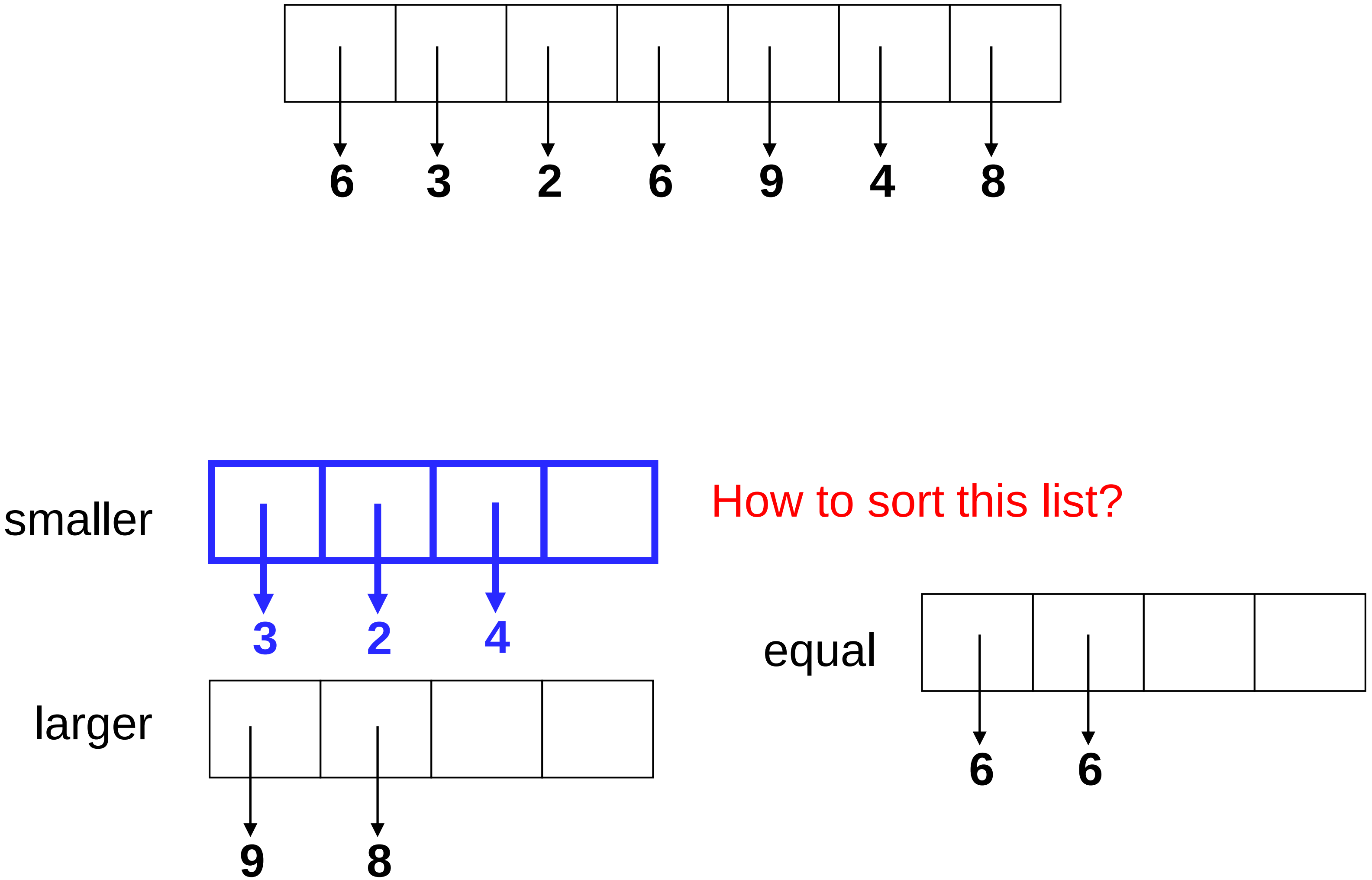
QuickSort


2
3
4
6
6
8
9
Copy data back to original list
smaller
2
3
4
larger
8
9
equal
6
6

QuickSort

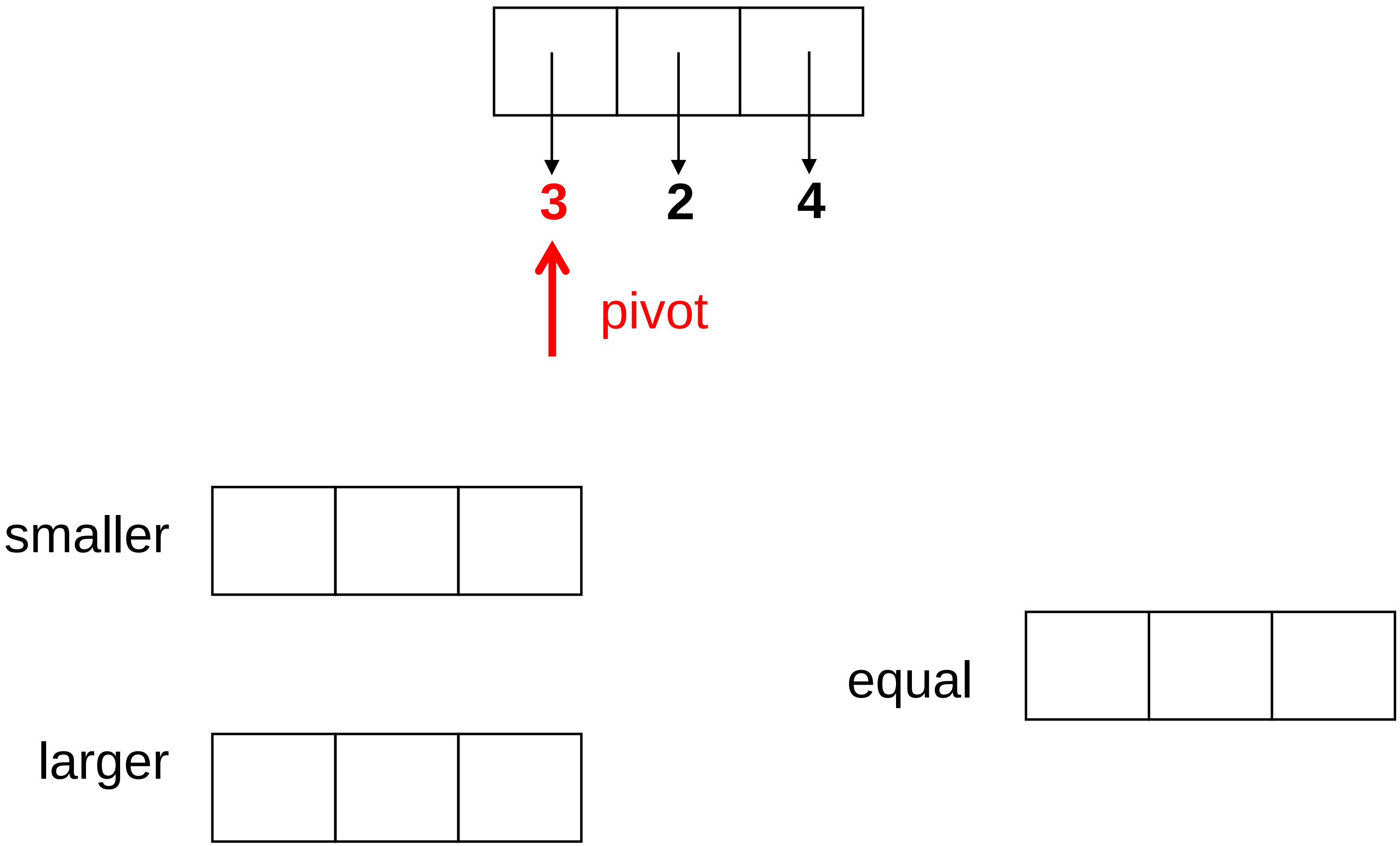


QuickSort

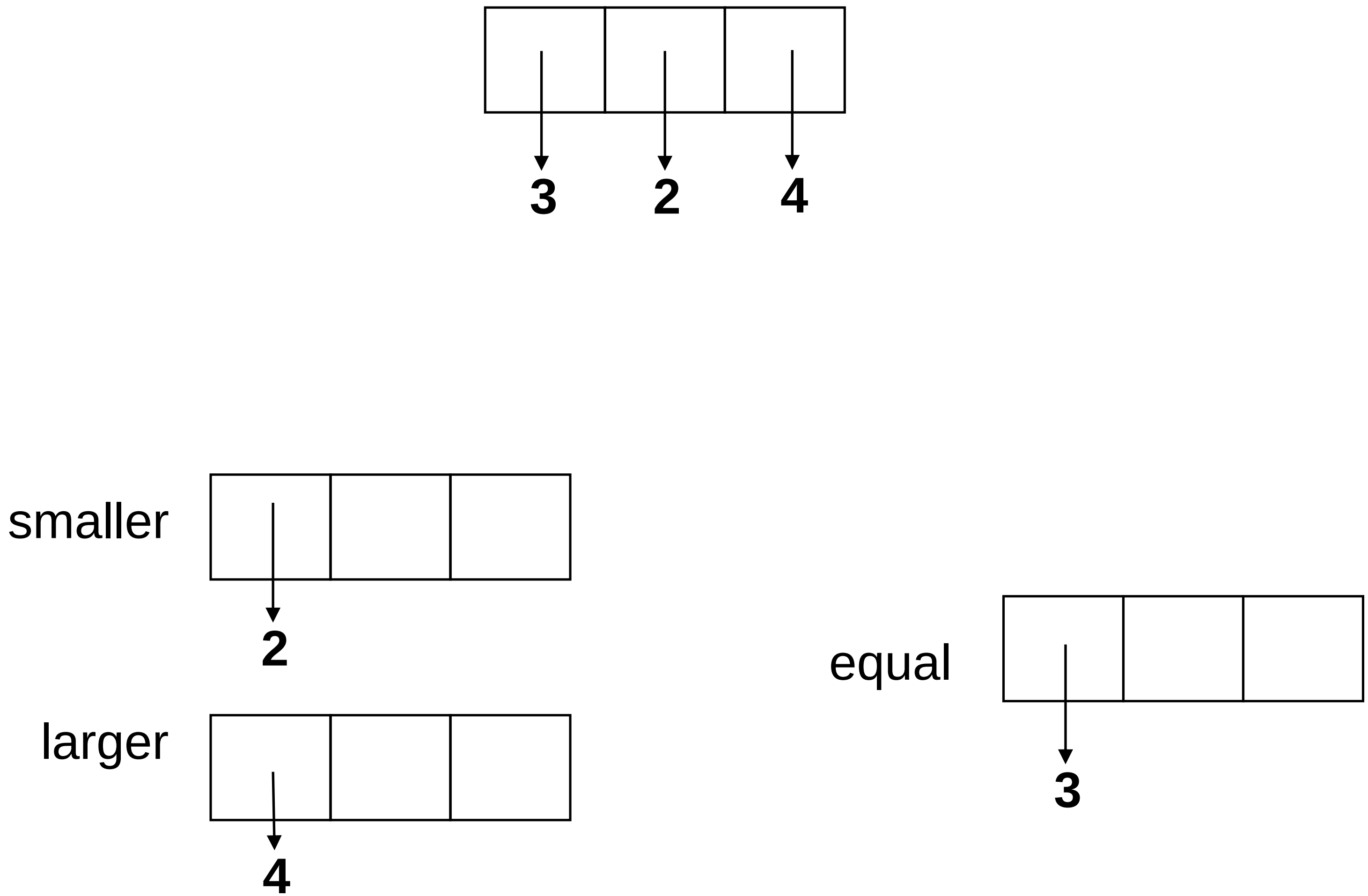


How to sort this list?

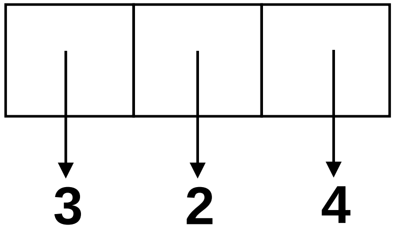
QuickSort

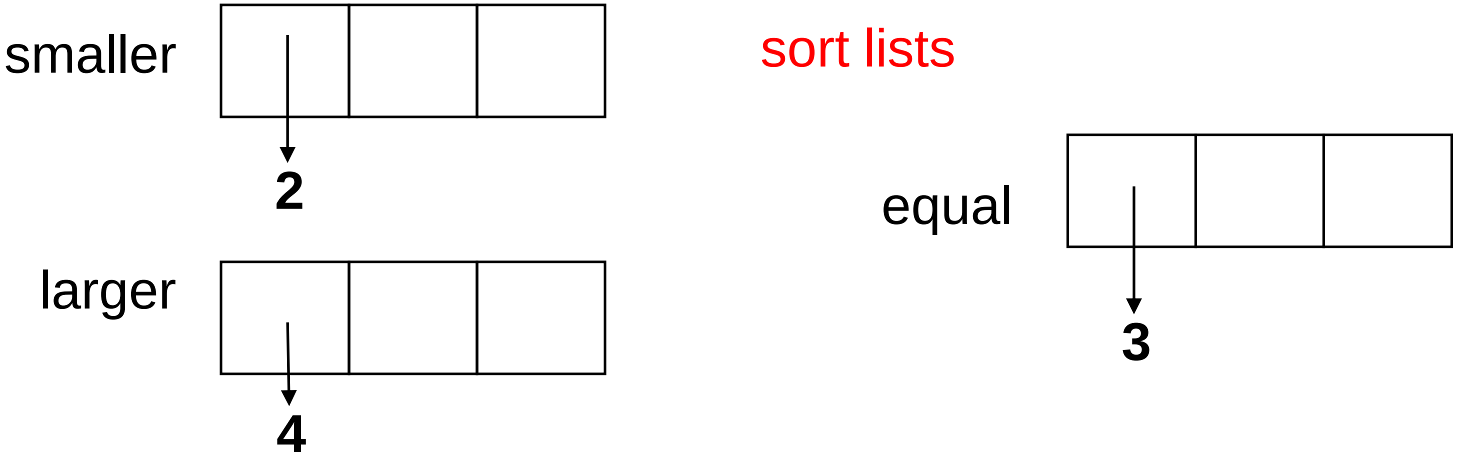


QuickSort

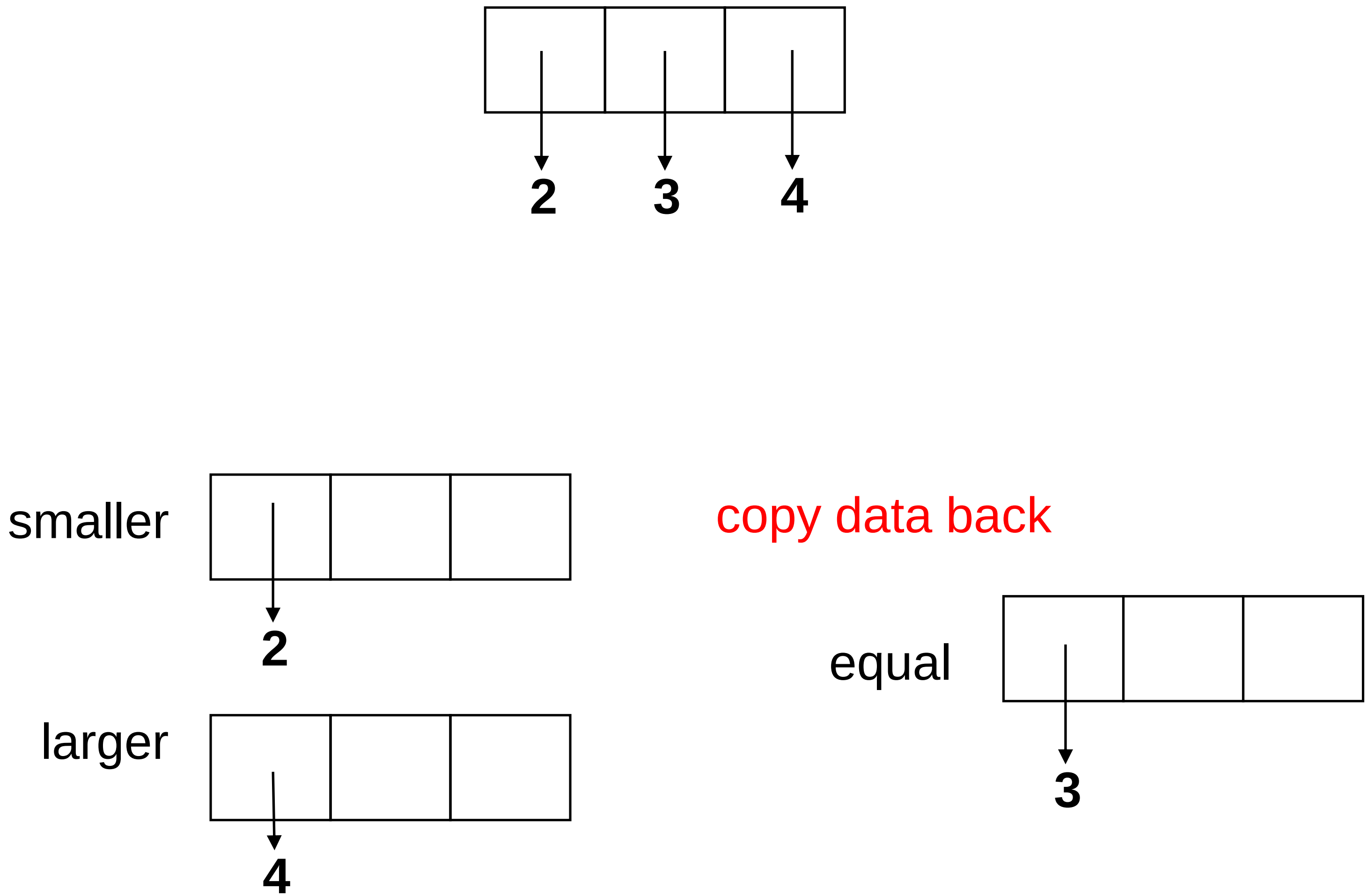


QuickSort





QuickSort



Algorithm quicksort(A,n)

In: Array A storing n values

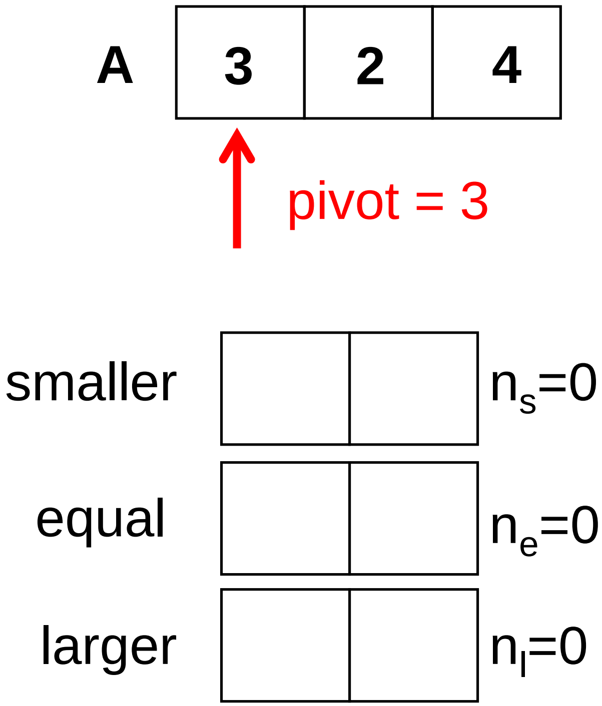
Out: {Sort A in increasing order}

If n > 1 **then** {

smaller, equal, larger = new arrays of size n

$n_s = n_e = n_l = 0$

pivot = A[0]



}

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If n > 1 **then** {

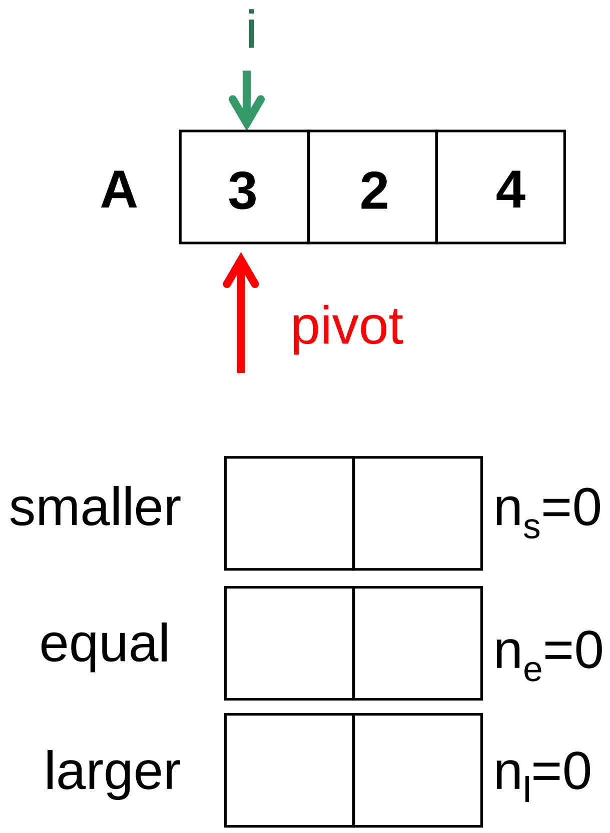
smaller, equal, larger = new arrays of size n

$n_s = n_e = n_l = 0$

pivot = A[0]

for i = 0 **to** n-1 **do** // Partition the values

}



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If n > 1 **then** {

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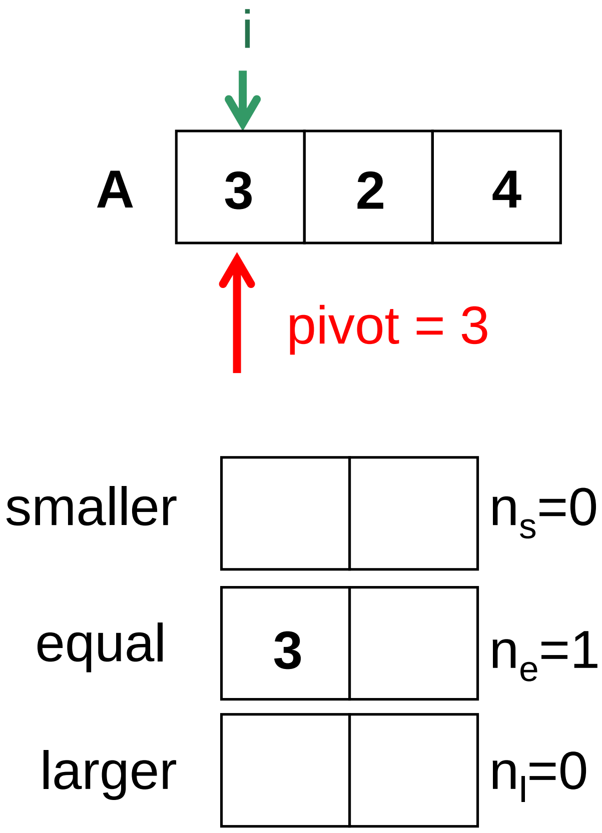
$n_s = n_e = n_l = 0$

pivot = A[0]

for i = 0 **to** n-1 **do** // Partition the values

if A[i] = pivot **then** equal[n_e++] = A[i]

}



Algorithm quicksort(A,n)

In: Array A storing n values

Out: {Sort A in increasing order}

If n > 1 **then** {

smaller, equal, larger = new arrays of size n

$n_s = n_e = n_l = 0$

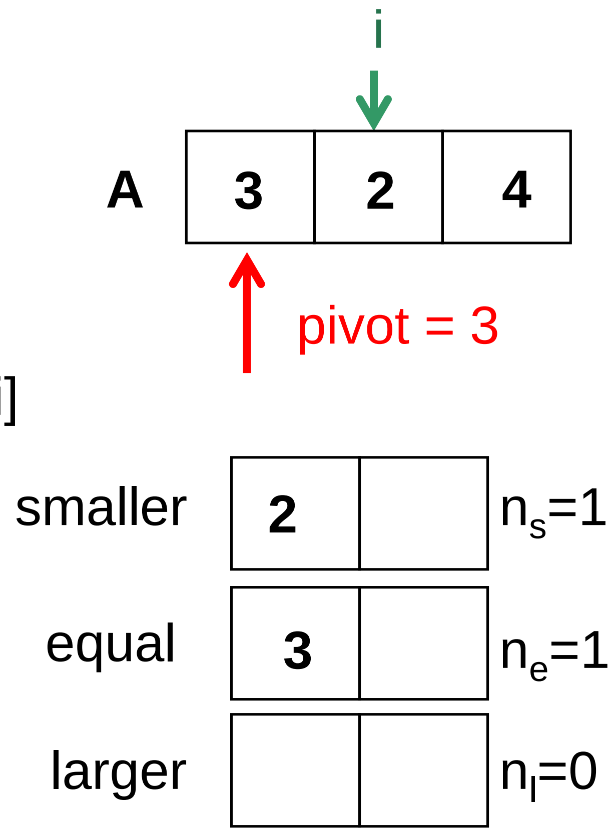
pivot = A[0]

for i = 0 **to** n-1 **do** // Partition the values

if A[i] = pivot **then** equal[n_e++] = A[i]

else if A[i] < pivot **then** smaller[n_s++] = A[i]

}



Algorithm quicksort(A,n)

In: Array A storing n values

Out: {Sort A in increasing order}

If n > 1 **then** {

  smaller, equal, larger = new arrays of size n

  $n_s = n_e = n_l = 0$

  pivot = A[0]

  for i = 0 **to** n-1 **do** // Partition the values

    if A[i] = pivot **then** equal[n_e++] = A[i]

    else if A[i] < pivot **then** smaller[n_s++] = A[i]

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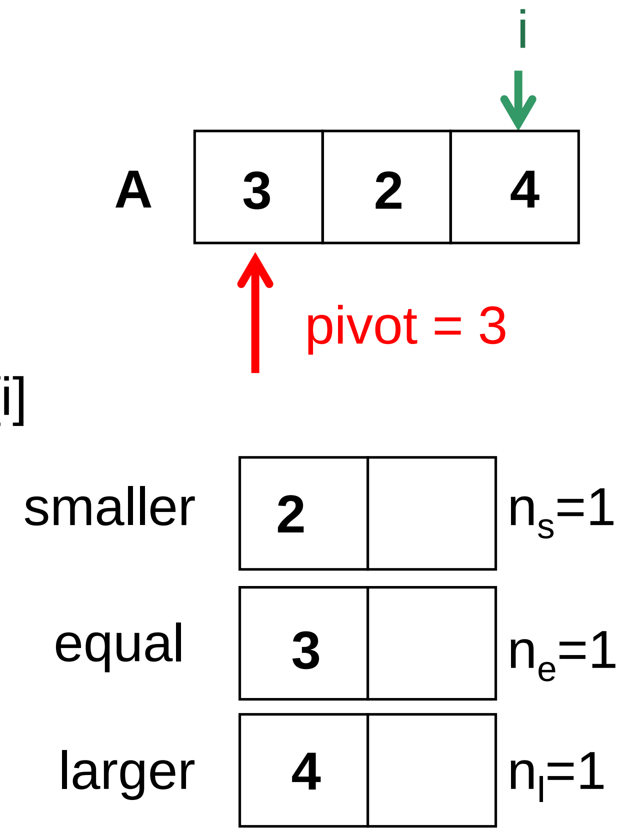


}

Algorithm quicksort(A,n)

In: Array A storing n values

Out: {Sort A in increasing order}

If n > 1 **then** {

  smaller, equal, larger = new arrays of size n

  $n_s = n_e = n_l = 0$

  pivot = A[0]

  for i = 0 **to** n-1 **do** // Partition the values

    if A[i] = pivot **then** equal[n_e++] = A[i]

    else if A[i] < pivot **then** smaller[n_s++] = A[i]

    else larger[n_l++] = A[i]

  quicksort(smaller,n_s)

}

A	**3**	**2**	**4**

Sort

smaller	**2**		n_s=1
equal	**3**		n_e=1
larger	**4**		n_l=1

Algorithm quicksort(A,n)

In: Array A storing n values

Out: {Sort A in increasing order}

If n > 1 **then** {

smaller, equal, larger = new arrays of size n

$n_s = n_e = n_l = 0$

pivot = A[0]

for i = 0 **to** n-1 **do** // Partition the values

if A[i] = pivot **then** equal[n_e++] = A[i]

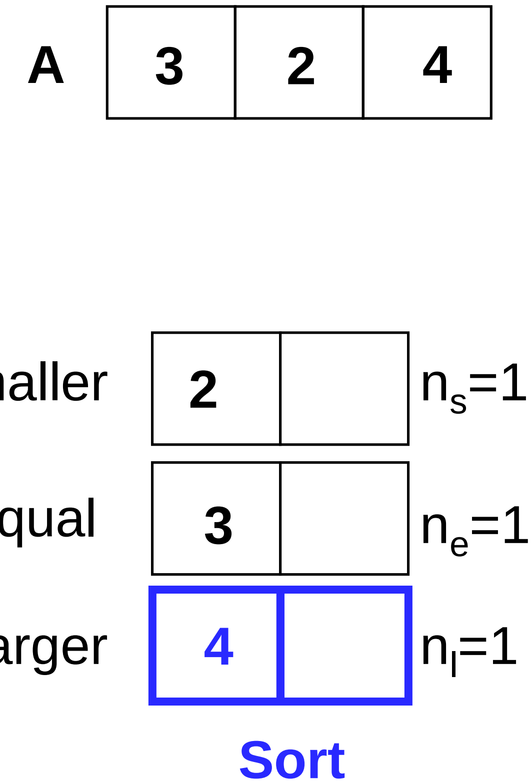
else if A[i] < pivot **then** smaller[n_s++] = A[i]

else larger[n_l++] = A[i]

quicksort(smaller,n_s)

quicksort(larger,n_l)

}



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In: Array A storing n values

Out: {Sort A in increasing order}

If n > 1 **then** {

smaller, equal, larger = new arrays of size n

$n_s = n_e = n_l = 0$

pivot = A[0]

for i = 0 **to** n-1 **do** // Partition the values

if A[i] = pivot **then** equal[n_e++] = A[i]

else if A[i] < pivot **then** smaller[n_s++] = A[i]

else larger[n_l++] = A[i]

quicksort(smaller,n_s)

quicksort(larger,n_l)

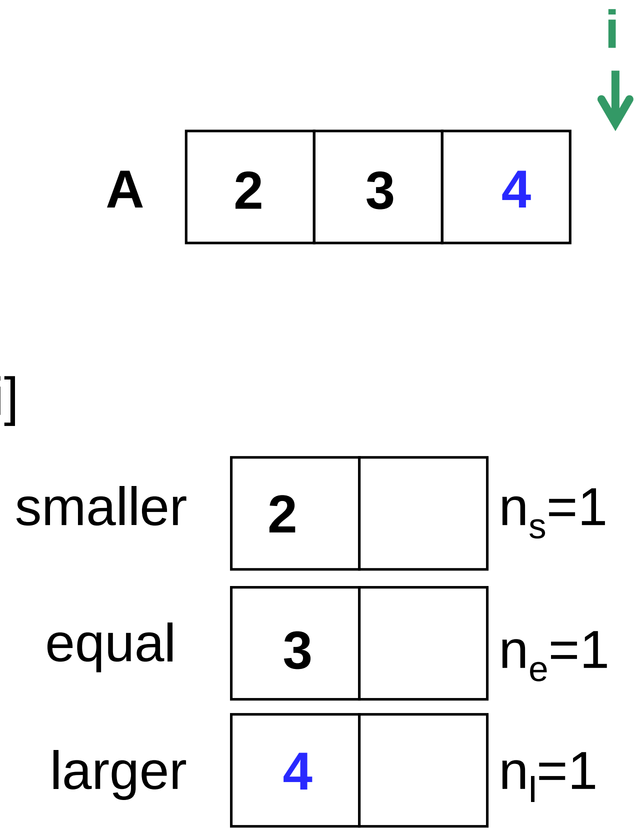
i = 0

for j = 0 **to** n_s **do** A[i++] = smaller[j]

for j = 0 **to** n_e **do** A[i++] = equal[j]

for j = 0 **to** n_l **do** A[i++] = larger[j]

}



Analysis of Quick Sort

- We will look at two cases for Quick Sort :
  - ***Worst case***
    - When the pivot element is the ***largest*** or ***smallest*** item in the container (why is this the worst case?)
  - ***Best case***
    - When the pivot element is the ***middle*** item (why is this the best case?)

Worst Case Analysis:

- We will count the number of operations needed to sort an initial container of **n** items, **T(n)**
- Assume that the pivot is the ***largest*** item in the container and **all values in the array are different**
- **n** ≤ **1**; the algorithm performs just one operation to test that n ≤ 1, so **T(0) = 1, T(1) = 1**
- **n > 1**; the pivot is chosen from the container (this needs a constant number c of operations) and then the **n** items are redistributed into three containers:
  - ***smaller*** is of size **n-1**
  - ***bigger*** is of size 0
  - **equal** is of size 1

  moving each item requires a constant number c' of operations, so this step performs **c + c'(n)** operations.

- Then we have two recursive calls:
  - Sort **smaller**, which is of size **n-1**
  - Sort **bigger**, which is of size **0**
- So, **T(n) = c + c'(n) + T(n-1) + T(0)**
  - But, the number of operations required to sort a container of size 0 is 1
  - And, the number of operations required to sort a container of size **k** in general is
    T(k) = c + c'(k) + (*the number of operations needed to sort a container of size* **k-1**)
    = c + c'(k) + T(k-1)

- So, the total number of operations **T(n)** performed by quicksort is

$$
\begin{aligned}
T(n) &= c + c'(n) + T(n-1) \\
&= c + c'(n) + c + c'(n) + T(n-2) \\
&= c + c'(n) + c + c'(n) + \ldots + c + c'(1) + T(0) \\
&= c(n) + c' \times n*(n+1)/2 + 1 \\
&= c'n^2 / 2 + n(c + c'/2) + 1
\end{aligned}
$$

- So, the ***worst case*** time complexity of Quick Sort is **$O(n^2)$**

Best Case Analysis

- The ***best case*** occurs when the pivot element is chosen so that the two new containers are as close as possible to having the same size
- It is beyond the scope of this course to do the analysis, but it turns out that the ***best case*** time complexity for Quick Sort is **$O(n \log_2 n)$**
- And it turns out that the ***average*** time complexity for Quick Sort is the same

Summary

- ***Insertion Sort*** is $\mathbf{O(n^2)}$
- ***Selection Sort*** is $\mathbf{O(n^2)}$
- ***Quick Sort*** is (in the average case) $\mathbf{O(n\log_2 n)}$

- Which one would you choose?