

The Tree ADT

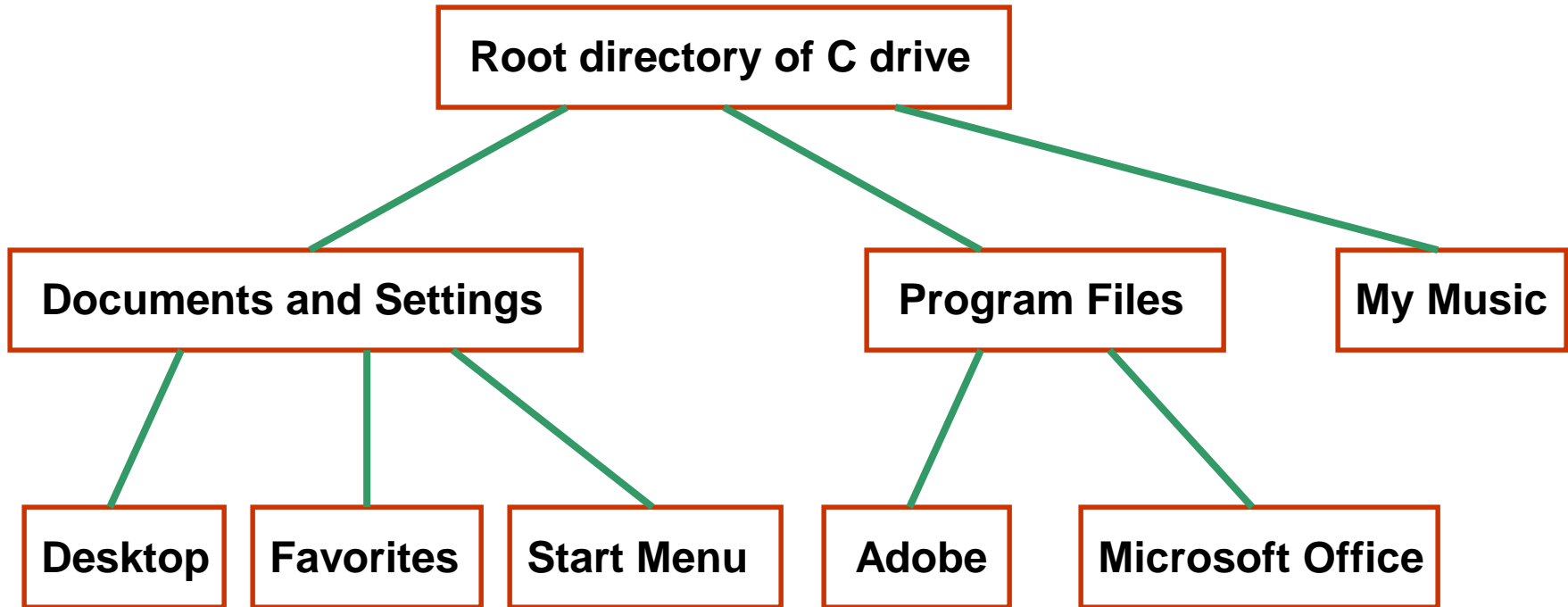
Objectives

- Define trees as data structures
- Discuss tree traversal algorithms
- Discuss a binary tree implementation

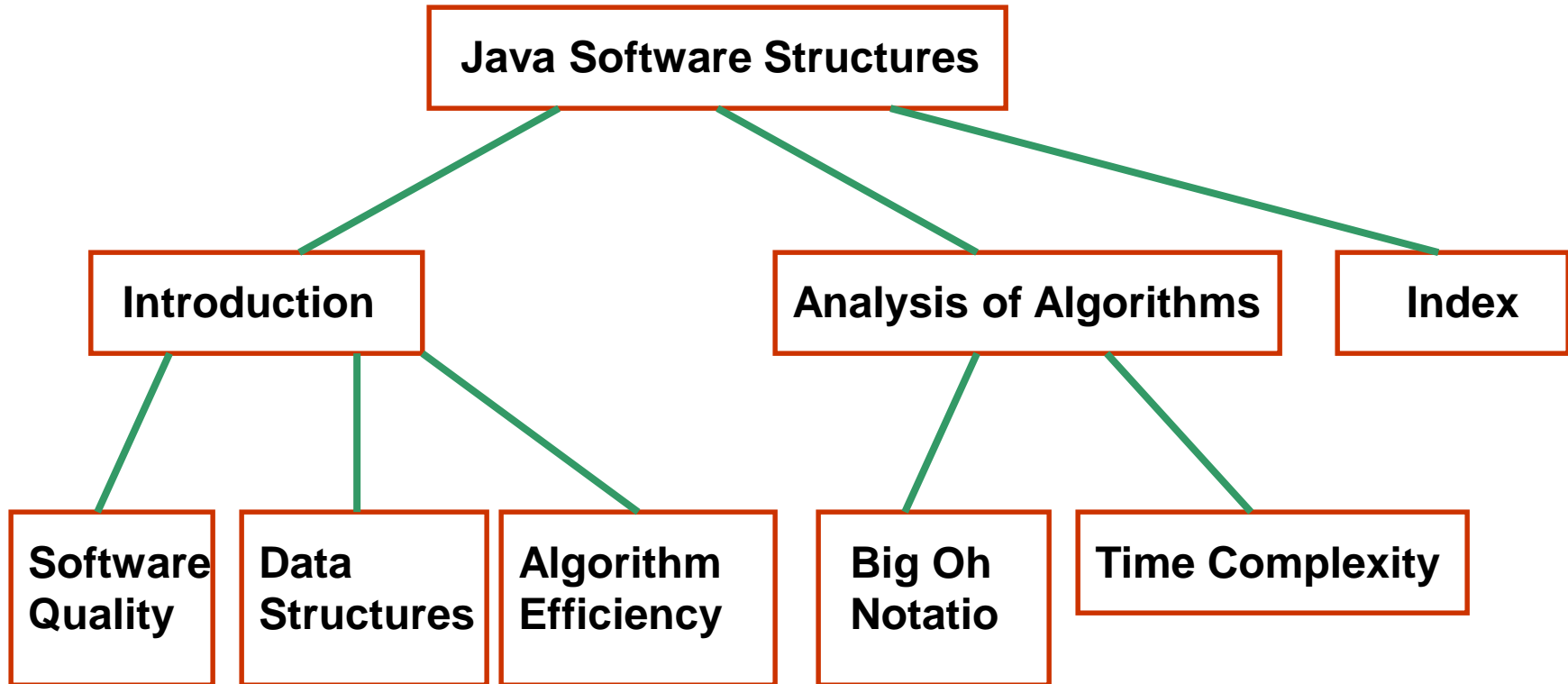
Trees

- A **tree** is a **nonlinear** abstract data type that stores elements in a hierarchy.
- **Examples** in real life:
 - Family tree
 - Table of contents of a book
 - Class inheritance hierarchy in Java
 - Computer file system (folders and subfolders)
 - Decision trees

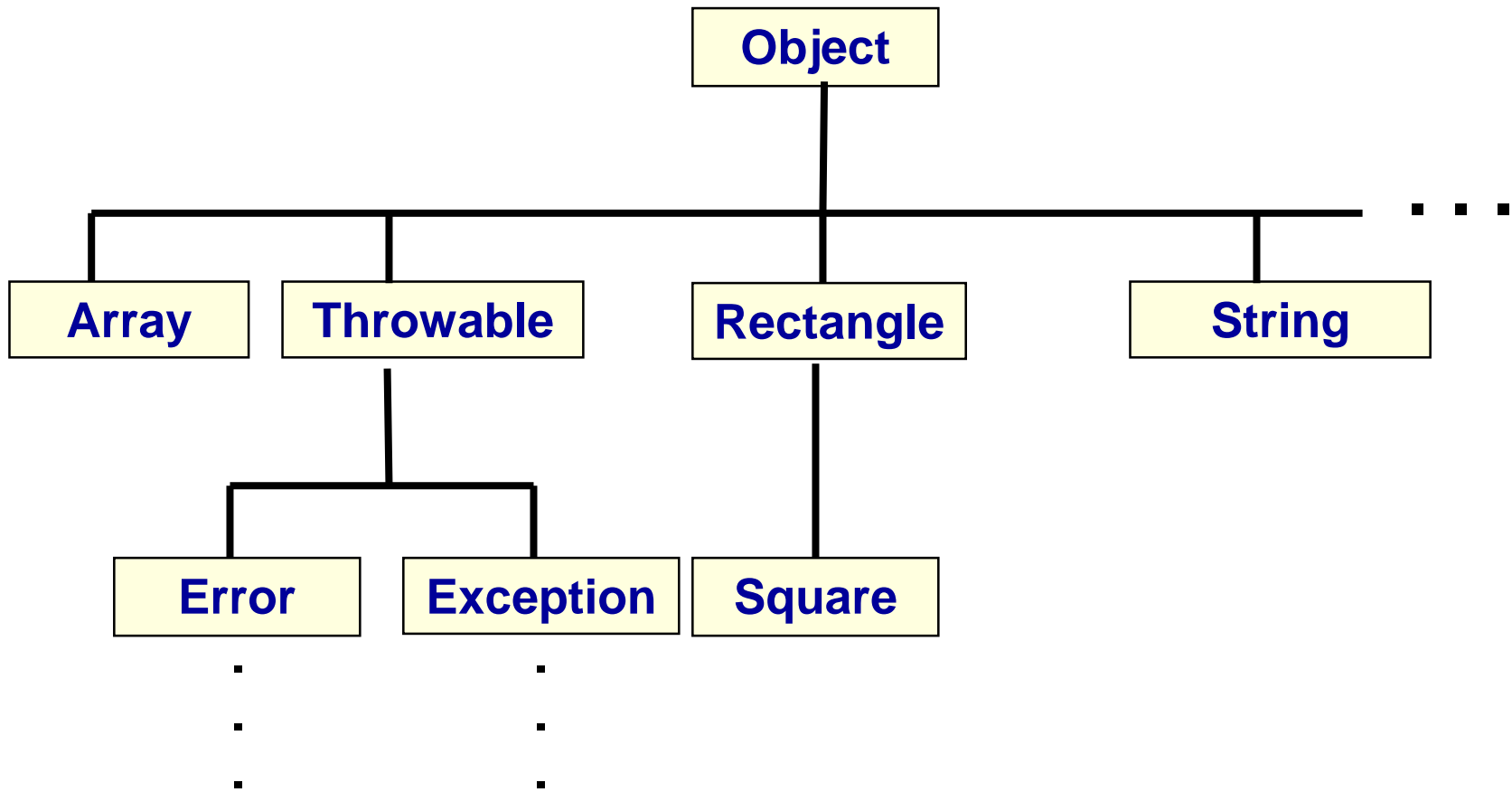
Example: Computer File System



Example: Table of Contents



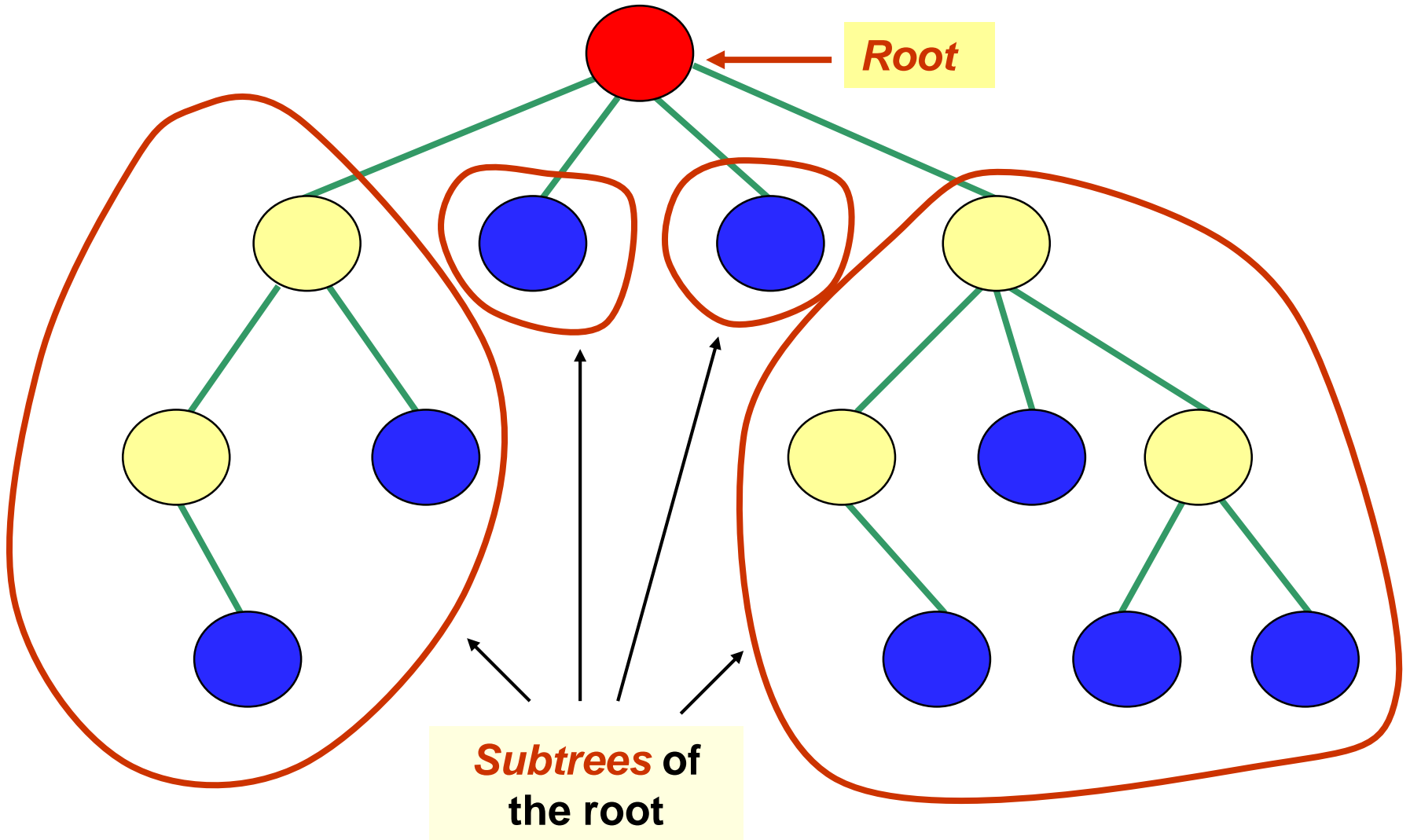
Example: Java's Class Hierarchy



Tree Definition

- A **tree** is a set of elements that either
 - it is empty, or
 - it has a distinguished element called the **root** and zero or more **trees** (called **subtrees** of the root)
- What kind of definition is this?
 - What is the base case?
 - What is the recursive part?

Tree Definition



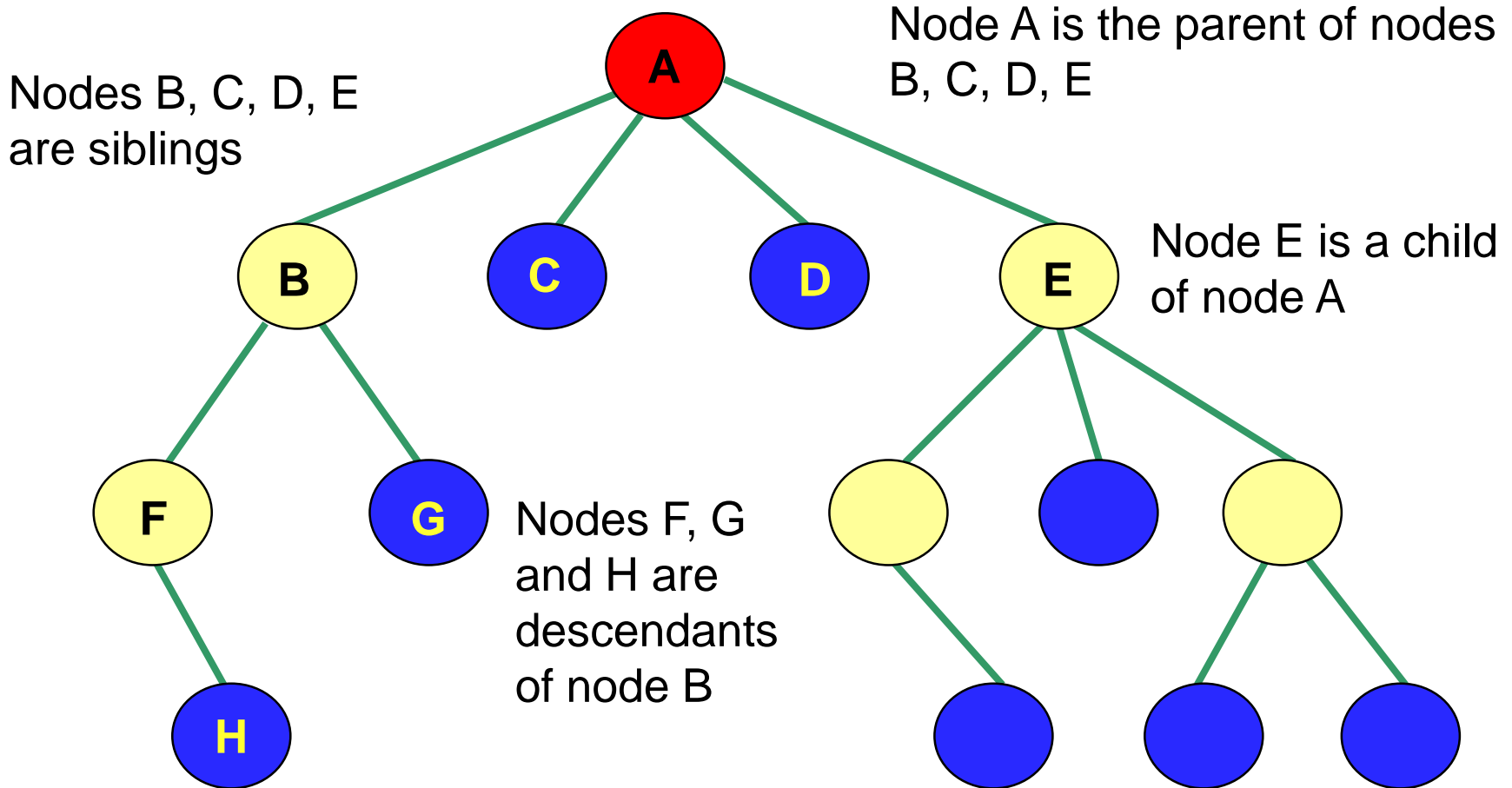
Tree Terminology

- **Nodes**: the elements in the tree
- **Edges**: connections between nodes
- **Root**: *the* distinguished element that is the origin of the tree
 - There is only one root node in a tree
- **Empty tree** has no nodes and no edges

Tree Terminology

- **Parent** or **predecessor**: the node directly above another node in the hierarchy
 - A node can have only one parent
- **Child** or **successor**: a node directly below another node in the hierarchy
- **Siblings**: nodes that have the same parent
- **Ancestors** of a node: its parent, the parent of its parent, etc.
- **Descendants** of a node: its children, the children of its children, etc.

Tree Terminology



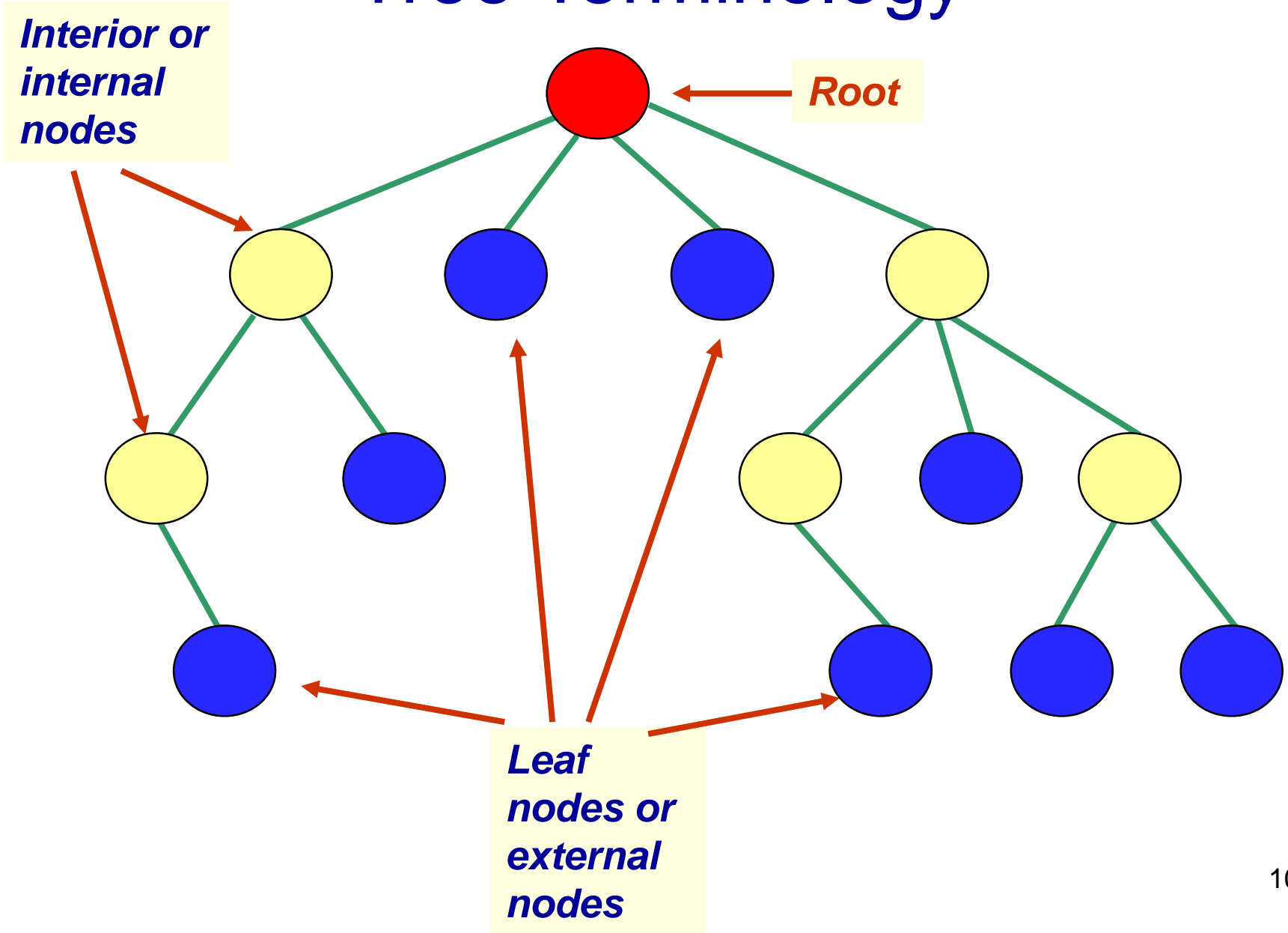
Nodes F, B, and A are ancestors of node H

All nodes, except A, are descendants of node A

Tree Terminology

- ***Leaf node***: a node without children
- ***Internal node***: a node with one or more children

Tree Terminology



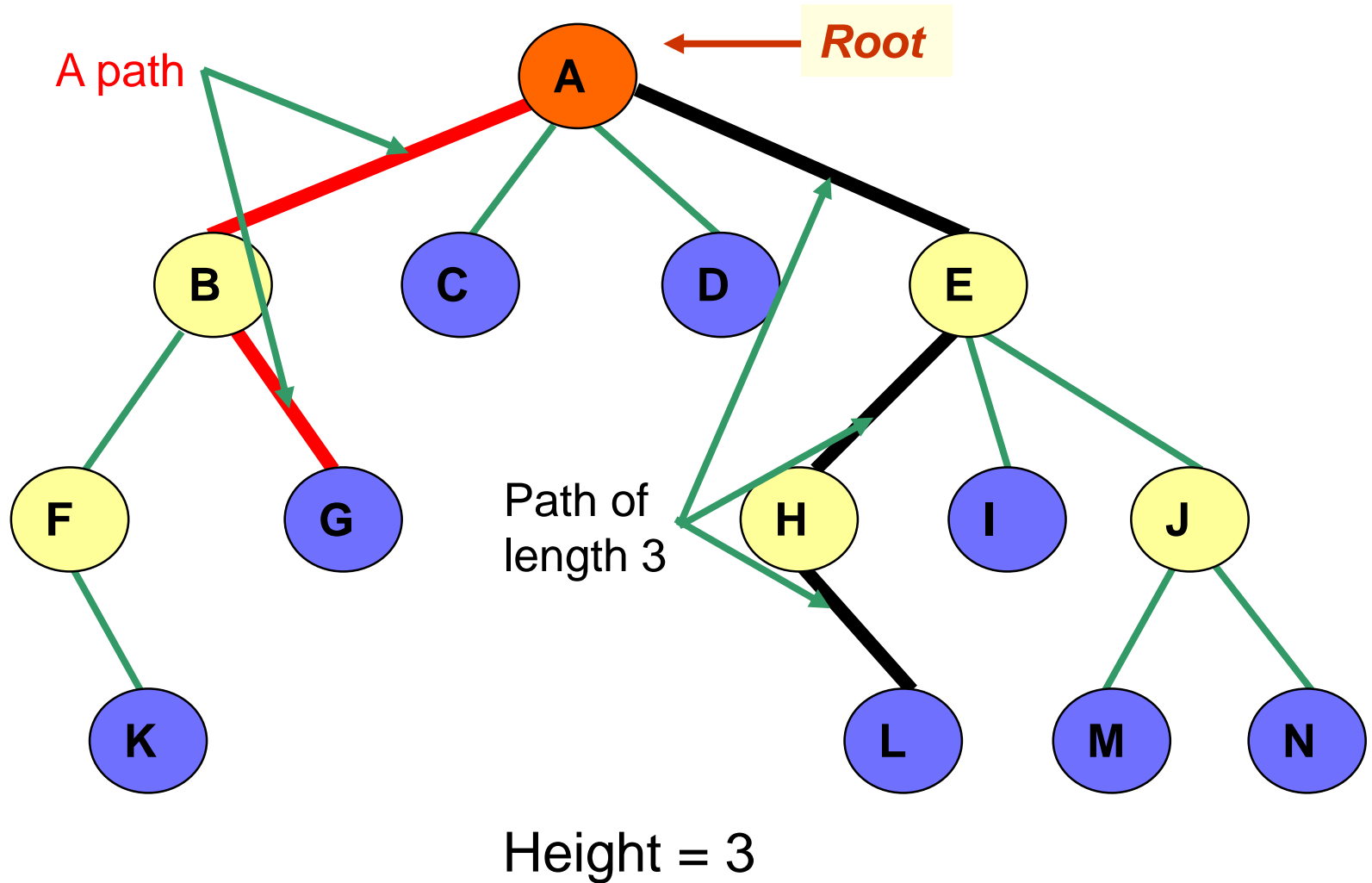
Discussion

- Does a leaf node have any children?
- Does the root node have a parent?
- How many parents does every node other than the root node have?

Height of a Tree

- A ***path*** is a sequence of edges leading from one node to another
- ***Length of a path***: number of edges on the path
- ***Height of a (non-empty) tree*** : length of the longest path from the root to a leaf
 - What is the height of a tree that has only a root node?

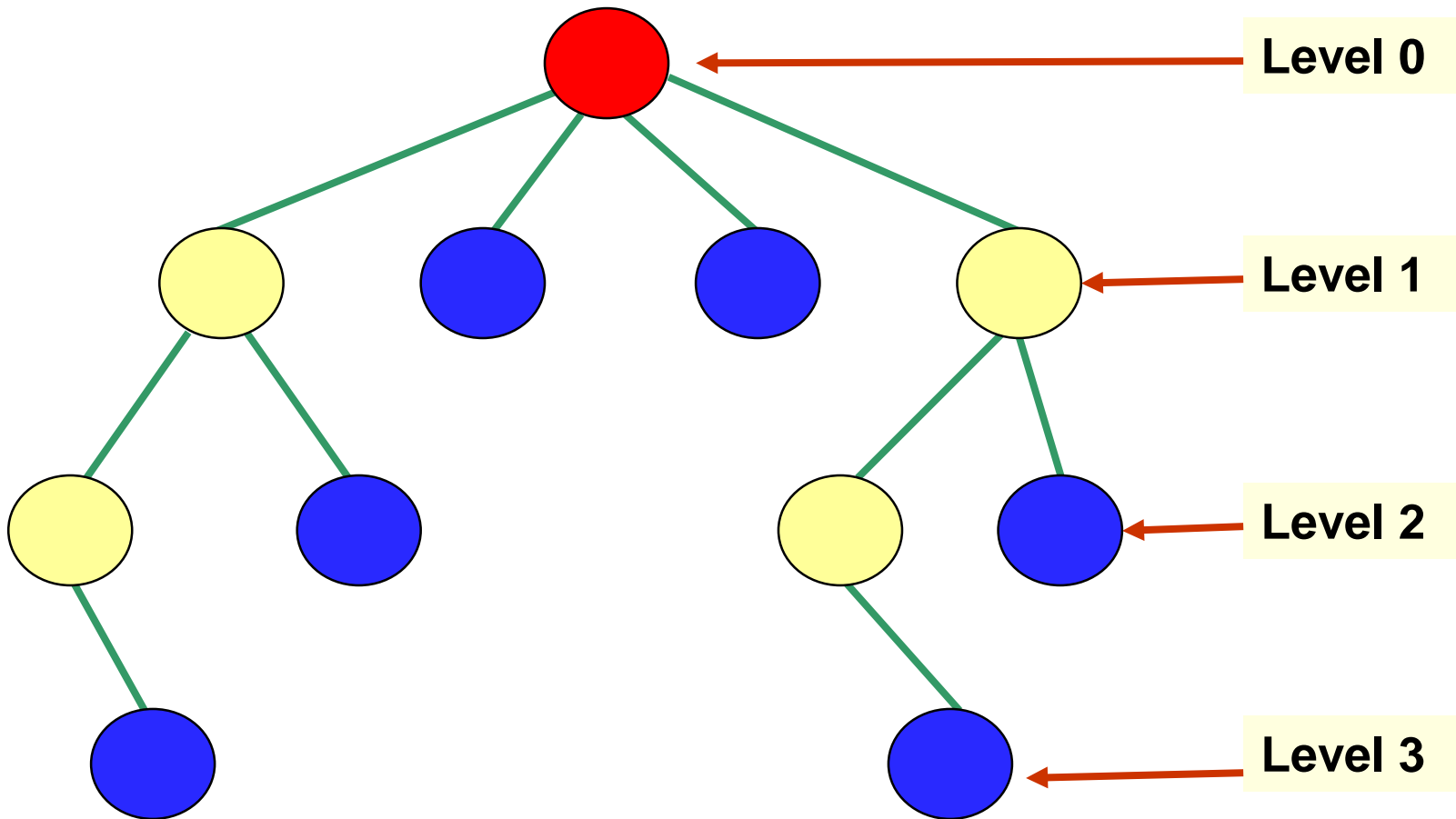
Tree Terminology



Level of a Node

- ***Level of a node***: number of **edges** between root and the node
- It can be defined ***recursively***:
 - Level of root node is **0**
 - Level of a node that is not the root node is ***level of its parent + 1***
- ***Question***: *What is the height of a tree in terms of levels?*

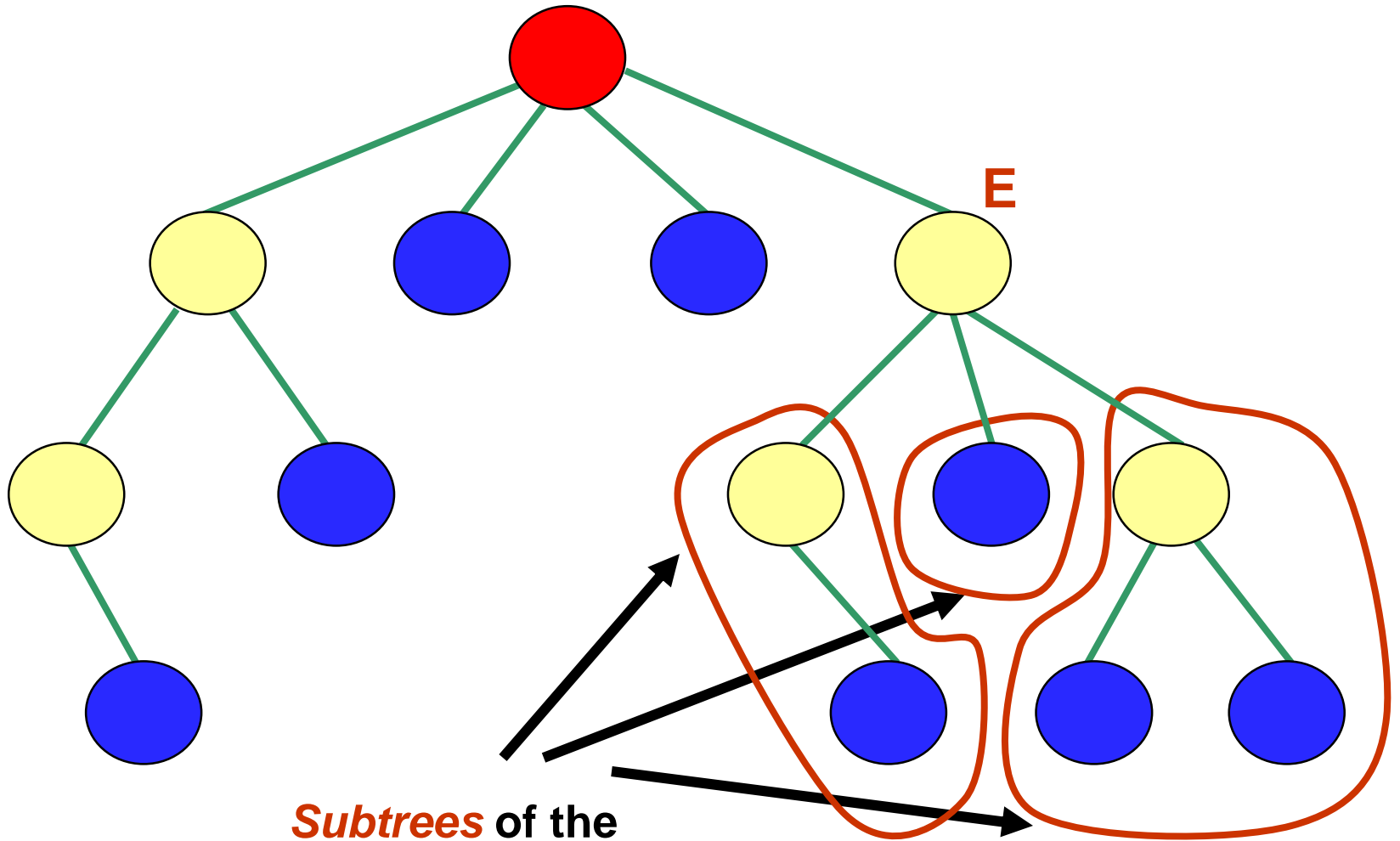
Level of a Node



Subtrees

- A ***subtree*** of a node consists of a child node and all its descendants
 - A subtree is itself a tree
 - A node may have many subtrees

Subtrees



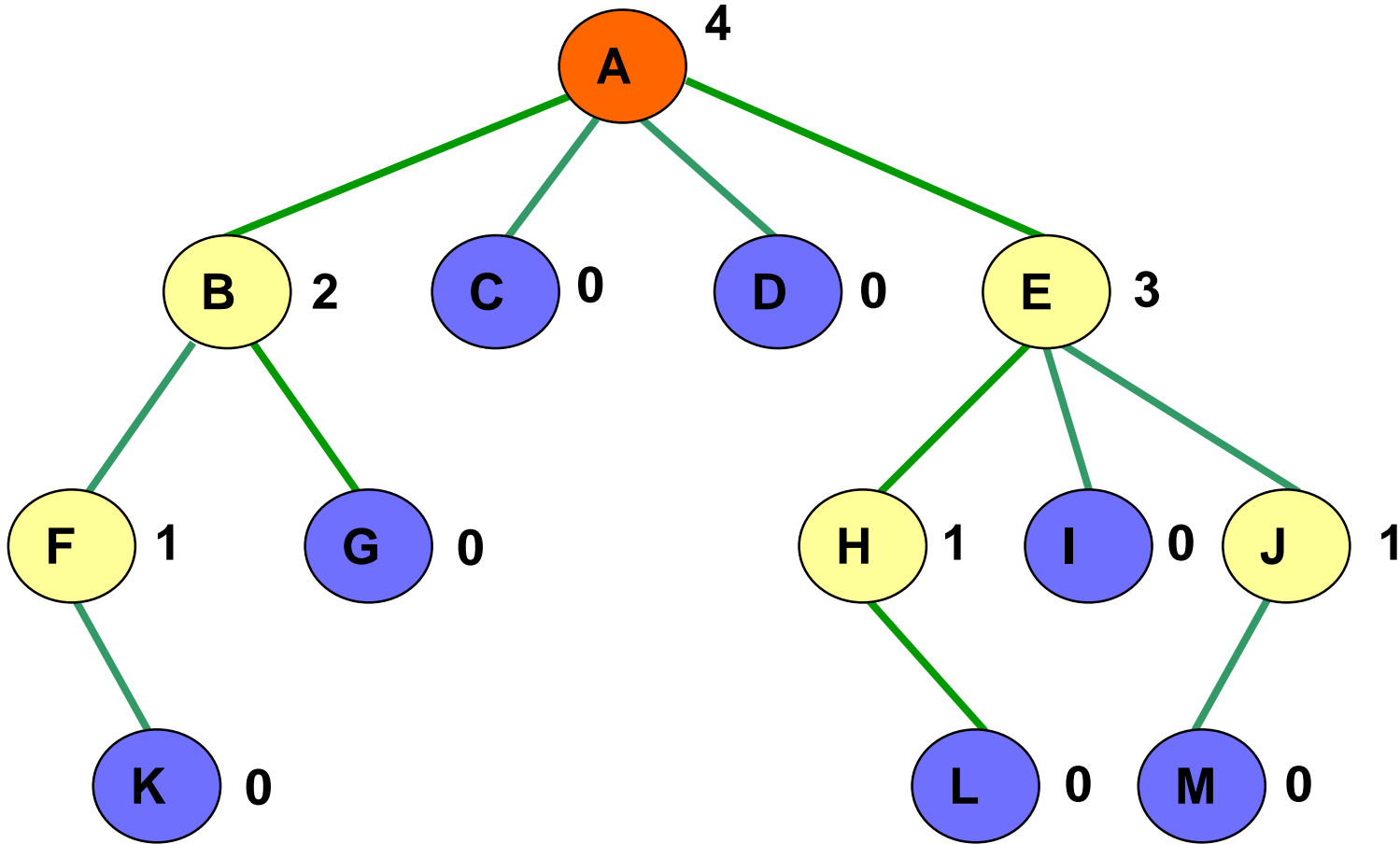
Subtrees of the
node labeled **E**

More Tree Terminology

- ***Degree*** of a ***node***: the number of children it has
- ***Degree*** of a ***tree***: the ***maximum*** of the degrees of the nodes of the tree

Degree

Degrees of the nodes are indicated beside the nodes



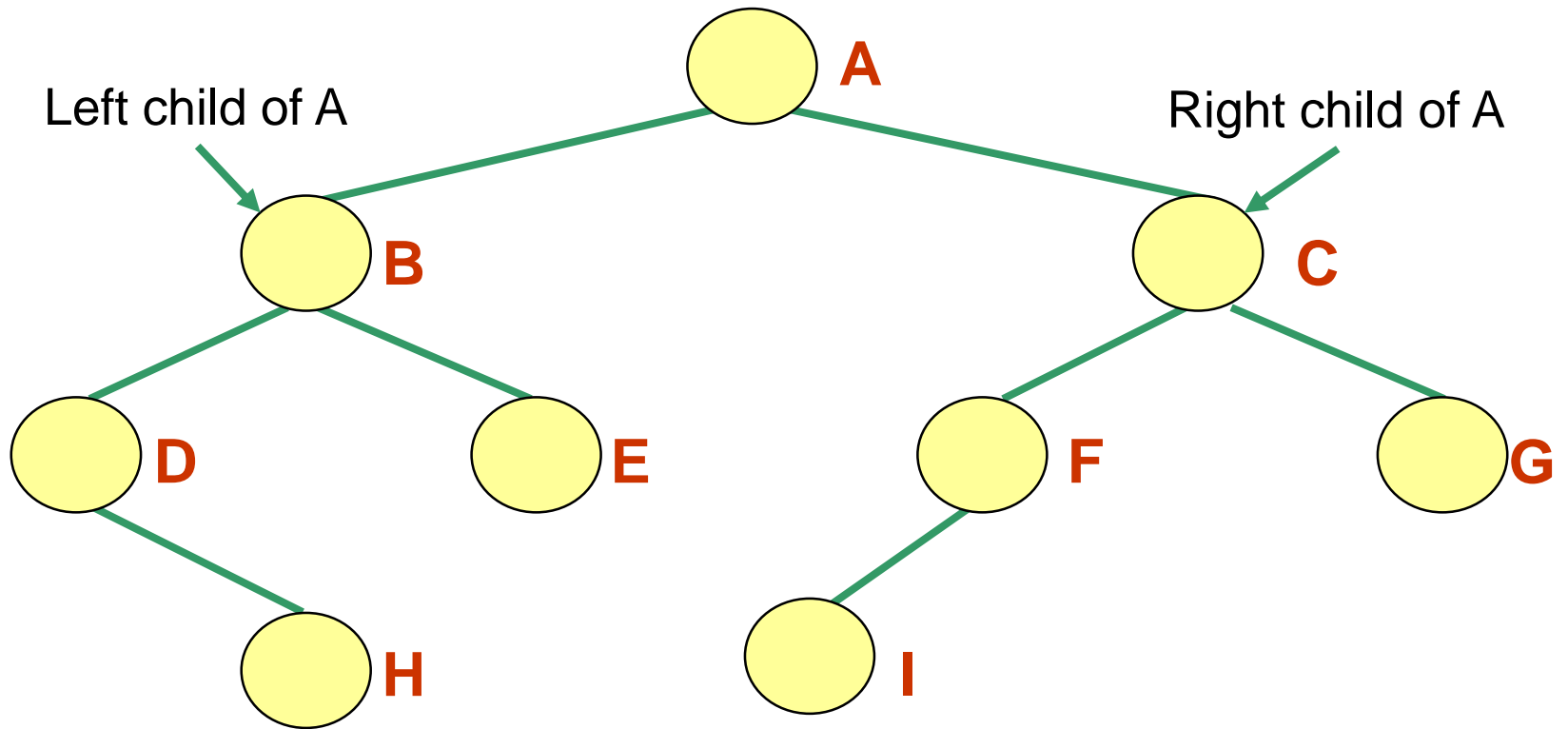
Binary Trees

- **General tree**: a tree each of whose nodes may have any number of children
- **n -ary tree**: a tree each of whose nodes may have no more than n children
- **Binary tree**: a tree each of whose nodes may have no more than **2** children
 - *i.e.* a binary tree is a tree with **degree 2**
 - The children of a node (if present) are called the **left child** and **right child**

Binary Trees

- ***Recursive definition*** of a ***binary tree***:
it is
 - the empty tree, or
 - a tree which has a root whose left and right subtrees are binary trees
- A binary tree is an **ordered** tree, ***i.e.*** it matters whether a child is a left or right child

Binary Tree



Tree Traversals

- A *traversal* of a tree starts at the root and visits each node of the tree once.
- Common tree traversals:
 - *preorder*
 - *inorder*
 - *postorder*
 - *level-order*

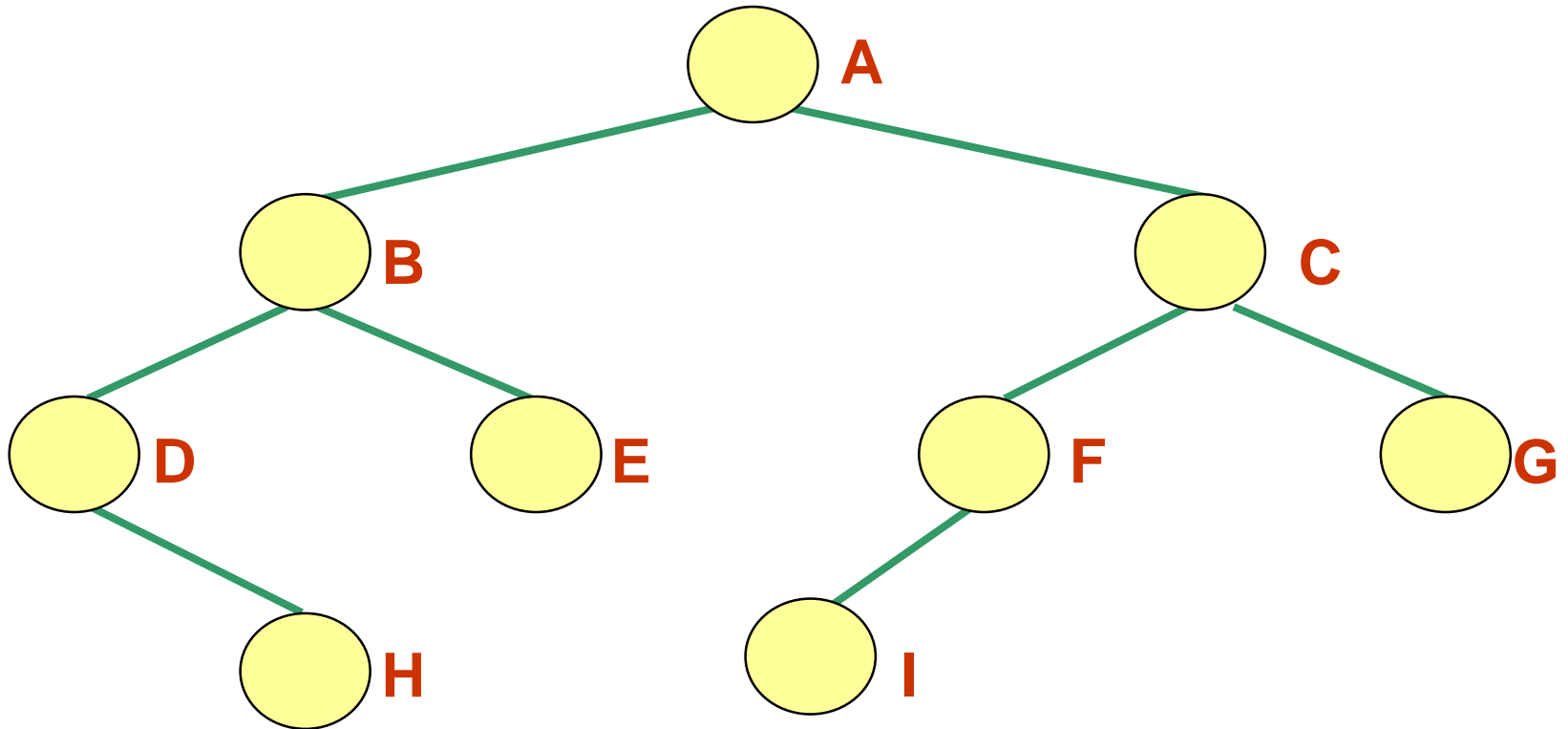
Preorder Traversal

- Start at the root
- Visit each node, followed by its children; we will visit the left child before the right one
- **Recursive algorithm** for preorder traversal:
 - If tree is not empty,
 - Visit root node of tree
 - Perform preorder traversal of its left subtree
 - Perform preorder traversal of its right subtree
- *What is the base case?*
- *What is the recursive part?*

Preorder Traversal

```
public void preorder (BinaryTreeNode<T> r) {  
    if (r != null) {  
        visit(r);  
        preorder (r.getLeftChild());  
        preorder (r.getRightChild());  
    }  
}
```

Preorder Traversal



In a preorder traversal of this tree the nodes are visited in the order
ABDHECFIG

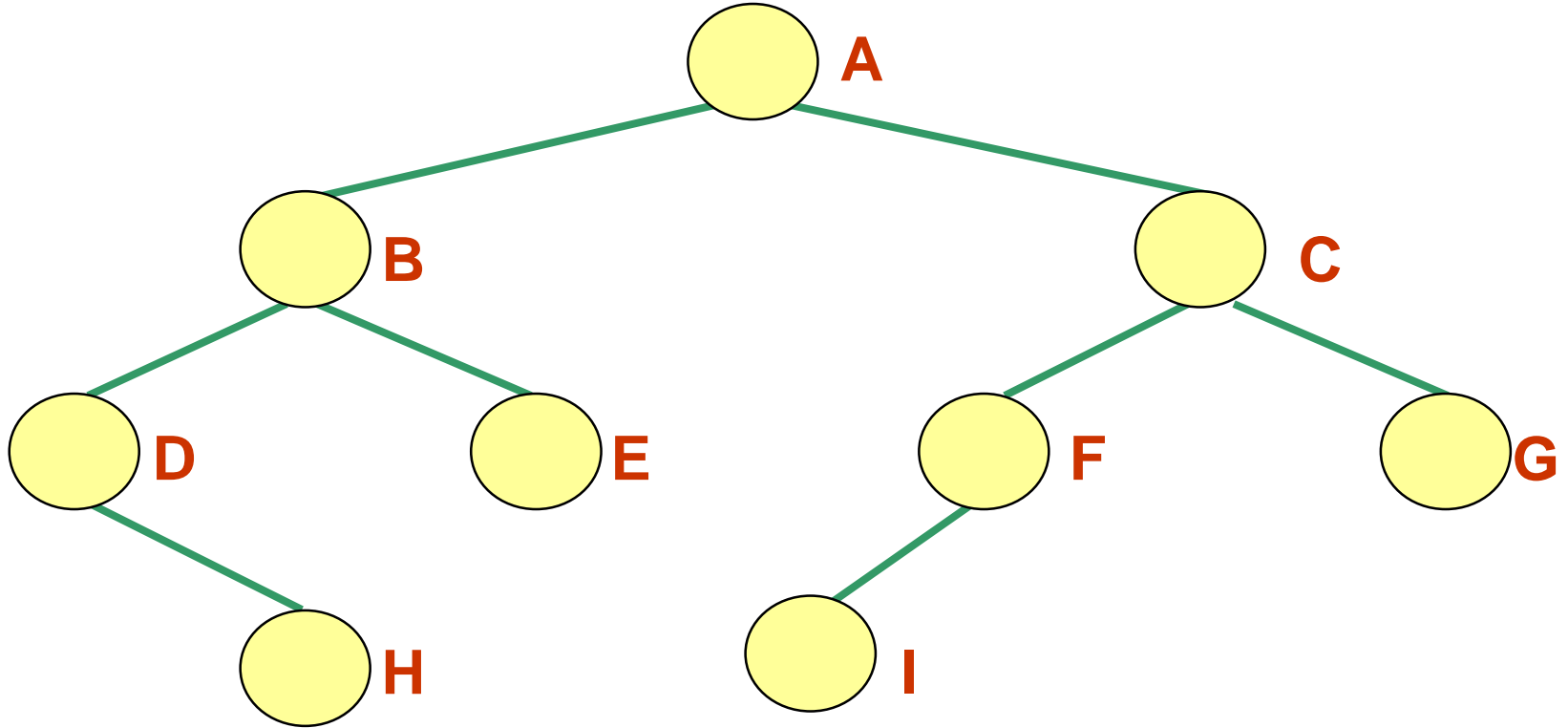
Inorder Traversal

- Start at the root
- Visit the left child of each node, then the node, then any remaining nodes
- ***Recursive algorithm*** for **inorder traversal**
 - If tree is not empty,
 - Perform **inorder traversal** of left subtree of root
 - Visit root node of tree
 - Perform **inorder traversal** of its right subtree

Inorder Traversal

```
public void inorder (BinaryTreeNode<T> r) {  
    if (r != null) {  
        inorder (r.getLeftChild());  
        visit(r);  
        inorder (r.getRightChild());  
    }  
}
```


Inorder Traversal



In an inorder traversal of this tree the nodes are visited in the order
DHBEAIFCG

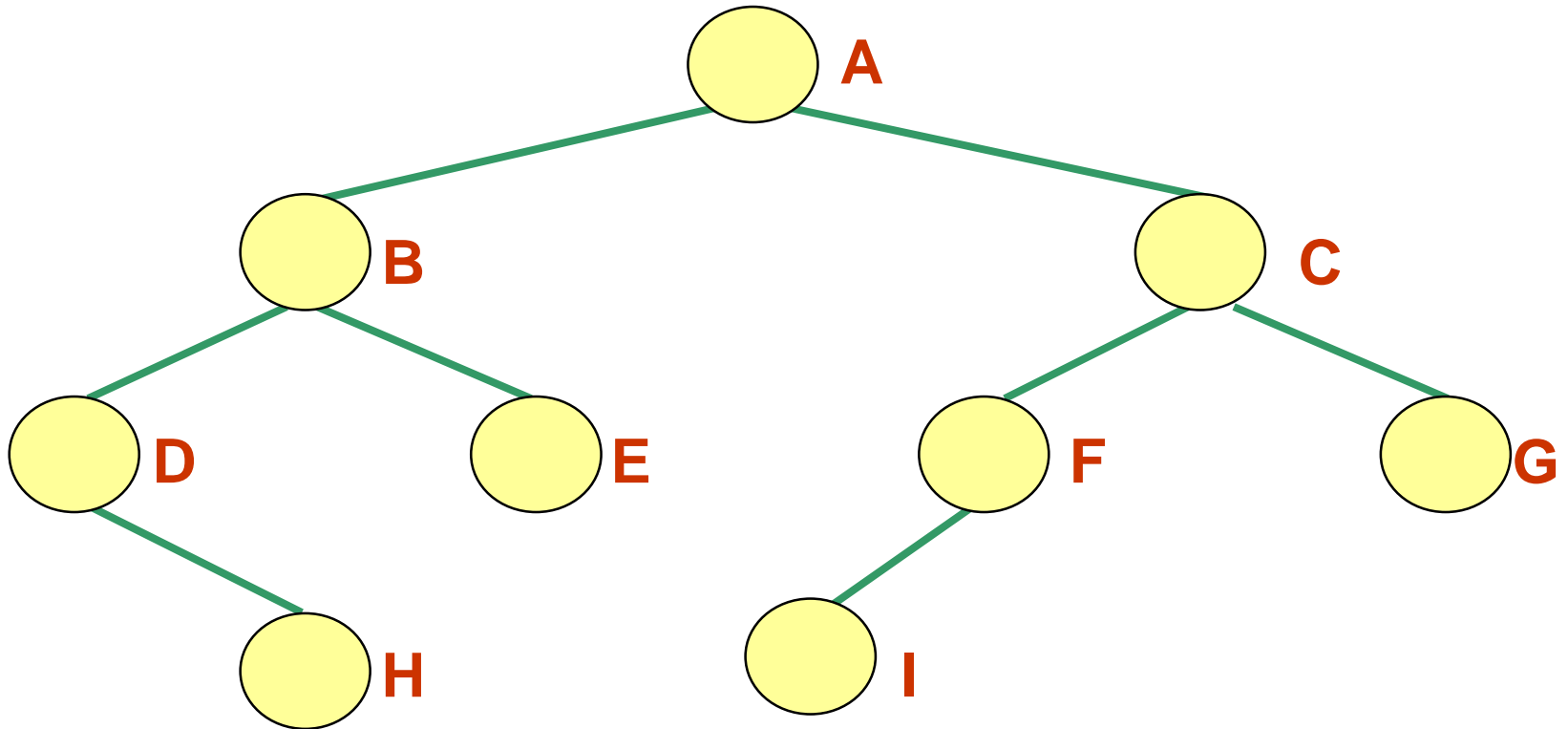
Postorder Traversal

- Start at the root
- Visit the children of each node, then the node
- *Recursive algorithm* for postorder traversal
 - If tree is not empty,
 - Perform postorder traversal of left subtree of root
 - Perform postorder traversal of right subtree of root
 - Visit root node of tree

Postorder Traversal

```
public void postorder (BinaryTreeNode<T> r) {  
    if (r != null) {  
        postorder (r.getLeftChild());  
        postorder (r.getRightChild());  
        visit(r);  
    }  
}
```

Postorder Traversal

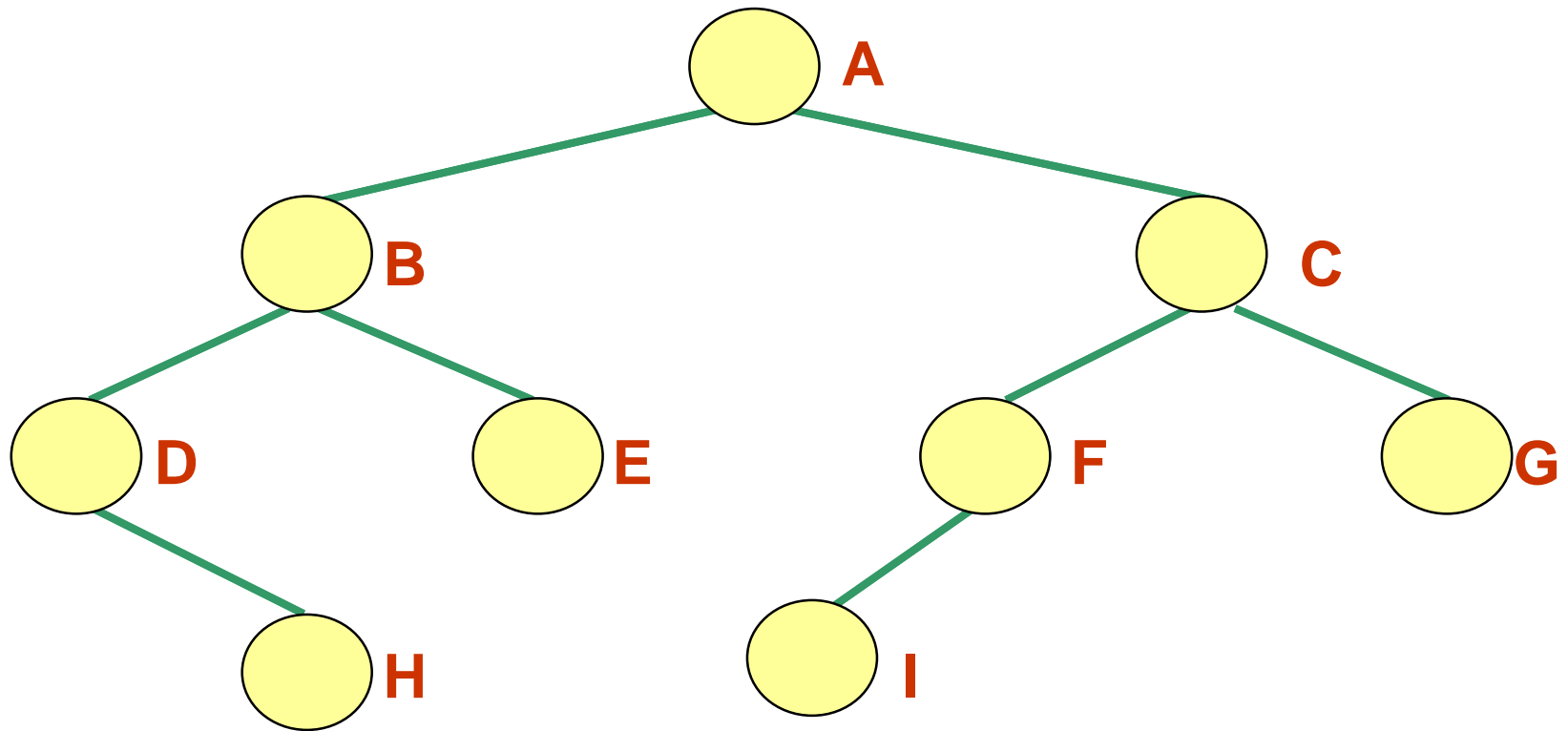


In an postorder traversal of this tree the nodes are visited in the order **HDEBIFGCA**

Level Order Traversal

- Start at the root
- Visit the nodes at each level, from left to right
- Is there a recursive algorithm for a level order traversal?

Level Order Traversal

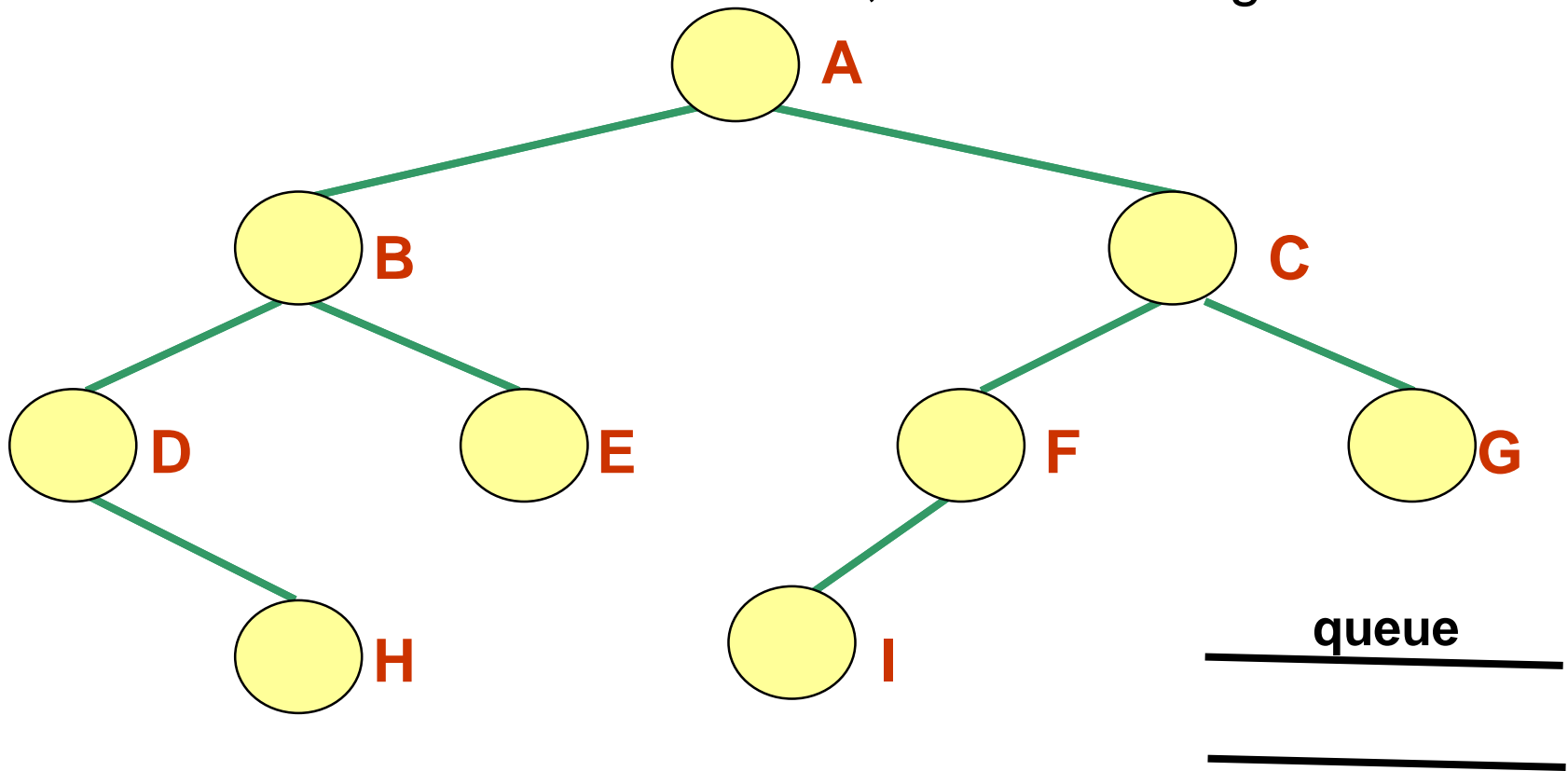


In a level order traversal of this tree the nodes will be visited in the order **ABCDEFGHI**

Level Order Traversal

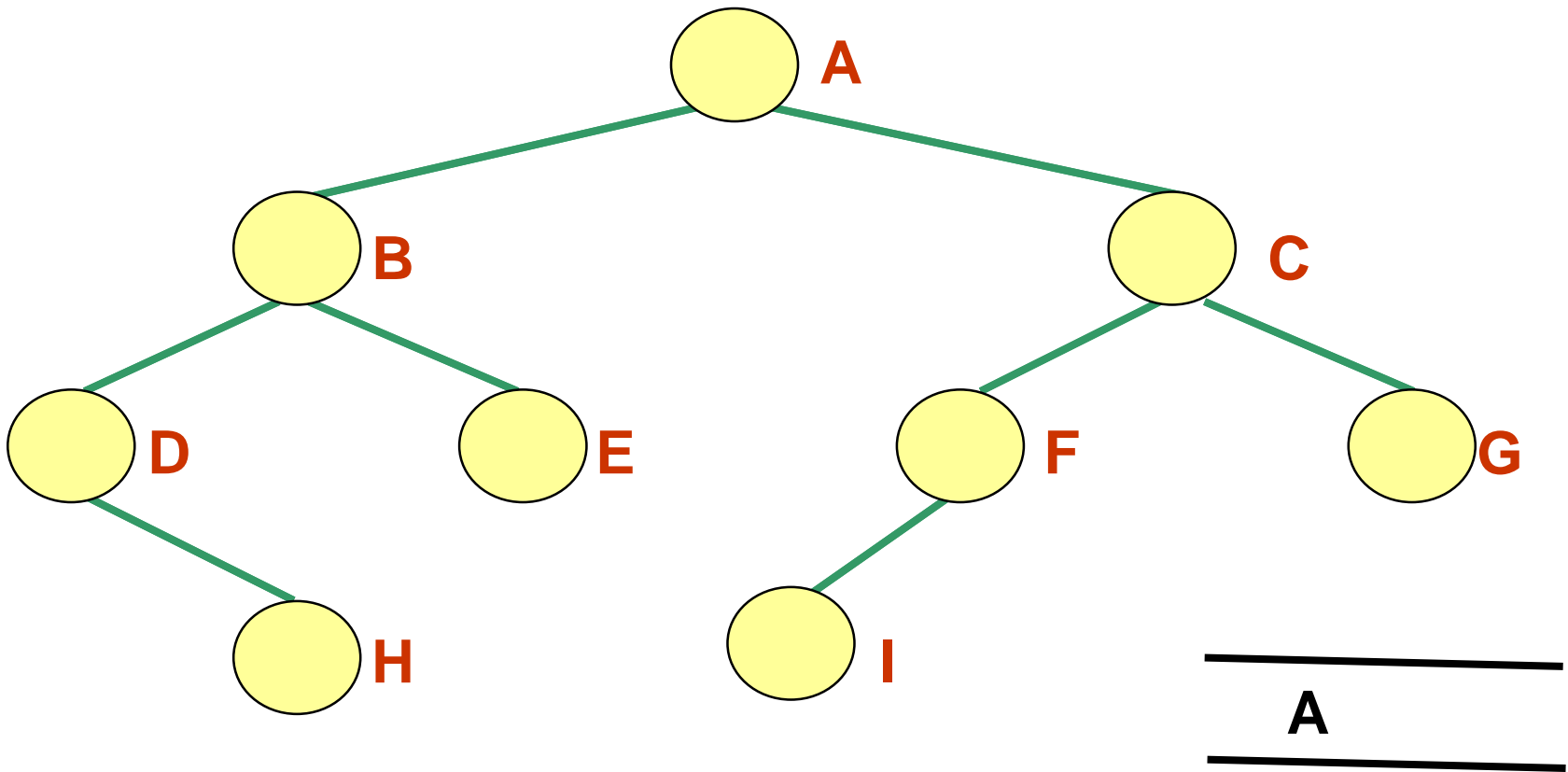
We need to use a queue:

- Start at the root
- Visit the nodes at each level, from left to right



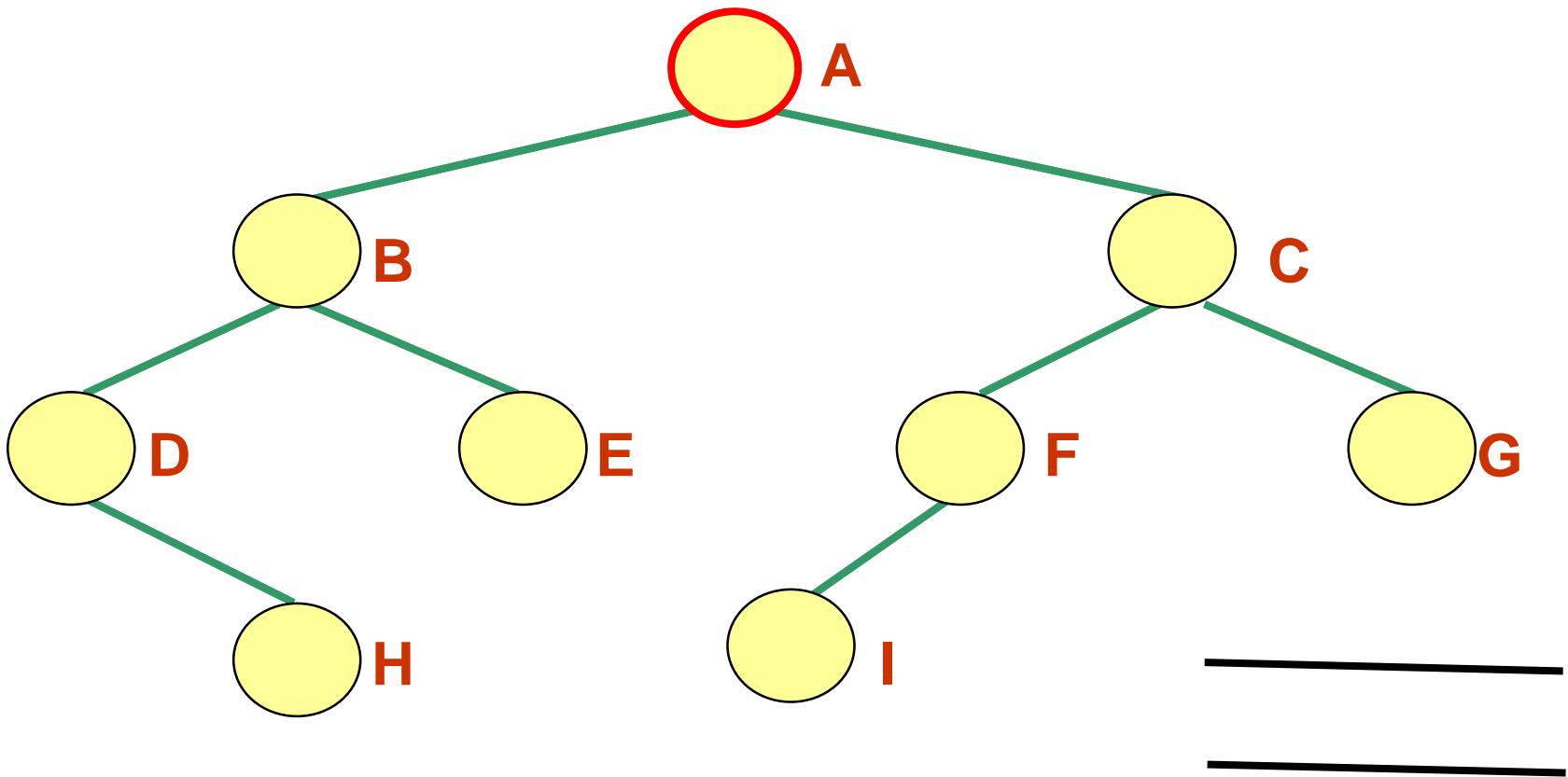
Level Order Traversal

- Put A in the queue



Level Order Traversal

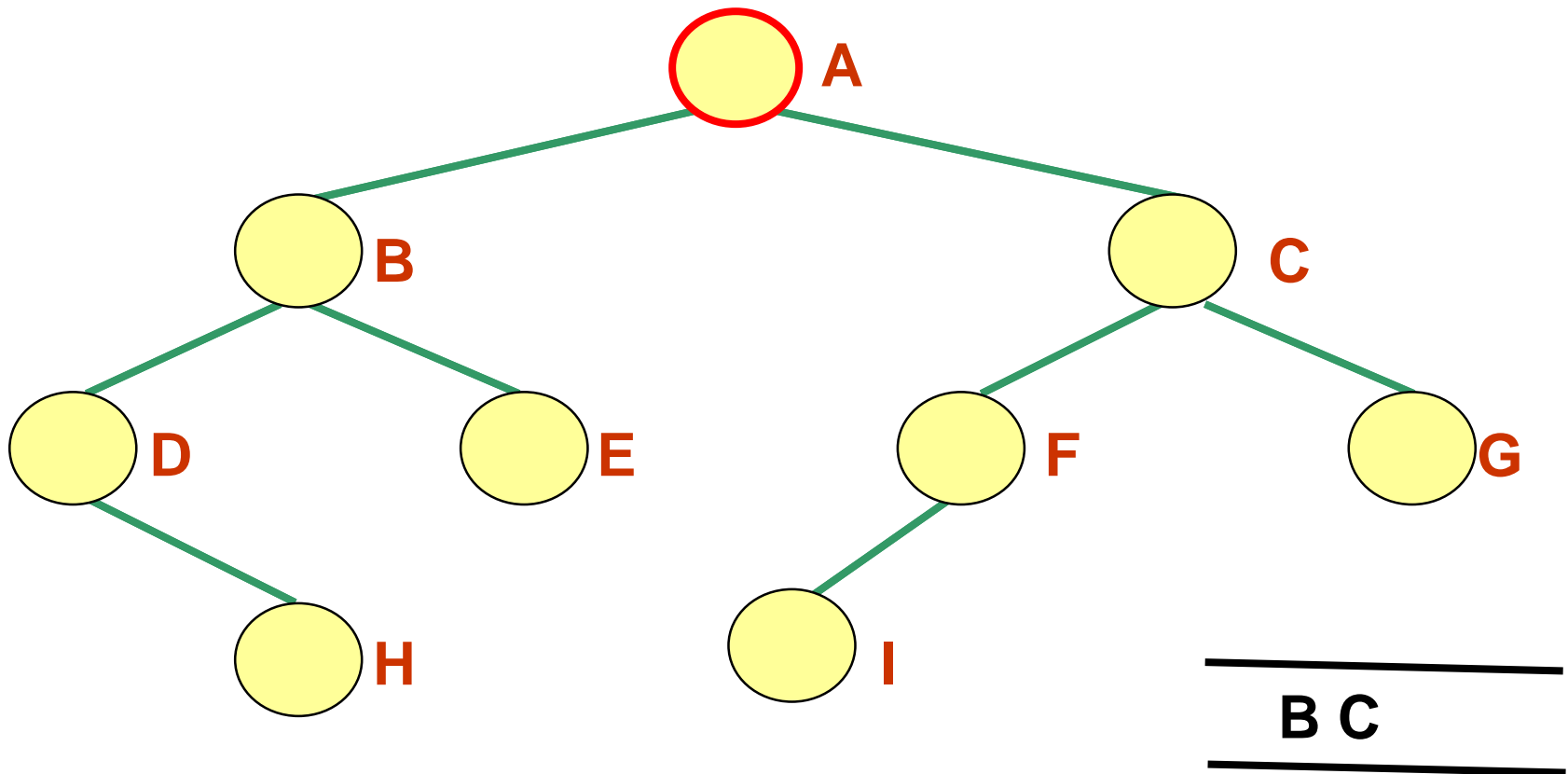
1. Remove the first node from the queue



A

Level Order Traversal

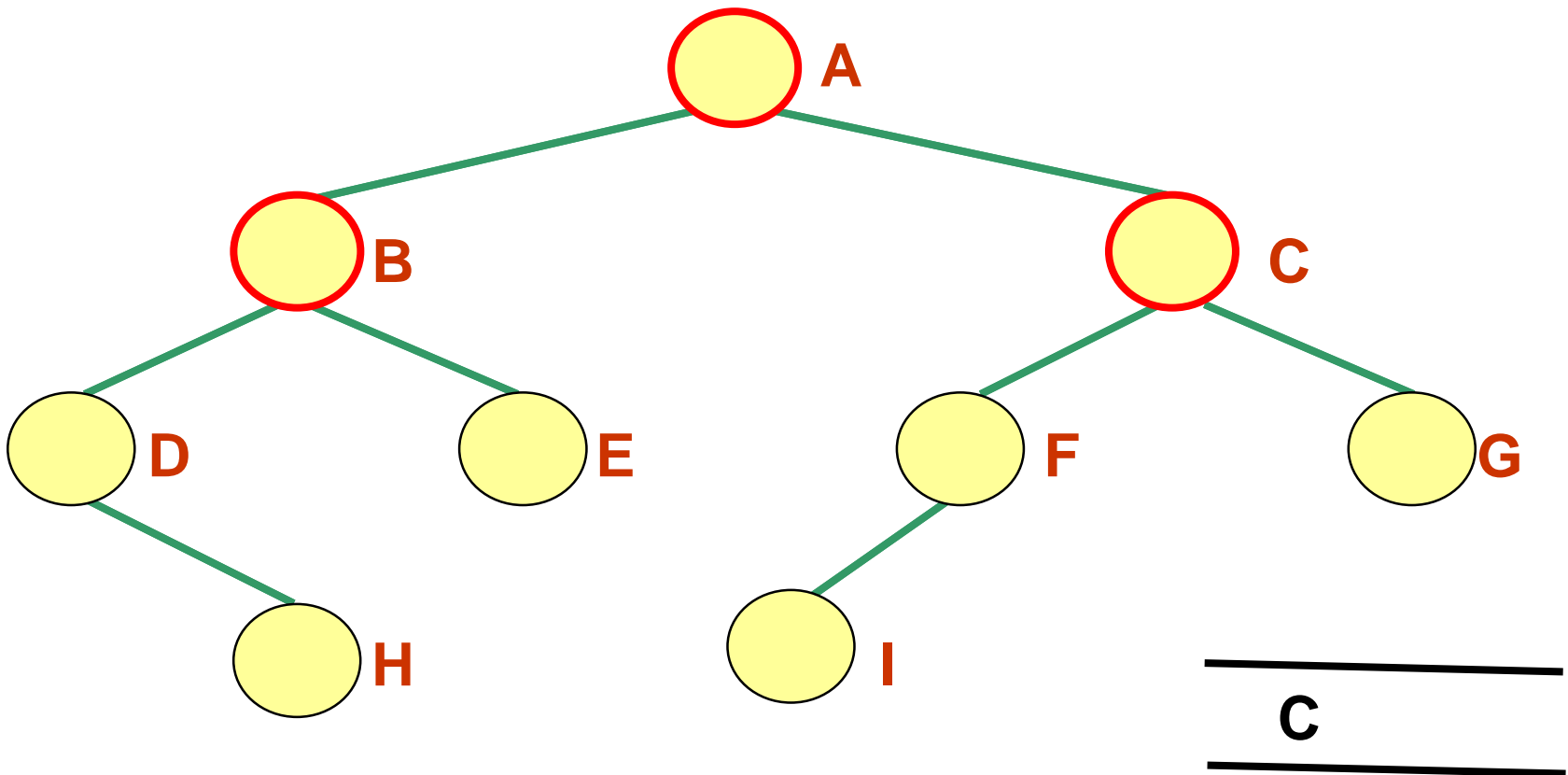
2. Enqueue all neighbours of the node that was removed from the queue



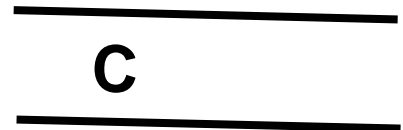
A

Level Order Traversal

Repeat steps 1 and 2 until the queue is empty

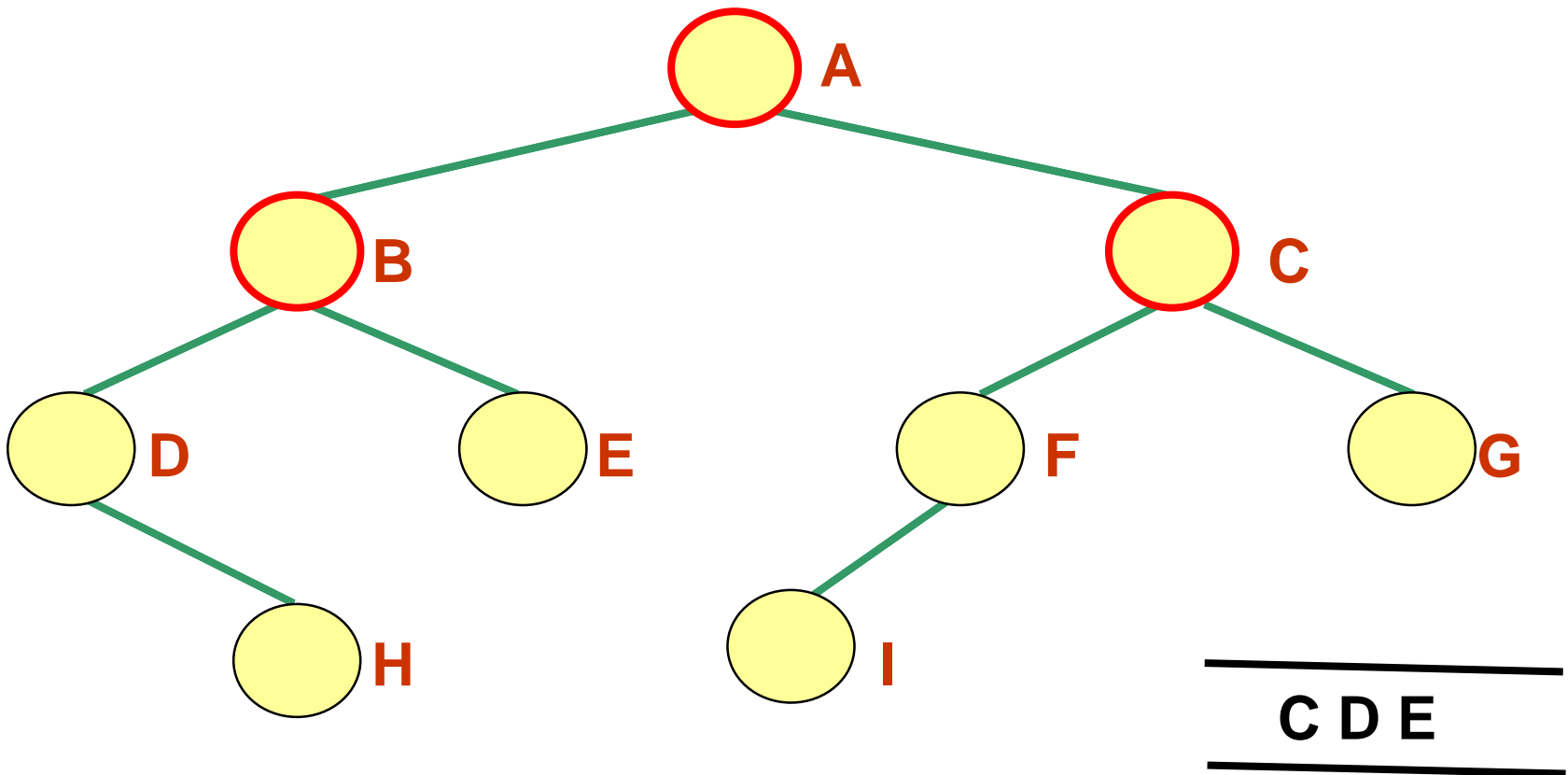


A B



Level Order Traversal

Repeat steps 1 and 2 until the queue is empty

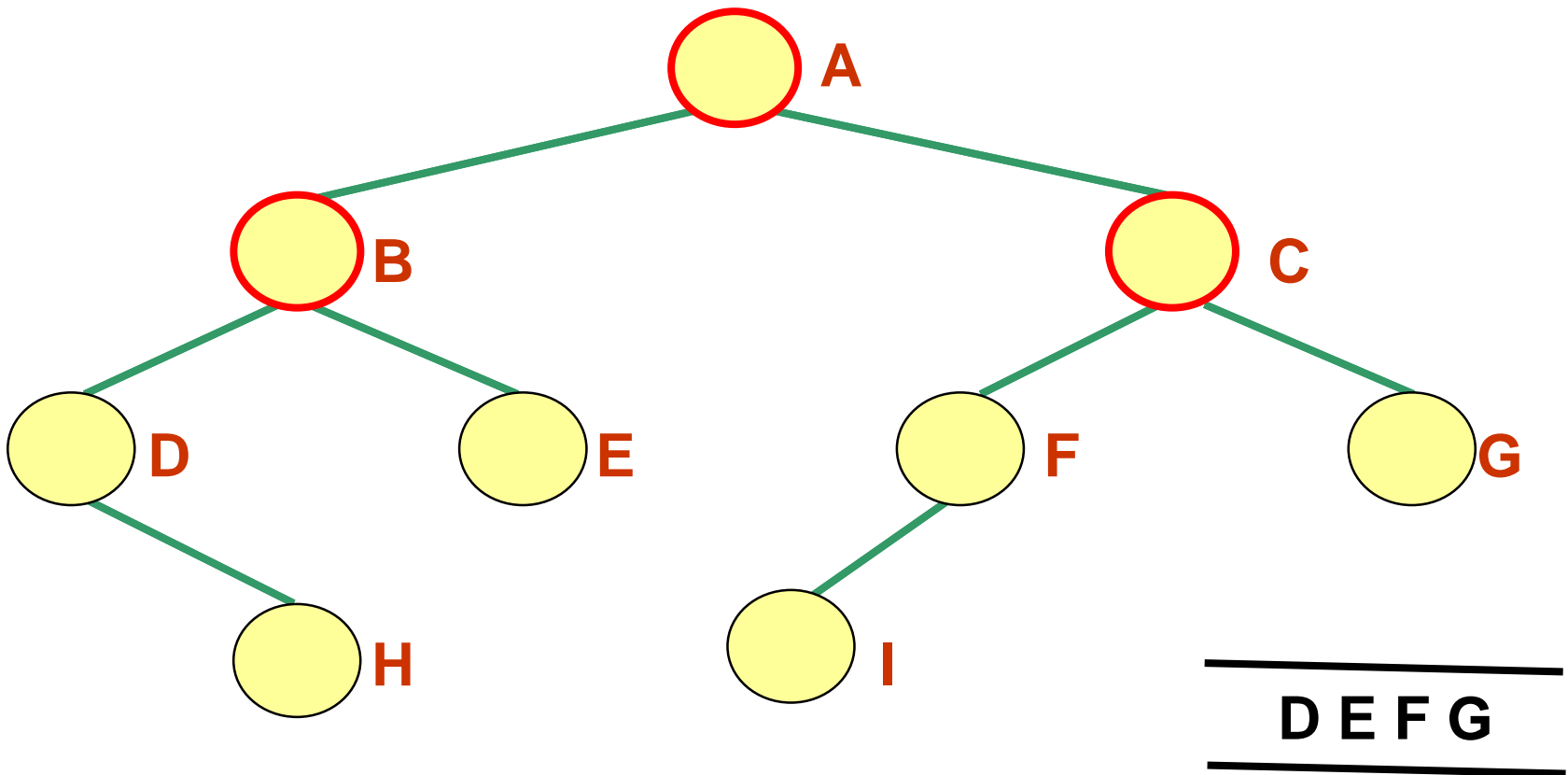


A B

C D E

Level Order Traversal

Repeat steps 1 and 2 until the queue is empty

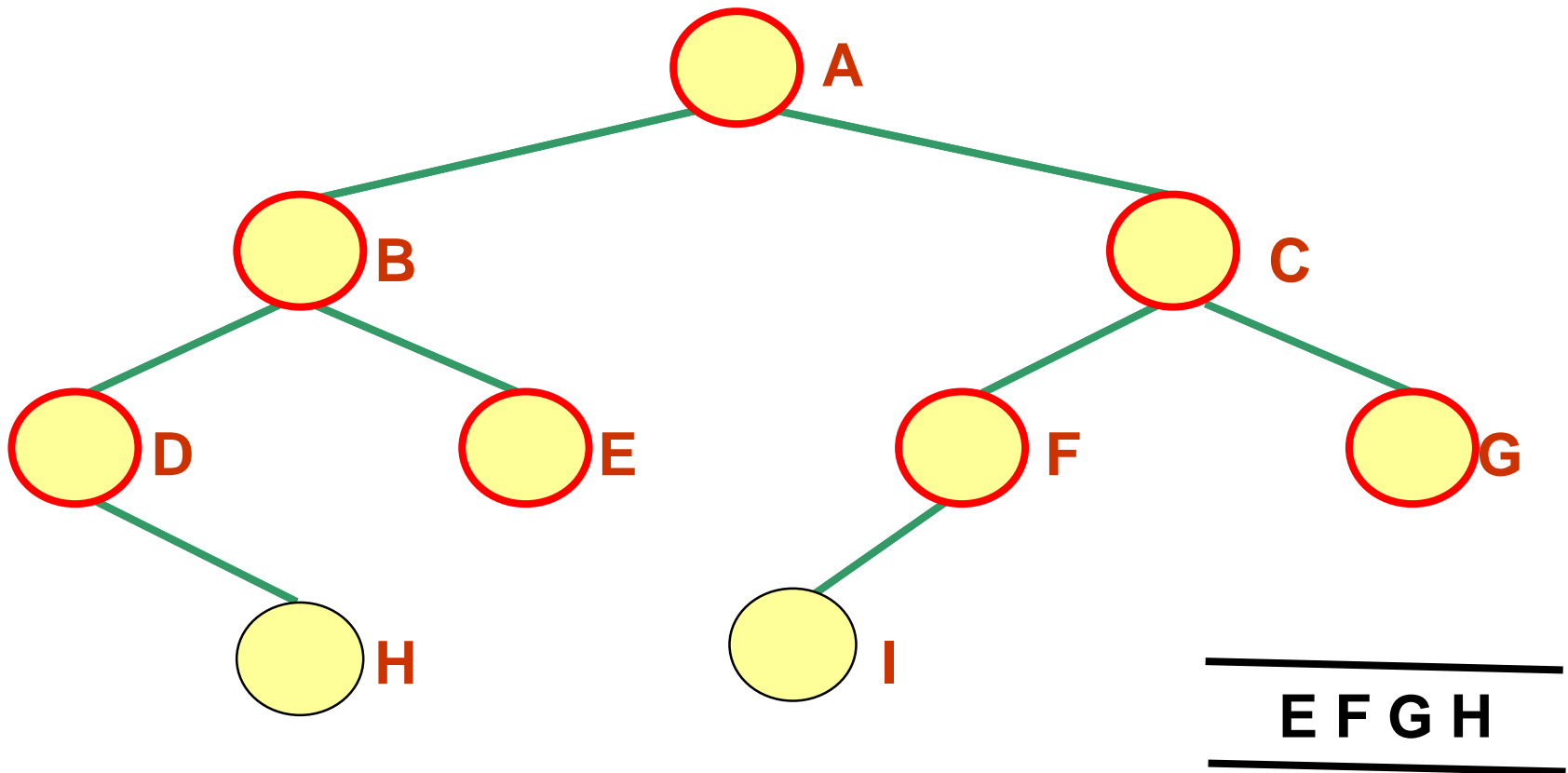


A B C

D E F G

Level Order Traversal

Repeat steps 1 and 2 until the queue is empty

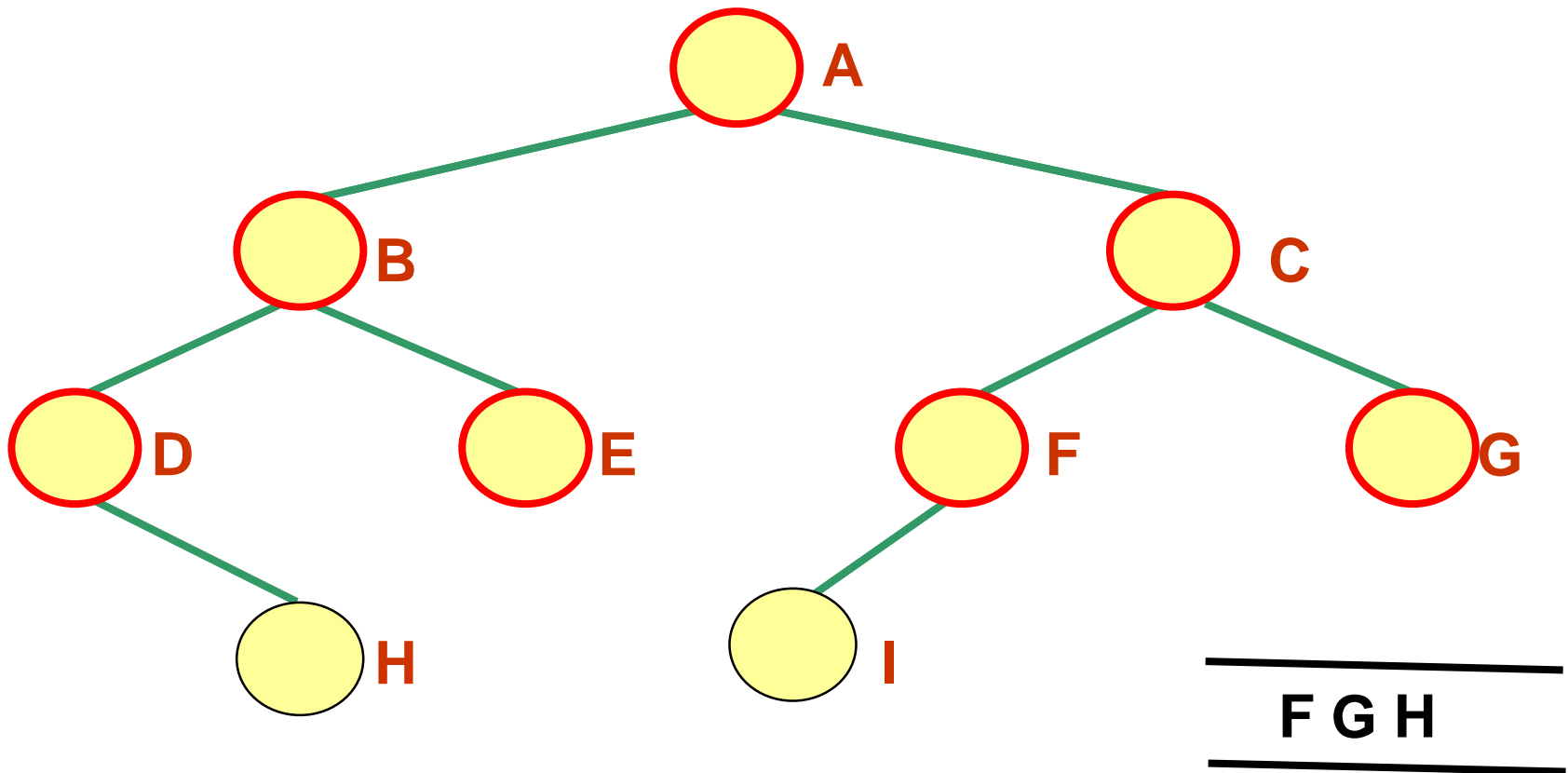


A B C D

EFGH

Level Order Traversal

Repeat steps 1 and 2 until the queue is empty

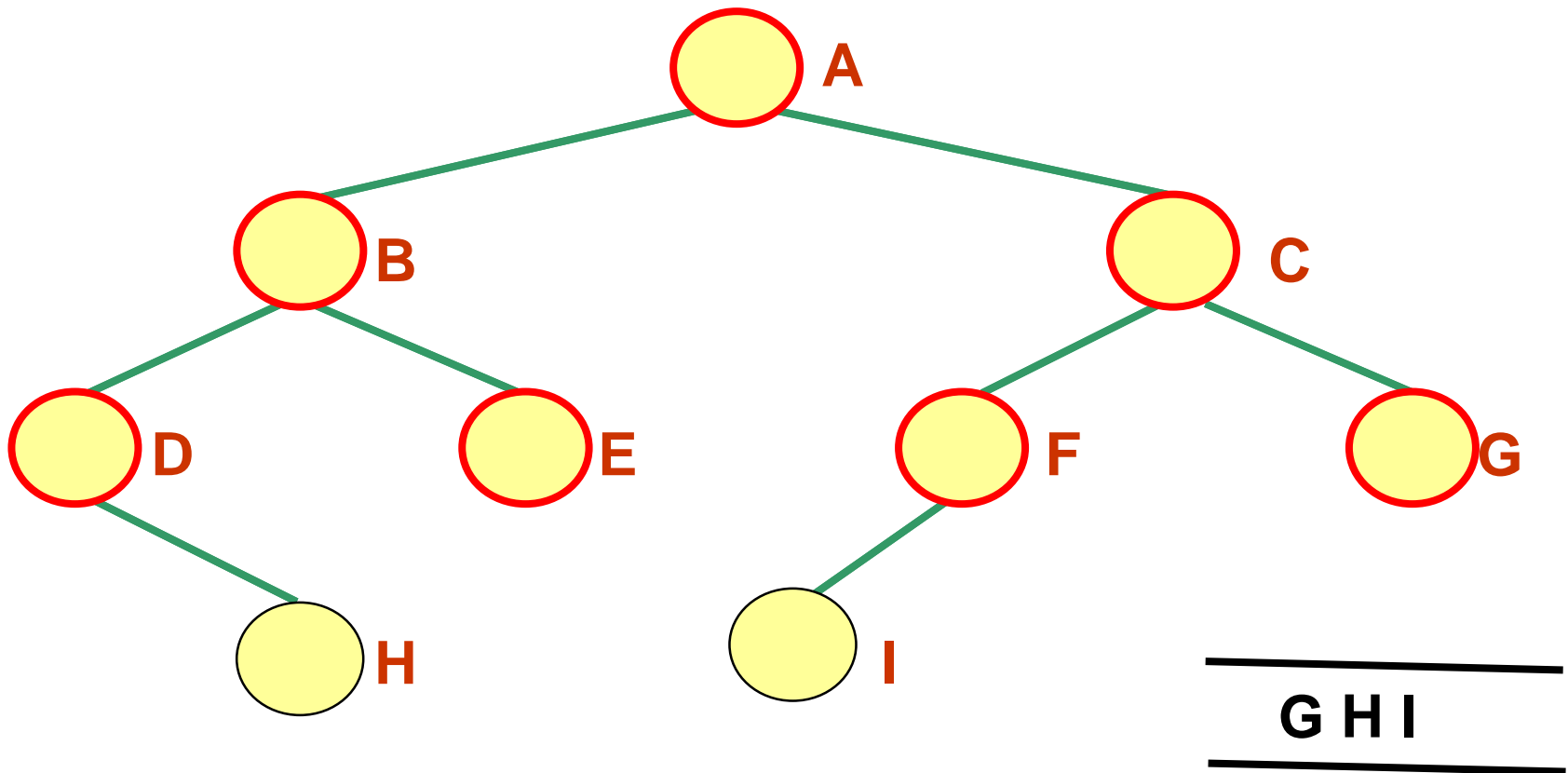


A B C D E

FGH

Level Order Traversal

Repeat steps 1 and 2 until the queue is empty

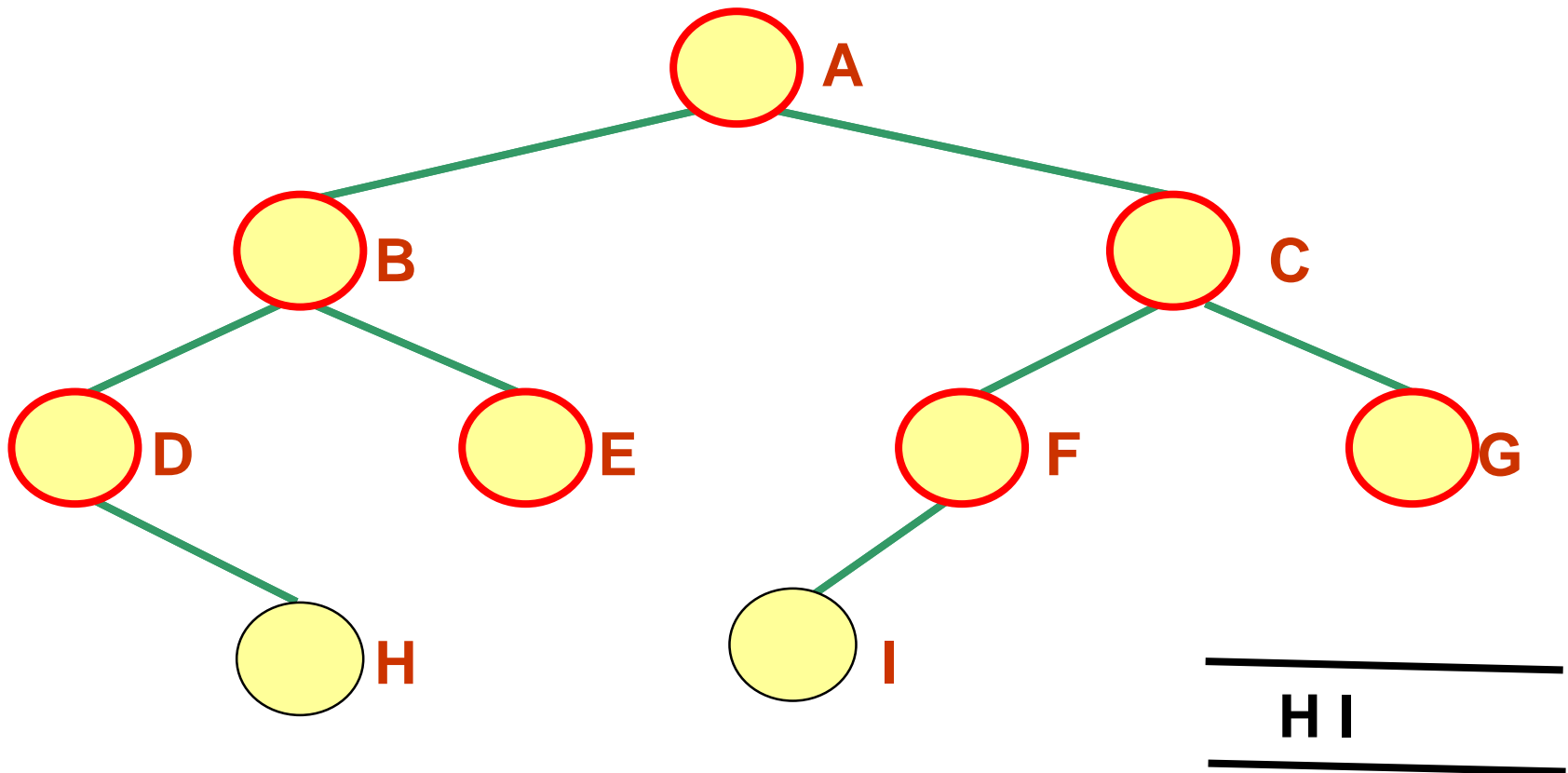


A B C D E F

G H I

Level Order Traversal

Repeat steps 1 and 2 until the queue is empty

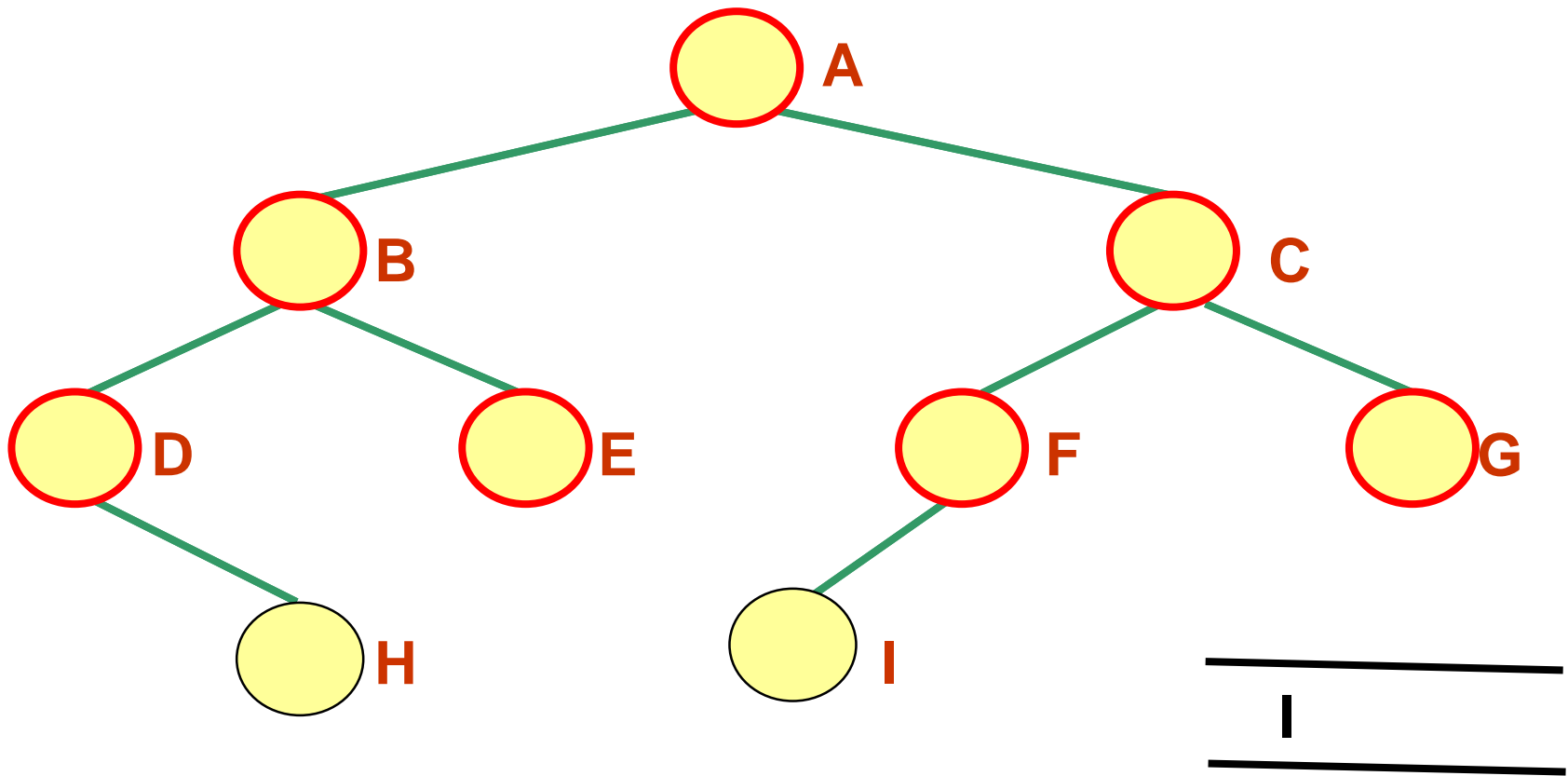


A B C D E F G

H I

Level Order Traversal

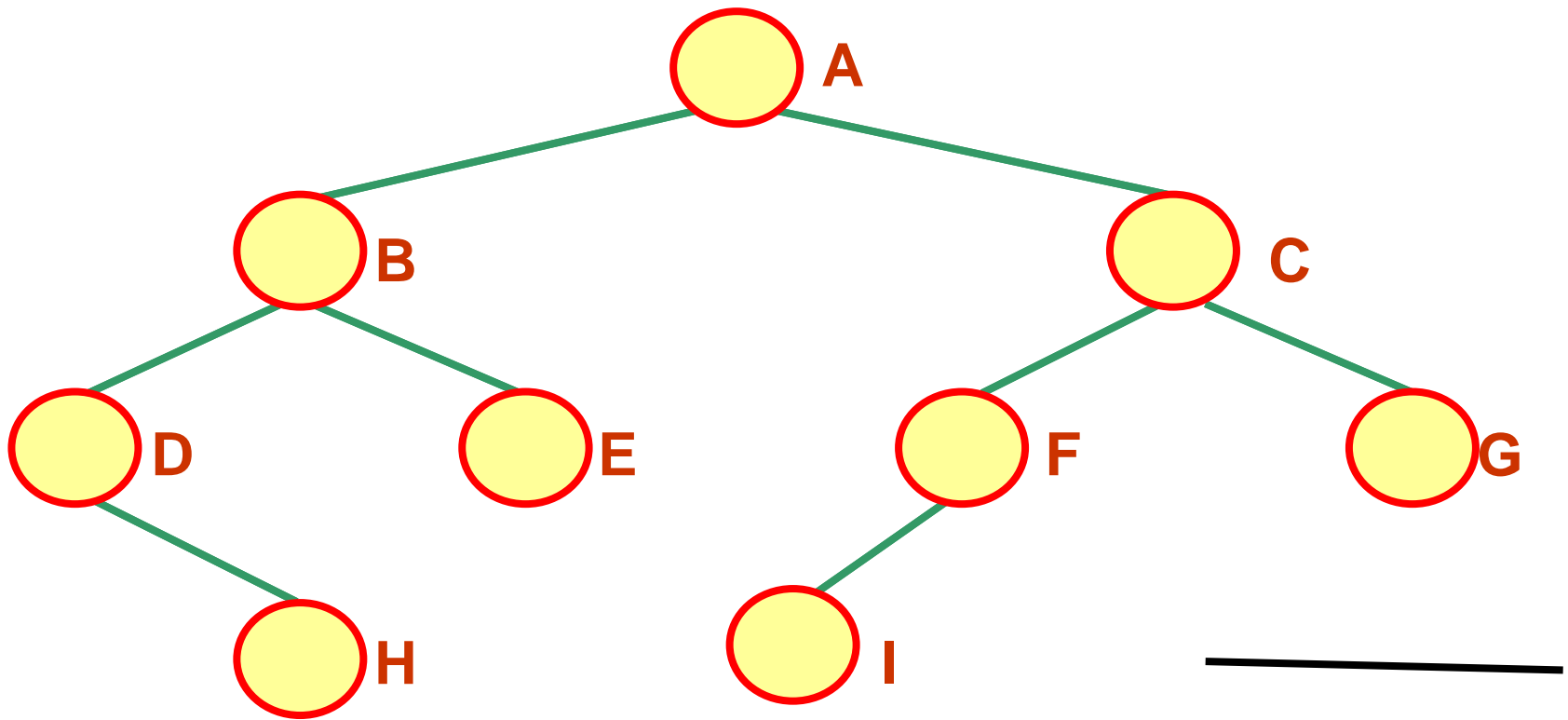
Repeat steps 1 and 2 until the queue is empty



A B C D E F G H

Level Order Traversal

The queue is empty. The algorithm terminates



Level order traversal: **A B C D E F G H I**

Level order Traversal

Algorithm levelOrder (root) {

Input: root of a binary tree

Output: Nothing, but visit the nodes of the tree in level order

if root = null **then return**

 Q = empty queue

 Q.enqueue(root);

while Q is not empty **do** {

 v = Q.dequeue();

 visit(v);

if v.leftChild() != null **then** Q.enqueue(v.leftChild());

if v.rightChild() != null **then** Q.enqueue(v.rightChild());

 }

}

Iterative Binary Tree Traversals

- In recursive tree traversals, the *Java execution stack* keeps track of where we are in the tree (by means of the activation records for each call)
- In *iterative traversals*, the programmer needs to keep track!
 - An iterative traversal uses a *container* to store references to nodes not yet visited
 - The order in which the nodes are visited will depend on the type of container being used (*stack*, *queue*, etc.)

An Iterative Traversal Algorithm

// *Assumption*: the tree is not empty

Create an empty container to hold references to nodes yet to be visited.

Put reference to the root node in the container.

While the container is not empty {

 Remove a reference **x** from the container.

 Visit the node **x** points to.

 Put references to non-empty children of **x** in the container.

}

Iterative Binary Tree Traversals

- Container is a **stack**: if we push the *right* child of a node before the *left* child, we get ***preorder*** traversal
- Container is a **queue**: if we enqueue the *left* child before the *right*, we get a ***level order traversal***

Operations on a Binary Tree

- What might we want to do with a binary tree?
 - Add an element
 - Remove an element
 - Is the tree empty?
 - Get size of the tree (i.e. how many elements)
 - Traverse the tree (in preorder, inorder, postorder, level order)

Possible Binary Tree Operations

Operation	Description
<code>getRoot</code>	Returns a reference to the root of the tree
<code>isEmpty</code>	Determines whether the tree is empty
<code>size</code>	Determines the number of elements in the tree
<code>find</code>	Returns a reference to the specified target, if found
<code>toString</code>	Returns a string representation of tree's contents
<code>iteratorInOrder</code>	Returns an iterator for an inorder traversal
<code>iteratorPreOrder</code>	Returns an iterator for a preorder traversal
<code>iteratorPostOrder</code>	Returns an iterator for a postorder traversal
<code>iteratorLevelOrder</code>	Returns an iterator for a levelorder traversal

What is an Iterator?

An *iterator* is an abstract data type that allows us to iterate through the elements of a collection one by one

Operations

- **next**: next element of the collection; ERROR if the element does not exist
- **hasNext**: true if there are more elements in the collection; false otherwise
- **remove**: removes the last element returned by the iterator

Iterator Interface

```
public interface Iterator<T> {  
    public boolean hasNext( );  
    public T next( );  
    public void remove( ); // (optional operation)  
}
```

This interface is in the `java.util` package of Java

Binary Tree ADT

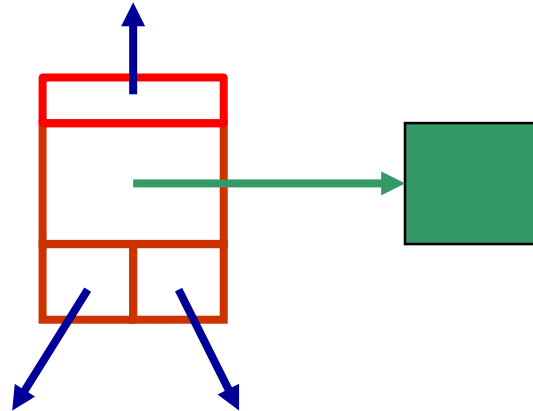
```
package binaryTree;
import java.util.Iterator;
public interface BinaryTreeADT<T> {
    public T getRoot ();
    public boolean isEmpty();
    public int size();
    public T find (T targetElement) throws
                                ElementNotFoundException;
    public String toString();
    public Iterator<T> iteratorInOrder();
    public Iterator<T> iteratorPreOrder();
    public Iterator<T> iteratorPostOrder();
    public Iterator<T> iteratorLevelOrder();
}
```

Linked Binary Tree Implementation

- To represent the binary tree, we will use a **linked** structure of nodes
 - **root**: reference to the node that is the **root** of the tree
 - **count**: keeps track of the number of nodes in the tree
- First, how will we represent a ***node of a binary tree?***

Linked Binary Tree Implementation

- A **binary tree node** will contain
 - a reference to a data element
 - references to its left and right children and to its parent



left and right children are binary tree nodes themselves

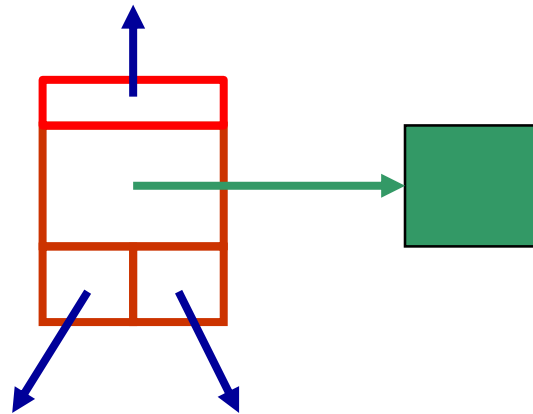
BinaryTreeNode class

- Represents a node in a binary tree
- Attributes:
 - **element**: reference to data element
 - **left**: reference to left child of the node
 - **right**: reference to right child of the node
 - **parent**: reference to the parent of the node

A BinaryTreeNode Object

protected T element;

protected BinaryTreeNode<T> left, right, parent;



Note that either or both of the **left** and **right** references could be null
What is the meaning of protected?

LinkedBinaryTree Class

- Attributes:

```
protected BinaryTreeNode<T> root;  
protected int count;
```

- The attributes are **protected** so that they can be accessed directly in any subclass of the `LinkedBinaryTree` class

LinkedBinaryTree Class

- Constructors:

//Creates empty binary tree

```
public LinkedBinaryTree() {  
    count = 0;  
    root = null;  
}
```

//Creates binary tree with specified element as its root

```
public LinkedBinaryTree (T element) {  
    count = 1;  
    root = new BinaryTreeNode<T> (element);  
}
```

/* Returns a reference to the specified target element if it is found in this binary tree.
Throws an ElementNotFoundException if not found. */

```
public T find(T targetElement) throws  
                ElementNotFoundException  
{  
    BinaryTreeNode<T> current =  
                findAgain( targetElement, root );  
    if ( current == null )  
        throw new ElementNotFoundException("binary tree");  
  
    return (current.element);  
}
```

Discussion

- What is **element** in this statement from the method?

```
return (current.element);
```

- If **element** were **private** rather than **protected** in `BinaryTreeNode.java`, what would be needed in order to access it?
- We will now look at the helper method **findAgain** ...

```
private BinaryTreeNode<T> findAgain(T targetElement,
                                     BinaryTreeNode<T> next) {
    if (next == null)
        return null;
    if (next.element.equals(targetElement))
        return next;
    BinaryTreeNode<T> temp =
        findAgain(targetElement, next.left);
    if (temp == null)
        temp = findAgain(targetElement, next.right);
    return temp;
}
```

Discussion

- What kind of method is `findAgain`?
 - What is the base case?
 - There are two!
 - What is the recursive part?

/* Performs an inorder traversal on this binary tree by calling a recursive inorder method that starts with the root.

Returns an inorder iterator over this binary tree */

```
public Iterator<T> iteratorInOrder() {  
    ArrayUnorderedList<T> tempList =  
        new ArrayUnorderedList<T>();  
  
    inorder (root, tempList);  
    return tempList.iterator();  
}
```

Discussion

- `iteratorInOrder` returns an iterator object
 - It will perform the iteration in ***inorder***
- But where is that iterator coming from?
`return tempList.iterator();`
- Let's now look at the helper method `inorder` ...

/* Performs a recursive inorder traversal.
Parameters are: the node to be used as the root
for this traversal, the temporary list for use in this
traversal */

```
protected void inorder (BinaryTreeNode<T> node,  
                        ArrayUnorderedList<T> tempList)  
{  
    if (node != null)  
    {  
        inorder (node.left, tempList);  
        tempList.addToRear(node.element);  
        inorder (node.right, tempList);  
    }  
}
```

Discussion

- Recall the *recursive algorithm* for inorder traversal:
 - If tree is not empty,
 - Perform **inorder traversal** of left subtree of root
 - Visit root node of tree
 - Perform **inorder traversal** of its right subtree
- That is exactly the order that is being implemented here!
 - What is “visiting” the root node here?

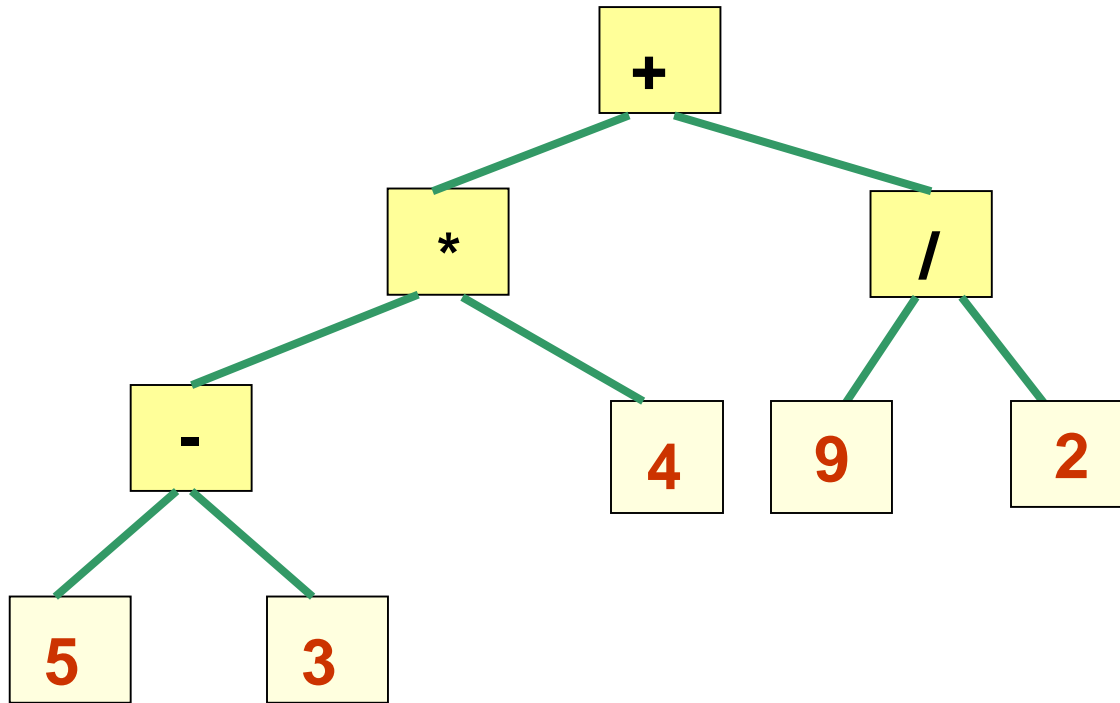
Discussion

- The data elements of the tree (i.e. items of type T) are being *temporarily* added to an unordered list, in *inorder* order
 - Why use an **unordered list**??
 - Why not? We already have this collection, with its **iterator** operation that we can use!

Using Binary Trees: Expression Trees

- Programs that manipulate or evaluate arithmetic expressions can use binary trees to hold the expressions
- An **expression tree** represents an arithmetic expression such as $(5 - 3) * 4 + 9 / 2$
 - Root node and interior nodes contain **operations**
 - Leaf nodes contain **operands**

Example: An Expression Tree

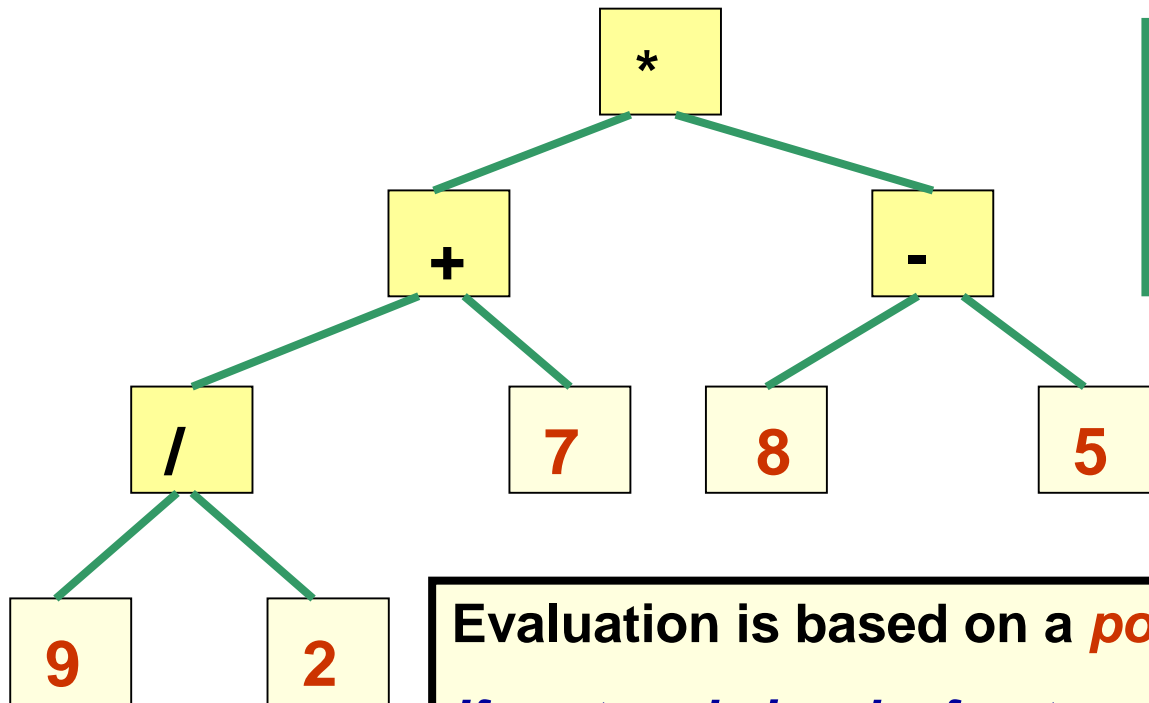


$(5 - 3) * 4 + 9 / 2$

Evaluating Expression Trees

- We can use an expression tree to *evaluate an expression*
 - We start the evaluation at the *bottom left*
 - What kind of traversal is this?

Evaluating an Expression Tree



This tree represents the expression

$$(9 / 2 + 7) * (8 - 5)$$

Evaluation is based on a *postorder* traversal:

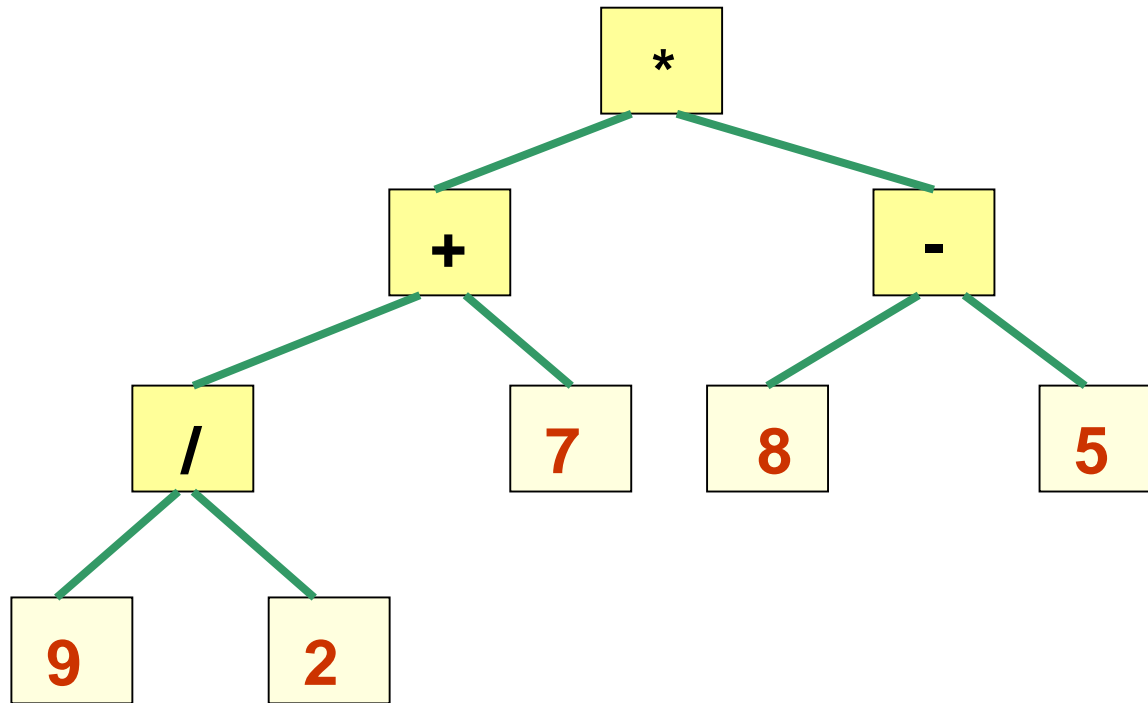
If root node is a leaf, return the associated value.

Recursively evaluate expression in left subtree.

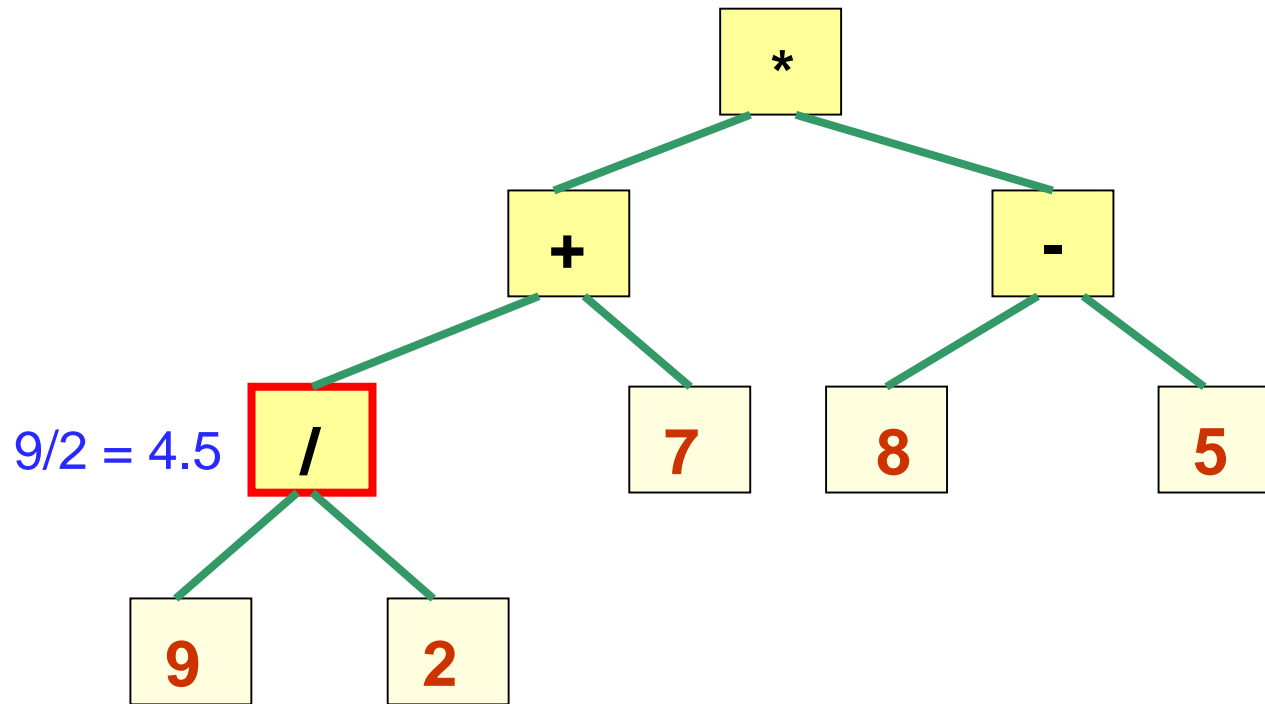
Recursively evaluate expression in right subtree.

Perform operation in root node on these two values, and return result.

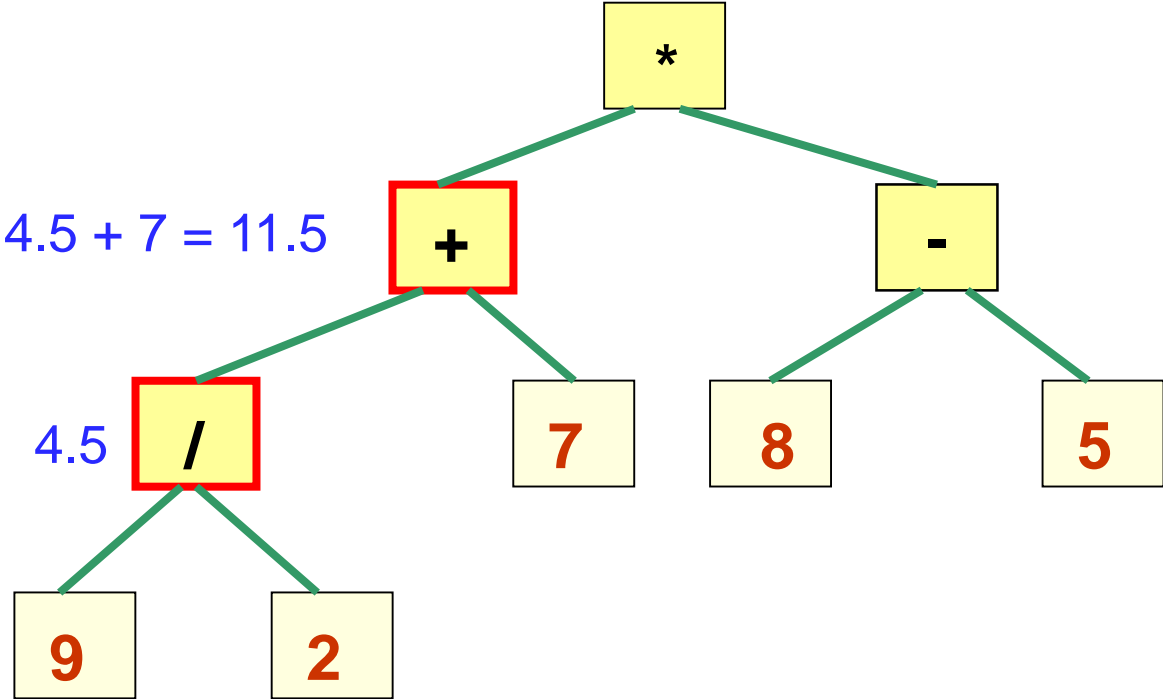
Evaluating an Expression Tree



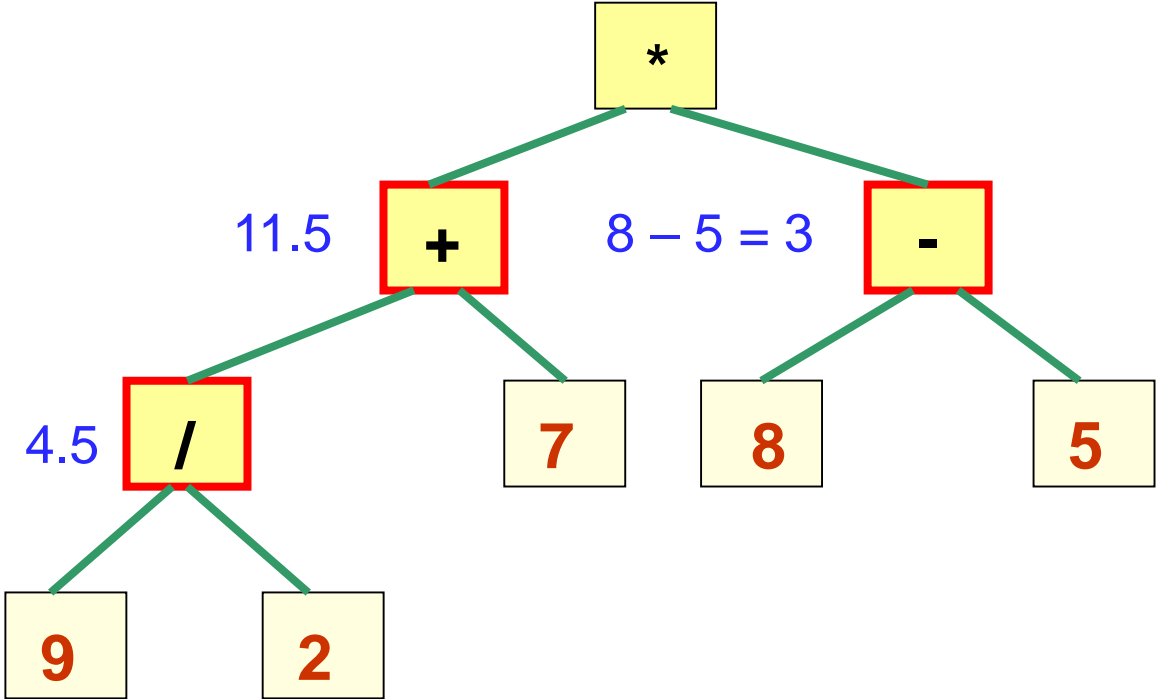
Evaluating an Expression Tree



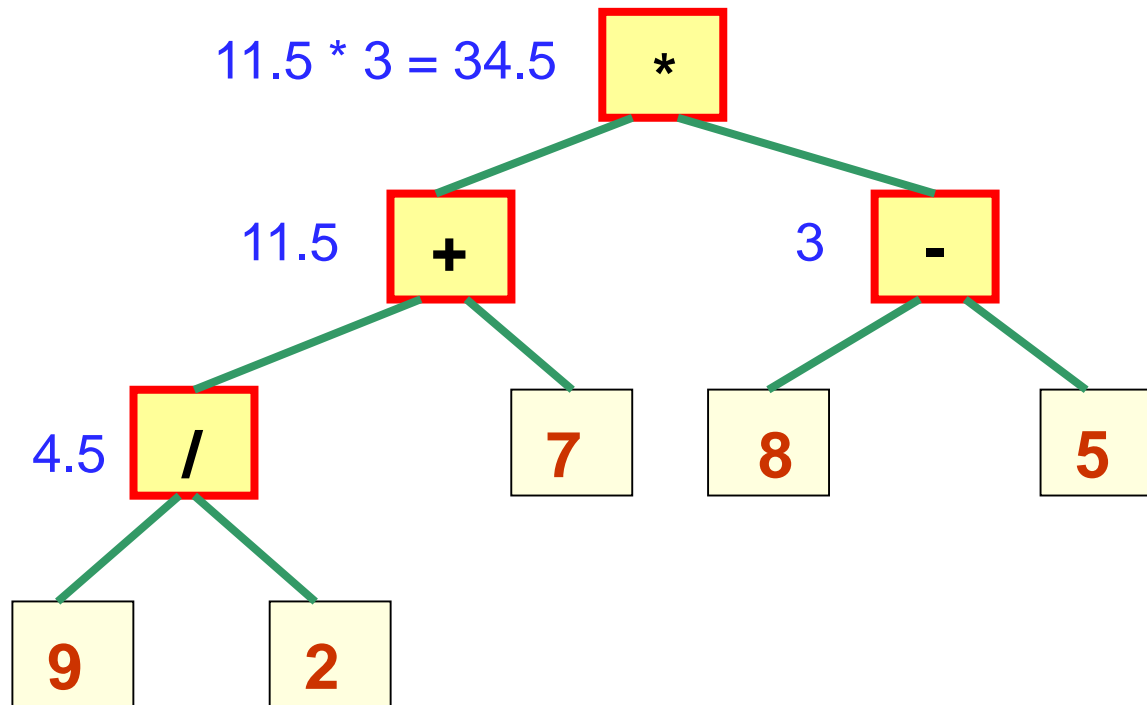
Evaluating an Expression Tree



Evaluating an Expression Tree



Evaluating an Expression Tree



Optional Notes: Building an Expression Tree

- Now we know how to evaluate an expression represented by an expression tree
- But, how do we *build* an expression tree?
 - We will build it from the **postfix form** of the expression
- **Exercise**: develop the algorithm by following the diagrams on the next pages

Building an Expression Tree

- The algorithm will use a stack of `ExpressionTree` objects
 - An `ExpressionTree` is a special case of a binary tree
 - The `ExpressionTree` constructor has 3 parameters:
 - Reference to data item
 - Reference to left child
 - Reference to right child
- *That's all you need to know to develop the algorithm!*

Build an expression tree from the postfix expression **5 3 - 4 * 9 +**

Symbol

5

Processing Step(s)

```
push(new ExpressionTree(5,null,null));
```

Expression Tree Stack (top at right)



Symbol

3

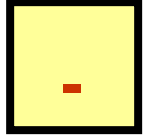
Processing Step(s)

```
push(new ExpressionTree(3,null,null));
```

Expression Tree Stack (top at right)



Symbol



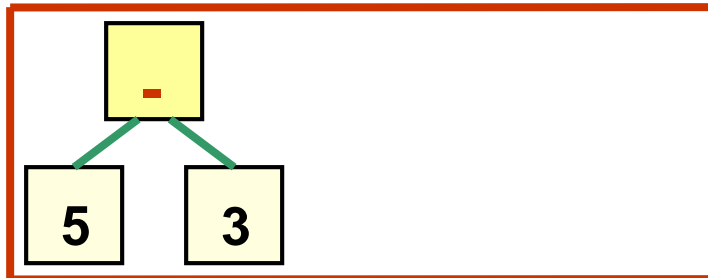
Processing Step(s)

op2 = pop

op1 = pop

push(new ExpressionTree(-,op1,op2));

Expression Tree Stack (top at right)



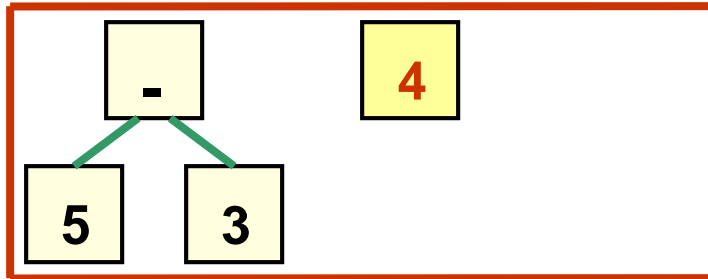
Symbol

4

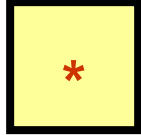
Processing Step(s)

```
push(new ExpressionTree(4,null,null));
```

Expression Tree Stack (top at right)



Symbol



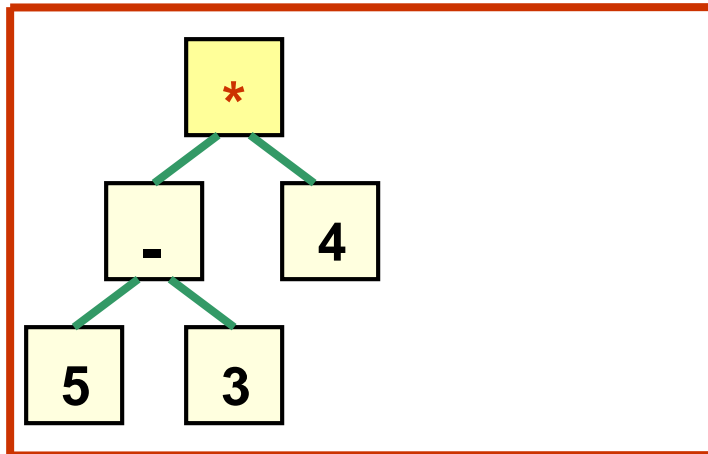
Processing Step(s)

op2 = pop

op1 = pop

push(new ExpressionTree(*,op1,op2));

Expression Tree Stack (top at right)



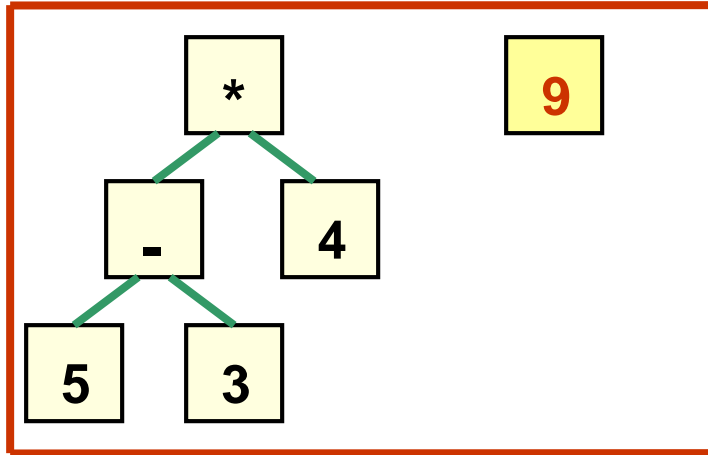
Symbol

9

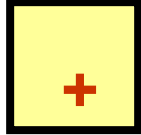
Processing Step(s)

`push(new ExpressionTree(9,null,null));`

Expression Tree Stack (top at right)



Symbol



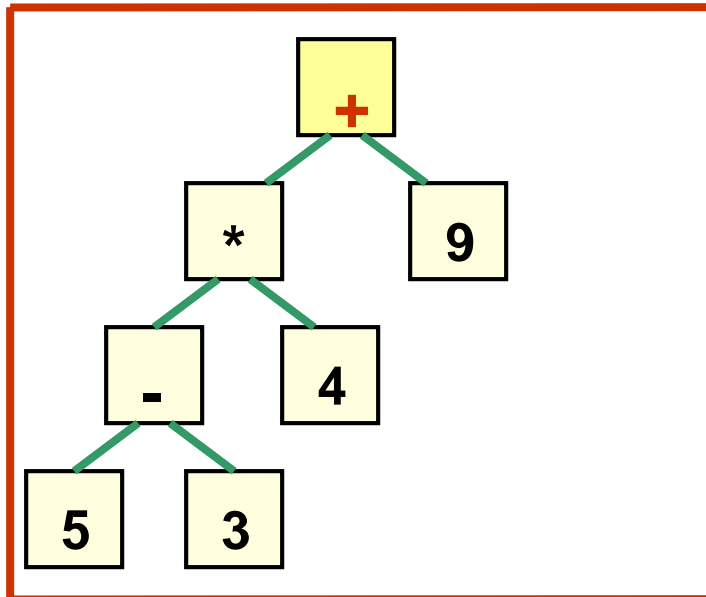
Processing Step(s)

op2 = pop

op1 = pop

push(new ExpressionTree(+,op1,op2));

Expression Tree Stack (top at right)



End of the expression has been reached, and the full expression tree is the only tree left on the stack