The University of Western Ontario
Department of Computer Science

Computer Science 209 b

Exercise Series 1

The times indicated are estimates you should not exceed in an exam situation. As the exercises are intended to assist you in learning the concepts and methods, you are likely to need more time than indicated when you attempt to solve an exercise problem.

Problem 1 (time estimate: 4 minutes).
Let $A = \{a, b, c\}$ and $B = \{x, y\}$. List all mappings of B into A.

Hint: You could arrange your answer as a table in which the first column contains the elements of B and each subsequent column contains a mapping given by the corresponding images of the elements of B.

Problem 2 (time estimate: 10 minutes).
Prove the following statement: *Let A and B be finite sets with $n = |B|$. There is a bijective mapping of the set A^B onto the set A^n.*

Hint: Give numbers to the elements of B, that is, $B = \{b_1, b_2, \ldots, b_n\}$. Then define a mapping $f : A^B \to A^n$ and show that f is bijective.

Reminder: A mapping is bijective if and only if it is injective (= one-to-one) and surjective (= onto).

Problem 3 (time estimate: 15 minutes).
Prove that there is a bijective mapping of $\mathbb{Z}$ onto $\mathbb{N}_0$.

Hint: "Fold" the line of integers at 0 onto the line of non-negative integers. From this intuition you would derive a formula defining the mapping and prove that the mapping is both surjective ('onto') and injective ('one-to-one').

Problem 4 (time estimate: 30 minutes).
Consider the context-free grammar

$$G = (\{S\}, \{a, b\}, P, S)$$

where P is the set consisting of the following rules:

$$S \rightsquigarrow aSa, S \rightsquigarrow bSb,$$
$$S \rightsquigarrow aa, S \rightsquigarrow bb.$$

For any word u, let u^{R} be the word obtained by reading u backward, that is, the *reversal* of u.

Show that $L(G)$ is exactly the set of words

$$\left\{ ww^{\mathrm{R}} \mid w \in \{a,b\}^{+} \right\}.$$

Problem 5 (time estimate: 15 minutes).
For the grammar of the previous problem, using pseudocode and no more than one short paragraph describe a program with the following behaviour: input any word $v \in \{a,b\}^{+}$; read v; at the end, output 'yes' if $v \in L(G)$ and 'no' otherwise.

Hint: Even if you did not solve the previous problem you may use the statement which is claimed there.

Problem 6 (time estimate: 5 minutes).
Prove, induction on n, that

$$\sum_{i=1}^{n} = \frac{n \cdot (n+1)}{2}.$$

Hint: There will be many proofs by induction in this course. You need to be completely familiar with this technique.