30. Constructive Logic

The logic we have discussed so far — except the multi-valued logic (ternary, fuzzy) — is called *classical logic.* In classical logic, the formula

$$\exists x\Big((x \text{ is a prime number}) \wedge x > 2^{100000000}\Big)$$

means

There is a prime number which is greater than $2^{100000000}$.

In this statement nothing is said about how to get such a number. In *constructive logic* this statement would only be true if such an x could actually be constructed; a proof of this statement could, for instance, consist of an algorithm by which to compute such an x. The following example shows the difference between the classical and constructive points of view.

Example 30.1 Consider the following statement:

There are irrational numbers a *and* b *such that* a^b *is rational.*

Classically, we prove this as follows: $\sqrt{2}$ is known to be irrational. A real number is either rational or irrational, but not both. Therefore, the number ${\sqrt{2}}^{\sqrt{2}}$ is either rational or irrational.

If it is rational then the statement is true with $a = b = \sqrt{2}$. If it is irrational then consider $a = {\sqrt{2}}^{\sqrt{2}}$ and $b = \sqrt{2}$. In this case,

$$a^b = \left({\sqrt{2}}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\left(\sqrt{2}^2\right)} = \sqrt{2}^2 = 2,$$

that is, a^b is rational.

We have proved that there are irrational numbers a and b such that a^b is rational. Even after this proof, however, we do not have actual values for a and b because we do not know whether ${\sqrt{2}}^{\sqrt{2}}$ is rational or irrational.

Our proof used classical logic and, in particular, the law that $(A \vee \neg A)$ is a tautology. Thus, in classical logic, if one proves that $\neg A$ is false, then one may conclude that A is true. In constructive logic, this does not work the same way; instead, one would still have to provide a proof of A.

Example 30.2 Suppose one proves

$$\neg\forall x \neg F(x),$$

that is, semantically, *it is not true that, for every x, $F(x)$ is not true.* Classically, one concludes

$$\exists x F(x),$$

that is, *there is an x satisfying F.* In constructive logic this conclusion is not acceptable unless x can actually be provided.

The formulæ in constructive logic are the same as in classical logic. However, the semantics and the deduction rules are different.

Whether classical or constructive logic is more adequate is a matter that is outside logic or mathematics. It is a philosophical question. One could argue that constructive logic is more adequate in Computer Science. On the other hand, in proving theorems about constructive logic, one uses classical logic at the metalanguage level.

The definition of interpretations of a formula A in constructive logic is significantly more complicated than that for classical logic. The intuitive meaning of the truth values 1 and 0 is modified as follows:

value	*classical*	*constructive*
1	true	true
0	false	not yet proven

Thus, instead of a single interpretation, we consider sequences of interpretations such that *later* interpretations will *never revoke an earlier statement about something being true,* but may *establish additional truth.* The semantics to be introduced is called the *Kripke semantics.* Note that, as a consequence, simple truth tables cannot be used here for the propositional part of logic.

Consider a set K of pairs $[I, s]$ where I is an interpretation and s is a value assignment within I to the free variable symbols. Let $\triangleright$ be a partial order on K, that is,

(1) $[I, s] \triangleright [I, s]$ for all $[I, s] \in K$ *(reflexivity),*

(2) for all $[I_1, s_1], [I_2, s_2] \in K$, if $[I_1, s_1] \triangleright [I_2, s_2]$ and $[I_2, s_2] \triangleright [I_1, s_1]$ then $[I_1, s_1] = [I_2, s_2]$ *(antisymmetry),* and

(3) for all $[I_1, s_1], [I_2, s_2], [I_3, s_3] \in K$, if $[I_1, s_1] \triangleright [I_2, s_2]$ and $[I_2, s_2] \triangleright [I_3, s_3]$ then $[I_1, s_1] \triangleright [I_3, s_3]$ *transitivity.*

We think of $\triangleright$ as meaning *before.* Then, with $[I_1, s_1], [I_2, s_2] \in K$ and $[I_1, s_1] \neq [I_2, s_2]$,

$$[I_1, s_1] \triangleright [I_2, s_2]$$

would mean that $[I_1, s_1]$ *has been defined before* $[I_2, s_2]$. For example,

- $[I_1, s_1](A) = 1$ means A *is true under* $[I_1, s_1]$;
- $[I_1, s_1](A) = 0$ means A *has not yet been assigned truth under* $[I_1, s_1]$.

Consider $[I, s] \in K$ where $I = (\mathbb{D}, \iota)$. Let A be a formula. In keeping with the intuitive meaning of $\triangleright$ we require the following properties:

(1) Let $[I', s'] \in K$ with $I' = (\mathbb{D}', \iota')$. If $[I, s] \triangleright [I', s']$ then

$$\mathbb{D} \subseteq \mathbb{D}',\ \iota(a) = \iota'(a),\ s(u) = s'(u),\ \iota(F) \subseteq \iota'(F),\ \iota(f) \subseteq \iota'(f)$$

for all constant symbols a, free variable symbols u, predicate symbols F, and function symbols f.

(2) If $[I', s'] \in K$, $[I, s] \triangleright [I', s']$, and A is an atom with $[I, s](A) = 1$, then $[I', s'](A) = 1$.

(3) If A has the form $A_1 \wedge A_2$ then

$$[I, s](A) = \min\{[I, s](A_1), [I, s](A_2)\}.$$

(4) If A has the form $A_1 \vee A_2$ then

$$[I, s](A) = \max\{[I, s](A_1), [I, s](A_2)\}.$$

Thus, for $\wedge$ and $\vee$ we use the same interpretations as before in the classical case.

(5) If A has the form $\neg A'$ then $[I, s](A) = 1$ if and only if, for all $[I', s'] \in K$ with $[I, s] \triangleright [I', s']$, one has $[I', s'](A') = 0$; otherwise, $[I, s](A) = 0$.

(6) If A has the form $A_1 \to A_2$ then $[I, s](A) = 1$ if and only if, for all $[I', s'] \in K$ with $[I, s] \triangleright [I', s']$, one has $[I', s'](A_2) = 1$ whenever $[I', s'](A_1) = 1$; otherwise, $[I, s](A) = 0$.

(7) If A has the form $A_1 \leftrightarrow A_2$ then $[I, s](A) = 1$ if and only if, for all $[I', s'] \in K$ with $[I, s] \triangleright [I', s']$, one has $[I', s'](A_1) = [I', s'](A_2)$; otherwise, $[I, s](A) = 0$.

Note that the wording of (5)–(7) uses classical logic — at the metalanguage level. Therefore, proofs about constructive logic will use classical logic at the metalanguage level.

(8) Suppose that A has the form $\exists x B(x)$ and let u be a free variable symbol not occurring in A. Then $[I, s](A) = 1$ if and only if there is a $d \in \mathbb{D}$ such that

$$[I, s_{[u=d]}](B(u)) = 1;$$

otherwise $[I, s](A) = 0$.

(9) Suppose that A has the form $\forall x B(x)$ and let u be a free variable symbol not occurring in A. Then $[I, s](A) = 1$ if and only if $[I', s'_{[u=d]}](B(u)) = 1$, for every $[I', s'] \in K$ with $[I, s] \triangleright [I', s']$ and every $d \in \mathbb{D}'$ where $I' = (\mathbb{D}', \iota')$; otherwise, $[I, s](A) = 0$.

In these notes, in the chapters on constructive logic, the notation $[I, s](A)$ always implies that the definition according (1)–(9) above is used. A pair $(K, \triangleright)$ with the properties above is called a *frame*. When specifying a frame we usually do not mention relations $[I, s] \triangleright [I', s']$ that are implied by reflexivity or transitivity.

Occasionally, it is necessary to evaluate the same formula both constructively and classically. In such a case,

$$[I, s]_{\text{class}}(A)$$

denotes the fact that A is evaluated in terms of $[I, s]$ classically, that is, according to the definitions in Section 18.

Example 30.3 Consider the four-element set

$$K = \{[I_1, s_1], [I_2, s_2], [I_3, s_3], [I_4, s_4]\}$$

with

$$[I_1, s_1] \triangleright [I_3, s_3] \triangleright [I_4, s_4] \quad \text{and} \quad [I_2, s_2] \triangleright [I_3, s_3].$$

For $i = 1, 2, 3, 4$, let $[I_i, s_i] = (\mathbb{D}_i, \iota_i)$. We consider the following domains:

$$\begin{aligned} \mathbb{D}_1 &= \{1, 2, \ldots, 10\}, \\ \mathbb{D}_2 &= \{-1, -2, \ldots, -15\}, \\ \mathbb{D}_3 &= \{n \mid n \in \mathbb{Z},\ -100 \leq n \leq 100\}, \end{aligned}$$

and

$$\mathbb{D}_4 = \{n \mid n \in \mathbb{Z},\ -1000 \leq n \leq 1000\}.$$

Consider the formula

$$\forall y \exists x \Big(G(x, y) \wedge H(x, y) \Big)$$

and let

$$\iota_i(G) = \{(d, d') \mid d, d' \in \mathbb{D}_i,\ d^2 \leq |d'|\}$$

and

$$\iota_i(H) = \{(d, d') \mid d, d' \in \mathbb{D}_i,\ d^2 + 1 \in \mathbb{D}_i,\ |d'| < (d + 1)^2\}.$$

Consider $j = 4$ first. As $[I_4, s_4]$ has no successor with respect to $\triangleright$, one has

$$[I_4, s_4](A) = [I_4, s_4]_{\text{class}}(A).$$

The value of $[I_4, s_4]_{\text{class}}(A)$ is 1 if and only if, for all $d' \in \mathbb{D}_4$, there is a $d \in \mathbb{D}_4$ such that

$$d^2 \leq |d'| < (d + 1)^2 \quad \text{and} \quad d^2 + 1 \in \mathbb{D}_4.$$

Given[22] d', let $d = \lfloor \sqrt{|d'|} \rfloor$. Then $d^2 \leq |d'| < (d + 1)^2$. Moreover, as $d' \in \mathbb{D}_4$ one has $|d'| \leq 1000$, hence, to satisfy

$$d^2 \leq |d'| < (d + 1)^2$$

one has $d \leq 31$ and $d^2 + 1 \leq 962 \in \mathbb{D}_4$ as required by the interpretation $\iota_4(H)$. Thus, $[I_4, s_4](A) = 1$.

Now consider $j = 3$. The value of $[I_3, s_3](A)$ is 1 if and only if, for $j = 3, 4$ and for all $d' \in \mathbb{D}_j$, there is a $d \in \mathbb{D}_j$ such that

$$d^2 \leq |d'| < (d + 1)^2 \quad \text{and} \quad d^2 + 1 \in \mathbb{D}_j.$$

[22] For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer not exceeding x and $\lceil x \rceil$ is the smallest integer no less than x.

We have already dealt with $j = 4$. For $j = 3$, consider $d' = 100$. Then d must be 10; for, if $d < 10$ then $(d+1)^2 \leq 100 = d'$. However, for $d = 10$ one has $d^2 + 1 = 101 \notin \mathbb{D}_3$. Hence $[I_3, s_3](A) = 0$.

Now consider $j = 1$. One has $[I_1, s_1](A) = 1$ if and only if, for $j = 1, 3, 4$ and all $d' \in \mathbb{D}_j$,

$$[I_j, s_{j,[u=d']}]\big(\exists x\, (G(x,u) \wedge H(x,u))\big) = 1,$$

where u is a free variable symbol not occurring in A. To get this, there has to exist a $d \in \mathbb{D}_i$ such that

$$[I_j, s_{j,[u=d',v=d]}]\big(G(v,u) \wedge H(v,u)\big) = 1,$$

where v is a free variable symbol not occurring so far.

In $\mathbb{D}_1$ we use the same approach as in $\mathbb{D}_4$, that is, we first compute the classical evaluation: given $d' \in \mathbb{D}_1$, define $d = \lfloor \sqrt{d'} \rfloor$. One has $d \leq 3$ and $d^2 + 1 \leq 10$. Moreover, $d^2 \leq d' < (d+1)^2$. Thus, for every $d' \in \mathbb{D}_1$ there is a $d \in \mathbb{D}_1$ such that

$$[I_1, s_{1,[u=d',v=d]}]_{\text{class}}\big(G(v,u) \wedge H(v,u)\big) = 1.$$

However, $[I_1, s_1](A) = 0$ because of the result above for $j = 3$. In fact, the computation of the classical evaluation above is redundant bcause of this.

The case of $j = 2$ is left as an exercise.

Definition 30.1 Let $\Sigma \subseteq \mathcal{L}^{\text{pred}}$ and $A \in \mathcal{L}^{\text{pred}}$. The set Σ is said to be *constructively satisfiable* if there are $(K, \triangleright)$ and $[I, s] \in K$ such that $[I, s](\Sigma) = 1$. The formula A is *constructively valid* if $[I, s](A) = 1$ for every $(K, \triangleright)$ and every $[I, s] \in K$. The formula A is a *constructive logical consequence* of Σ if, for all $(K, \triangleright)$ and all $[I, s] \in K$, one has $[I, s](A) = 1$ whenever $[I, s](\Sigma) = 1$. In this case we write $\Sigma \models_{\text{c}} A$.

Theorem 30.1 *Consider $A \in \mathcal{L}^{\text{pred}}$, $(K, \triangleright)$, and $[I, s], [I', s'] \in K$. If $[I, s] \triangleright [I', s']$ and $[I, s](A) = 1$ then $[I', s'](A) = 1$.*

Proof: Omitted. □

Example 30.4 We show that $A \models_{\text{c}} \neg\neg A$. Suppose that $(K, \triangleright)$ and $[I, s] \in K$ are such that $[I, s](A) = 1$. Assume that $[I, s](\neg\neg A) = 0$. Then there is a $[I', s'] \in K$ with $[I, s] \triangleright [I', s']$ and $[I', s'](\neg A) = 1$. Hence, for all $[I'', s''] \in K$ with $[I', s'] \triangleright [I'', s'']$ one has $[I'', s''](A) = 0$. But $[I, s] \triangleright [I'', s'']$ and $[I, s](A) = 1$, a contradiction by Theorem 30.1.

Example 30.5 In constructive logic one cannot, in general, conclude A from $\neg\neg A$. Suppose, we have $(K, \triangleright)$ and $[I, s] \in K$ such that $[I, s](\neg\neg A) = 1$. Then, for all $[I', s'] \in K$ with $[I, s] \triangleright [I', s']$, one has $[I', s'](\neg A) = 0$. Therefore, for some $[I'', s''] \in K$ with $[I', s'] \triangleright [I'', s'']$, one has $[I'', s''](A) = 1$.

Thus, *eventually,* the value of A will be 1. However, we cannot conclude that $[I, s] = [I'', s'']$ and, therefore, $[I, s](A)$ could be 0. Such a situation would arise, for instance, when A has the form $\forall x \exists y B(x, y)$. In this case, the domain of I may not contain the elements required by the existential quantifier in all situations, and this could be different in a larger domain, that of I'' say.

In such a situation we would have

$$[I, s](\neg A) = [I, s](A) = 0 \quad \text{and} \quad [I, s](\neg\neg A) = 1.$$

In classical logic this is impossible.

Example 30.5 shows that, in constructive logic, the formula $A \vee \neg A$ is not valid and that proof by contradiction is not possible in general. Note that proof by contradiction uses the following schema: To prove A we assume that $\neg A$ is true. Then we show that this leads to a contradiction from which we can conclude that $\neg\neg A$ is true. In classical logic one can then conclude that A is true; not so constructively.

31. Formal Deduction in Constructive Logic

In Table 31.1 we give the rules c_1–c_{13} for formal deduction in constructive logic. To these one adds the rules q_1–q_6. The rules c_1–c_{13} are nearly the same as the rules r_1–r_{13}. There is only one difference: Rule r_3 has been replaced by the rule c_3 which says that from contradictory statements any conclusion can be obtained. The old rule r_{13}, which is a consequence of r_1–r_{11}, is now the main rule for negation; it is not a consequence of rules c_1–c_{12}.

c_1)	$\dfrac{}{f \vdash f}$	*(reflexivity)*	c_2)	$\dfrac{\Sigma \vdash f}{\Sigma, \Sigma' \vdash f}$	*(introduction of premisses)*
c_3)	$\dfrac{\Sigma \vdash f \quad \Sigma \vdash \neg f}{\Sigma \vdash g}$		c_4)	$\dfrac{\Sigma \vdash f \to g \quad \Sigma \vdash f}{\Sigma \vdash g}$	*(elimination of $\to$)*
c_5)	$\dfrac{\Sigma, f \vdash g}{\Sigma \vdash f \to g}$	*(introduction of $\to$)*	c_6)	$\dfrac{\Sigma \vdash f \wedge g}{\Sigma \vdash f}$	*(elimination of $\wedge$)*
$c_{6'}$)	$\dfrac{\Sigma \vdash f \wedge g}{\Sigma \vdash g}$	*(elimination of $\wedge$)*	c_7)	$\dfrac{\Sigma \vdash f \quad \Sigma \vdash g}{\Sigma \vdash f \wedge g}$	*(introduction of $\wedge$)*
c_8)	$\dfrac{\Sigma, f \vdash h \quad \Sigma, g \vdash h}{\Sigma, f \vee g \vdash h}$	*(elimination of $\vee$)*	c_9)	$\dfrac{\Sigma \vdash f}{\Sigma \vdash f \vee g}$	*(introduction of $\vee$)*
$c_{9'}$)	$\dfrac{\Sigma \vdash f}{\Sigma \vdash g \vee f}$	*(introduction of $\vee$)*	c_{10})	$\dfrac{\Sigma \vdash f \leftrightarrow g \quad \Sigma \vdash f}{\Sigma \vdash g}$	*(elimination of $\leftrightarrow$)*
$c_{10'}$)	$\dfrac{\Sigma \vdash f \leftrightarrow g \quad \Sigma \vdash g}{\Sigma \vdash f}$	*(elimination of $\leftrightarrow$)*	c_{11})	$\dfrac{\Sigma, f \vdash g \quad \Sigma, g \vdash f}{\Sigma \vdash f \leftrightarrow g}$	*(introduction of $\leftrightarrow$)*
c_{12})	*If* $f \in \Sigma$ *then* $\Sigma \vdash f$	*(consequence of r_1, r_2)*	c_{13})	$\dfrac{\Sigma, f \vdash g \quad \Sigma, f \vdash \neg g}{\Sigma \vdash \neg f}$	*(introduction of $\neg$)*

Table 31.1. Deduction rules for propositional logic (constructive).

The definition of formal deduction and proof for constructive logic is identical with that for classical logic, except that now the new set of rules is to be used. For $\Sigma \subseteq \mathcal{L}^{\text{pred}}$

and $A \in \mathcal{L}^{\text{pred}}$, we write $\Sigma \vdash_{\text{c}} A$ to denote the fact that there is a proof of A from Σ using rules c_1–c_{13} and q_1–q_6.

Theorem 31.1 *Let* $\Sigma \subseteq \mathcal{L}^{\text{pred}}$ *and* $A \in \mathcal{L}^{\text{pred}}$. $\Sigma \models_{\text{c}} A$ *if and only if* $\Sigma \vdash_{\text{c}} A$.

Proof: Omitted. □

This result is, again, a statement of soundness and completeness. It allows one to move freely between formal deduction and logial consequence.

Theorem 31.2 *Let* $\Sigma \subseteq \mathcal{L}^{\text{pred}}$ *and* $A \in \mathcal{L}^{\text{pred}}$. *If* $\Sigma \models_{\text{c}} A$ *then* $\Sigma \models A$.

Proof: The rules c_1–c_{13} are identical with or consequences of rules r_1–r_{13}. Therefore, if $\Sigma \vdash_{\text{c}} A$ then $\Sigma \vdash A$. The claim follows by Theorem 31.1 and Theorem 22.4. □

[1]	$A \vdash_{\text{c}} \neg\neg A$	[2]	$A \to B \vdash_{\text{c}} \neg B \to \neg A$
[3]	$A \to B \vdash_{\text{c}} \neg\neg A \to \neg\neg B$	[4]	If $A \vdash_{\text{c}} B$ then $\neg B \vdash_{\text{c}} \neg A$
[5]	If $A \vdash_{\text{c}} B$ then $\neg\neg A \vdash_{\text{c}} \neg\neg B$	[6]	$\emptyset \vdash_{\text{c}} \neg(A \wedge \neg A)$
[7]	$\emptyset \vdash_{\text{c}} \neg\neg(A \vee \neg A)$	[8]	$\neg(A \vee B) \dashv\vdash_{\text{c}} \neg A \wedge \neg B$
[9]	$A \vee B \vdash_{\text{c}} \neg(\neg A \wedge \neg B)$	[10]	$\neg A \vee \neg B \vdash_{\text{c}} \neg(A \wedge B)$
[11]	$A \wedge B \vdash_{\text{c}} \neg(\neg A \vee \neg B)$	[12]	$A \vee B \vdash_{\text{c}} \neg A \to B$
[13]	$\neg A \vee B \vdash_{\text{c}} A \to B$	[14]	$\neg(A \wedge B) \dashv\vdash_{\text{c}} A \to \neg B$
[15]	$A \wedge B \vdash_{\text{c}} \neg(A \to \neg B)$	[16]	$A \to B \vdash_{\text{c}} \neg(A \wedge \neg B)$
[17]	$A \wedge \neg B \vdash_{\text{c}} \neg(A \to B)$	[18]	$\neg\exists x A(x) \dashv\vdash_{\text{c}} \forall x \neg A(x)$
[19]	$\exists x A(x) \vdash_{\text{c}} \neg\forall x \neg A(x)$	[20]	$\forall x A(x) \vdash_{\text{c}} \neg\exists x \neg A(x)$
[21]	$\exists x \neg A(x) \vdash_{\text{c}} \neg\forall x A(x)$	[22]	$\neg\neg\forall x A(x) \vdash_{\text{c}} \forall x \neg\neg A(x)$

Table 31.2. Theorems of $\mathcal{L}^{\text{pred}}$, to be done as exercises. A, B, and C denote arbitrary formulæ in $\mathcal{L}^{\text{pred}}$, t, $t_1, t_2, \ldots, t_3$ are arbitrary terms, Q_1 and Q_2 are arbitrary quantifier symbols; *x not in A* means the condition or assumption that x does not occur in A; *partial replacement* means that some, but not necessarily all occurrences of a term are replaced by a variable symbol or vice versa.