Binary Search Trees
Ordered Dictionary (Map) ADT

- get (k): record with key k
- put (k, data): add record (k, data)
- remove (k): delete record with key k
- smallest(): record with smallest key
- largest(): record with largest key
- predecessor(k): record with largest key less than k
- successor(k): record with smallest key greater than k
Total Order

The keys stored in an ordered dictionary must come from a **total order**: given any two keys $k$ and $k'$ we can always determine whether

- $k = k'$
- $k > k'$, or
- $k < k'$

For example total orders are defined over the integers, real numbers, characters, Strings, ...
Search Tables

A search table is an ordered map implemented by means of a sorted sequence

- We store the items in an array sorted by key

Performance:

- Searches take $O(\log n)$ time, using binary search
- Inserting a new item takes $O(n)$ time, since in the worst case we have to shift $n - 1$ items to make room for the new item
- Removing an item takes $O(n)$ time, since in the worst case we have to shift $n - 1$ items to compact the items after the removal

The search table is effective only for ordered maps of small size or for maps on which searches are the most common operations, while insertions and removals are rarely performed (e.g., credit card authorizations)
A binary search tree is a proper binary tree storing records in its internal nodes such that for each internal node $u$:

- Every key in the left subtree of $u$ is smaller than $key(u)$.
- Every key in the right subtree of $u$ is larger than $key(u)$. 

\[
\begin{align*}
&k_1 & & < k & & > k & & k_2 \\
\end{align*}
\]
A binary search tree is a proper binary tree storing records in its internal nodes such that for each internal node \( u \):
- Every key in the left subtree of \( u \) is smaller than \( \text{key}(u) \).
- Every key in the right subtree of \( u \) is larger than \( \text{key}(u) \).

External nodes do not store records.
A binary search tree is a proper binary tree storing records in its internal nodes such that for each internal node $u$:

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Not a binary search tree
Inorder Traversal

An inorder traversal of a binary search tree visits the keys in increasing order.

1, 2, 4, 6, 8, 9
Search

To search for a key $k$, we trace a downward path starting at the root.

The next node visited depends on the comparison of $k$ with the key of the current node.

If we reach a leaf, the key is not found.

Example: get(4):
- Call $\text{TreeSearch}(4, \text{root})$

```
Algorithm $\text{TreeSearch}(k, v)$
    if $v$ is a leaf then
        return $v$
    if $k < \text{key}(v)$ then
        return $\text{TreeSearch}(k, \text{left}(v))$
    else if $k = \text{key}(v)$ then
        return $v$
    else { $k > \text{key}(v)$ }
    return $\text{TreeSearch}(k, \text{right}(v))$
```
Insertion

To perform operation \( \text{put}(k, o) \), we search for key \( k \) (using TreeSearch).

Assume \( k \) is not already in the tree, and let \( w \) be the leaf reached by the search.

We insert \( k \) at node \( w \) and expand \( w \) into an internal node.

Example: insert 5
Deletion

- To perform operation \( \text{remove}(k) \), we search for key \( k \)
- Assume key \( k \) is in the tree, and let let \( v \) be the node storing \( k \)
- If node \( v \) has a leaf child \( w \), we remove \( v \) and \( w \) from the tree with operation \( \text{removeExternal}(w) \), which removes \( w \) and its parent
- Example: remove 4

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We consider the case where the key \( k \) to be removed is stored at a node \( v \) whose children are both internal

- we find the internal node \( w \) that follows \( v \) in an inorder traversal
- we copy \( \text{key}(w) \) into node \( v \)
- we remove node \( w \) and its left child \( z \) (which must be a leaf) by means of operation \( \text{removeExternal}(z) \)

Example: remove 3
Performance

Consider an ordered dictionary with $n$ items implemented by means of a binary search tree of height $h$

- the space used is $O(n)$
- methods get, put and remove take $O(h)$ time

The height $h$ is $O(n)$ in the worst case and $O(\log n)$ in the best case