Notes on Complexity Analysis

1 Power Functions

For any \( a < b \) s.t. \( a \geq 1 \) and \( b \geq 1 \), it holds that \( n^b \) is not \( O(n^a) \). In other words, \( n^b \) has a larger growth rate than \( n^a \).

**Proof:**

Argue by contradiction. Suppose \( n^b \) is \( O(n^a) \). Then there exist positive constants \( n_0, c \) s.t.

\[
 n^b \leq c \cdot n^a, \quad \forall n \geq n_0
\]

Dividing both sides of the inequality by \( n^a \) we get:

\[
 n^{b-a} \leq c, \quad \forall n \geq n_0.
\]

Now \( b-a \) is a positive number since \( b > a \). This \( n^{b-a} \) is a function that increases in \( n \) and the above inequality cannot hold. To make the proof precise, take \( m = 1 + \max\{n_0, c^{\frac{1}{b-a}}\} \). Then \( m \geq n_0 \) and plugging it into the inequality above, we get a contradiction.

2 Exponent Functions

For any \( a < b \) s.t. \( a \geq 1 \) and \( b \geq 1 \), it holds that \( b^n \) is not \( O(a^n) \). In other words, \( b^n \) has a larger growth rate than \( a^n \).

**Proof:**

Argue by contradiction. Suppose \( b^n \) is \( O(a^n) \). Then there exist positive constants \( n_0, c \) s.t.

\[
 b^n \leq c \cdot a^n, \quad \forall n \geq n_0
\]

Dividing both sides by \( a^n \) we get:

\[
 \left(\frac{b}{a}\right)^n \leq c, \quad \forall n \geq n_0.
\]

This cannot hold since \( \left(\frac{b}{a}\right)^n \) is a function that increases with \( n \), due to the fact that \( \frac{b}{a} > 1 \). To make the proof precise, \( m = 1 + \max\{n_0, \log_a c\} \) and we get a contradiction.