Assignment #1
Due: Oct. 1, 2019, by 23:55
Submission: on the OWL web site of the course

Format of the submission. You must submit a single file which must be in PDF format. All other formats (text or Microsoft word format) will be ignored and considered as null. You are strongly encouraged to type your solutions using a text editor. To this end, we suggest the following options:

1. Microsoft word and convert your document to PDF
2. the typesetting system \LaTeX; see https://www.latex-project.org/ and https://en.wikipedia.org/wiki/LaTeX#Example to learn about \LaTeX; see https://www.tug.org/begin.html to get started
3. using a software tool for typing mathematical symbols, for instance http://math.typeit.org/
4. using a Handwriting recognition system such as those equipping tablet PCs

Hand-writing and scanning your answers is allowed for first assignment only.

1. if you go this route please use a scanning printer and do not take a picture of your answers with your phone,
2. if the quality of the obtained PDF is too poor, your submission will be ignored and considered as null.

Problem 1 (Proving theorems!) [30 marks] For each of the following statements, translate it into predicate logic and prove it, if the statement is true, or disprove it, otherwise:

1. for any positive integer, there exists a second positive the square of which is equal to the first integer,
2. for any positive integer, there exists a second positive integer which is greater or equal to the square of the the first integer,
3. for any positive integer, there exists a second positive which is greater or equal to the square of the the first integer, but smaller than the cube of the the first integer.

Solution 1
1. Using predicate logic, the proposition writes:

\[(\forall n > 0)(\exists m > 0) \ m^2 = n,\]

where the domain of discourse is the set of the integers. This proposition is false. Indeed, its negation, namely

\[(\exists n > 0)(\forall m > 0) \ m^2 \neq n,\]

is true. For instance, the positive integer \( m = 3 \) is not the square of another (positive) integer.

2. Using predicate logic, the proposition writes:

\[(\forall n > 0)(\exists m > 0) \ m \geq n^2,\]

where the domain of discourse is the set of the integers. This proposition is true. Indeed, consider an arbitrary positive integer \( n \). We have: \( n^2 + 1 \geq n^2 \), therefore we can choose \( m = n^2 + 1 \), which proves the proposition.

3. Using predicate logic, the proposition writes:

\[(\forall n > 0)(\exists m > 0) \ (n^2 \leq m < n^3).\]

This proposition is false. Indeed, consider \( n = 1 \). In this special case, we have \( n^2 = n^3 \). Hence, we cannot find a positive integer \( m \) such that we have \( n^2 \geq m < n^3 \).

**Problem 2 (Who is who?) [20 marks]** In this next episode of Star Trek, Captain James T. Kirk and the crew of the Starship Enterprise lands on a strange new world. This planet is indeed inhabited by three types of strange creatures:

- the Vulcans, who always tell the truth,
- the Gorns, who always lie, and
- the Tholians, who can lie or tell the truth at will.

When landing on this planet, Captain Kirk is approached by three creatures wearing different colored spacesuits, one in blue, one in red and one in green. Captain Kirk knew that he would be welcome by one individual of each type, but because of their spacesuits, he cannot tell who is from which type. The creatures speak in the following order:

1. The creature wearing blue says, "I am a Vulcan."
2. The creature wearing red says, "Blue speaks the truth."
3. The creature wearing green says, "Blue is a Tholian."
Who is a Vulcan, who is a Gorn, who is a Tholian? Justify your answer.

**Solution 2** Among the 3 creatures (namely Blue, Red, Green) one is a Vulcan, one is a Gorn and one is a Tholian. There are six possible cases:
- VGT: Blue is Vulcan, Red is Gorn (and thus Green is Tholian),
- VTG: Blue is Vulcan, Red is Tholian (and thus Green is Gorn),
- GVT: Blue is Gorn, Red is Vulcan (and thus Green is Tholian),
- GTV: Blue is Gorn, Red is Tholian (and thus Green is Vulcan),
- TGV: Blue is Tholian, Red is Gorn (and thus Green is Vulcan),
- TVG: Blue is Tholian, Red is Vulcan (and thus Green is Gorn).

We denote by $B$, $R$, and $G$ the facts that “Blue tells the truth”, “Red tells the truth” and “Green tells the truth”, respectively. We make a few observations:
- if $B$ holds then $R$ must hold as well (otherwise, there would be a contradiction) while $G$ must be false (indeed, there is at least one liar); this is realized with VTG and only with this case.
- if $B$ does not hold, then $R$ must be false too (otherwise, there would be a contradiction) while $G$ must be true (indeed, there is at least one correct claim); this is realized with GTV and TGV.

Poor Capitain Kirk. He still does not know who is how. Very embarrassing.

**Problem 3 (Deciding consistency)** [20 marks] In the previous episode of *Star Trek*, the *Starship Enterprise* was destroyed and a new starship has been designed by Spock by Captain Kirk. Is this set of requirements consistent?

1. The starship is flying at light speed if and only if it is not operating normally.
2. If the starship is not flying at light speed, then its shield is active.
3. If the system ( = starship) is operating normally, then the nuclear core is functioning.
4. The nuclear core is functioning or the shield is active.
5. If the starship is flying at light speed, then the nuclear core is functioning.

Justify your answer.

**Solution 3** We define the following Boolean variables:
- $L$: the starship is flying at light speed,
- $O$: the starship is operating normally,
- $S$: the shield is active,
- $N$: the nuclear core is functioning.
Then, the above 5 requirements translate respectively as follows:
1. \( L \leftrightarrow \neg O \),
2. \( \neg L \to S \),
3. \( O \to N \),
4. \( N \lor S \),
5. \( L \to N \).

We rewrite the above 5 propositions so as to replace the connectives \( \leftrightarrow \) and \( \to \) with \( \lor \) and \( \neg \). This leads to:
1. \( (\neg L \lor \neg O) \land (L \lor O) \),
2. \( L \lor S \),
3. \( \neg O \lor N \),
4. \( N \lor S \),
5. \( \neg L \lor N \).

It follows that we should decide whether the following formula is satisfiable:
\[
(\neg L \lor \neg O) \land (L \lor O) \land (L \lor N) \land (\neg O \lor N) \land (N \lor S) \land (\neg L \lor N).
\]

Remember that each of the above parenthesized expressions is called a clause. We proceed as in class:
1. assigning a Boolean value (true or false) to one of the variables \( L \), \( O \), \( S \), \( N \)
2. trying to deduce assignments for the other variables, unless a contradiction is reached.

We try \( L \) = false, which satisfies the first clause. It follows (from the second clause) that \( O \) = true. Then, with the fourth clause, we deduce \( N \) = true. We also deduce with the third clause that \( S \) = true must hold. Finally, we observe that with these choices the fifth and the sixth clauses are satisfied. Therefore the 5 requirements are consistent. Good job, Captain Kirk!

**Problem 4 (Deciding satisfiability) [20 marks]** Let \( p, q, r, t \) be three Boolean variables. For each of the following propositional formulas determine whether it is satisfiable or not. Justify your answer.

1. \( (\neg q \land \neg p \land \neg r) \lor (q \land \neg p) \lor (\neg q \land \neg r) \)
2. \( (\neg q \land \neg p \land \neg r \land t) \lor (q \land \neg p \land \neg r \land \neg t) \lor (\neg q \land p \land \neg r \land \neg t) \lor (\neg q \land \neg p \land r \land \neg t) \).

**Solution 4** Each of the propositional formulas is a disjunction of conjunctions. Hence, in order to satisfy each of these two propositional formulas, it is sufficient to satisfy one of its conjunctions. For the first one, we can pick \( \neg q \land \neg p \land \neg r \) which is true as soon as \( q, p, r \) are all false. For the second one, we can pick \( \neg q \land \neg p \land \neg r \land t \) which is true as soon as \( q, p, r, t \) are false, false, false, true.