Format of the submission. You must submit a single file which must be in PDF format. All other formats (text or microsft word format) will be ignored and considered as null. You are required to type your solutions using a text editor. To this end, we suggest the following options:

1. Microsoft word and convert your document to PDF
2. the typesetting system LaTeX; see https://www.latex-project.org/ and https://en.wikipedia.org/wiki/LaTeX#Example to learn about LaTeX; see https://www.tug.org/begin.html to get started
3. using a software tool for typing mathematical symbols, for instance http://math.typeit.org/
4. using a Handwriting recognition system such as those equipping tablet PCs

Hand-writing and scanning your answers is not allowed for this assignment and the subsequent ones.

Problem 1 (Pigeonhole principle and combinatorial proofs) [25 marks]

1. Let $S$ be a subset of $\mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers) such that $S$ has at least 3 elements. Prove that there exist at least two elements $s, y$ of $S$ so that $x + y$ is even.

2. Let $S$ be a subset of $\mathbb{N} \times \mathbb{N}$ such that $S$ has at least 5 elements. Prove that there exist at least two elements $(x_1, x_2)$ and $(y_1, y_2)$ in $S$ so that $x_1 + y_1$ and $x_2 + y_2$ are both even.

3. Prove that, in the previous question, one can not replace that 5 by 4 while preserving the same conclusion.

Solution 1

1. The sum of two integers of the same parity gives an even integer. We classify the elements of $S$ into two classes even and odd integers. Since there are at least three elements in $S$ one of the two classes contains at least two elements. There are therefore in $S$ two integers of the same parity.

2. We reproduce the same reasoning but this time we make 4 classes.
   (a) PP couples whose two components are even
(b) PI the couples whose first component is even and the second component is odd.
(c) IP the couples whose first component is odd and the second component is even.
(d) II couples whose two components are odd.

As soon as we have 5 elements, we are sure that there is at least one class that contains two elements. Now the sum of two elements of a class gives a couple whose two components are even. This confirms that from 5 elements we are sure of the existence of the two desired couples.

3. considering the part $S = (0, 0), (0, 1), (1, 0), (1, 1)$ we see that 4 couples do not allow to ensure existence.

**Problem 2 (Conditional probability)**  [25 marks] Prove that, for any three events $A, B, C$, from some sample space, each event having positive probability, we have

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|(A \cap B)).$$

**Solution 2** By definition of a conditional probability, we have:

$$P(C|(A \cap B)) = \frac{P(A \cap B \cap C)}{P(A \cap B)},$$

which gives:

$$P(A \cap B \cap C) = P(A \cap B)P(C|(A \cap B)).$$

By definition again of a conditional probability, we have:

$$P(B|A) = \frac{P(A \cap B)}{P(A)},$$

that is,

$$\frac{1}{P(A \cap B)} = P(B|A)P(A).$$

From where we derive:

$$P(A \cap B \cap C) = P(B|A)P(A)P(C|(A \cap B)).$$

**Problem 3 (Conditional probability)**  [25 marks] In London, half of the days have some rain. The weather forecaster is correct $2/3$ of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted that it won’t rain, are both equal to $2/3$. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability $1/3$. Find
1. the probability that Pickwick has no umbrella, given that it rains,
2. the probability that it doesn’t rain, given that he brings his umbrella.

Solution 3  

Problem 4 (Bernoulli Trials)  [25 marks] A multiple-choice exam consists of 20 questions. Each question offers 4 choices, one and only one of them being correct. The exam is passed if at least 16 questions are answered correctly. What is the probability that someone answering randomly passes the exam.

Solution 4

Let $P(k)$ the probability of answering correctly exactly $k$ questions out of 16. We have

$$P(k) = \binom{20}{k}(1/16)^k(15/16)^{20-k}$$

We are asled to compute $P(16) + P(17) + P(18) + P(19) + P(20)$

In Maple, we obtain the following

```maple
> sum(binomial(20,k) * (1/16)^k * (15/16)^(20-k), k=16..20);
62292169
------------------------
302231454903657293676544

> evalf(%);
-15
0.2061074980 10
```