Problem 1 Show that if the sets $A$, $B$ and $C$ satisfy the following relations simultaneously:

1. $A \cup B = C$,
2. $(A \cup C) \cap B = C$,
3. $(A \cap C) \cup B = A$

then they are equal.

Solution 1 The first hypothesis, namely $A \cup B = C$ implies that both $A$ and $B$ are contained in $C$, that is

$A \subseteq C$ and $B \subseteq C$.

Now looking at the third hypothesis, namely $(A \cap C) \cup B = A$, we see that by applying distributivity of $\cup$ over $\cap$ we obtain:

$$(A \cup B) \cap (C \cup B) = A.$$ 

Using the first hypothesis, we deduce:

$$C \cap (C \cup B) = A.$$ 

Now using distributivity of $\cap$ over $\cup$, we further deduce:

$$C \cup (C \cap B) = A.$$ 

Since $C \cap B$ is contained in $C$, we have $C \cup (C \cap B) = C$ and we derive this equality:

$$C = A.$$ 

It remains to be proved that $A = B$ holds. Using $C = A$ within the the second hypothesis, namely $(A \cup C) \cap B = C$, we obtain:

$$(A \cup A) \cap B = A,$$

that is,

$$A \cap B = A.$$ 

This latter equality implies that every element of $A$ is in the intersection of $A$ and $B$, thus, every element of $A$ is in $B$. In other words, we have $A \subseteq B$. Now, remember that we saw $B \subseteq C$ and $C = A$, which imply $B \subseteq A$. Finally, we have $B = A$ and the three sets are indeed equal.
Problem 2 Prove the following set identities:
1. \((B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A,\)
2. \((A \setminus B) \setminus (B \setminus C) = A \setminus B.\)

Solution 2 Recall that \(B \setminus A\) means \(B \cap \overline{A}\).
1. We prove that \((B \setminus A) \cup (C \setminus A)\) can be rewritten as \((B \cup C) \setminus A\). First, we use the fact \(B \setminus A = B \cap \overline{A}\) and \(C \setminus A = C \cap \overline{A}\) hold. Hence, we have:
\[
(B \setminus A) \cup (C \setminus A) = (B \cap \overline{A}) \cup (C \cap \overline{A})
\]
Next, we apply distributivity of \(\cup\) over \(\cap\), leading to:
\[
(B \setminus A) \cup (C \setminus A) = (B \cup C) \cap \overline{A},
\]
from which we finally derive
\[
(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A.
\]
2. We have the following chain of equalities:
\[
(A \setminus B) \setminus (B \setminus C) = (A \setminus B) \cap (B \cup C) \setminus (B \cap C)
\]
\[
= (A \setminus B) \cap (B \cup C) \cap \overline{B} \cup (A \setminus B) \cap (B \cap C)
\]
\[
= (A \setminus B) \cap (A \setminus B) \cup (A \setminus B) \cap (B \cap C)
\]
\[
= A \setminus B.
\]

Problem 3 In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

Solution 3 Denote by \(D, M, L\) the sets of students who have eaten durians, mangoes and litchis. The key observation is that the sets
\[
D \cap M \cap L, D \cap M \cap \overline{L}, D \cap M \cap L, D \cap M \cap \overline{L}, D \cap M \cap L, D \cap M \cap \overline{L}, \overline{M} \cap L, \overline{D} \cap \overline{M} \cap \overline{L}
\]
are pairwise disjoint. In fact, they form a partition of \(D \cup M \cup L\). From the hypotheses and this observation, we have
1. \(|D \cap M \cap L| = 5,\)
2. \(|D \cap M \cap \overline{L}| = 34 - 5 = 29,\)
3. \( |D \cap M \cap L| = 16 - 5 = 11, \)
4. \( |D \cap M \cap L| = 88 - (29 + 11 + 5) = 43, \)
5. \( |\overline{D} \cap M \cap L| = 12 - 5 = 7, \)
6. \( |\overline{D} \cap M \cap L| = 73 - (7 + 29 + 5) = 32, \)
7. \( |\overline{D} \cap M \cap L| = 46 - (11 + 7 + 5) = 23, \)
8. \( |\overline{D} \cap M \cap L| = 200 - (5 + 29 + 11 + 43 + 7 + 32 + 23) = 50. \)

**Problem 4** Prove that for the sets \( A, B, C, D, \) we have:

\[(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).\]

Does equality hold?