Problem 1  
1. Find all integers $x$ such that $0 \leq x < 15$ and $4x + 9 \equiv 13 \mod 15$. Justify your answer.
2. Find all integers $x$ and $y$ such that $0 \leq x < 15$, $0 \leq y < 15$, $x + 2y \equiv 4 \mod 15$ and $3x - y \equiv 10 \mod 15$. Justify your answer.

Solution 1  
1. We have $4 \times 4 \equiv 1 \mod 15$. That is, 4 is the inverse of 4 modulo 15. We multiply by 4 each side of:

$$4x + 9 \equiv 13 \mod 15,$$

leading to:

$$x + 4 \times 9 \equiv 4 \times 13 \mod 15,$$

that is:

$$x \equiv 4(13 - 9) \mod 15,$$

which finally yields: $x \equiv 1 \mod 15$.

2. We eliminate $y$ in order to solve for $x$ first. Multiplying $3x - y \equiv 10 \mod 15$ by 2 yields $6x - 2y \equiv 5 \mod 15$. Adding this equation side-by-side with $x + 2y \equiv 4 \mod 15$ yields $7x \equiv 9 \mod 15$. Since $7 \times 13 \equiv 1 \mod 15$, we have $x \equiv 9 \times 13 \mod 15$, that is, $x \equiv 12 \mod 15$. Substituting $x$ with 12 into $3x - y \equiv 10 \mod 15$ yields $y \equiv 11 \mod 15$.

Problem 2 Let $a, b, q, r$ be non-negative integer numbers such that $b \neq 0$ and we have

$$a \equiv \frac{b}{q} \mod r \quad (1)$$

Thus we have: $a = bq + r$ as well as $0 \leq r < b$. Prove that we have:

$$q = \left\lfloor \frac{a}{b} \right\rfloor. \quad (2)$$

Solution 2 From $a = bq + r$ and $0 \leq r < b$ we derive

$$bq \leq bq + r < b(q + 1), \quad (3)$$
thus

\[ qa \leq a < b(q + 1), \]  

(4)

that is

\[ q \leq a/b < q + 1, \]

(5)

which means:

\[ q = \lfloor \frac{a}{b} \rfloor. \]

(6)

**Problem 3** Let \( a, b, q_1, r_1, q_2, r_2 \) be non-negative integer numbers such that \( b \neq 0 \) and we have

\[
\begin{array}{c|c}
 a & b \\
 q_1 & r_1 \\
 q_2 & r_2 \\
\end{array}
\]

(7)

Thus we have: \( a = bq_1 + r_1 = bq_2 + r_2 \) as well as \( 0 \leq r_1 < b \) and \( 0 \leq r_2 < b \). Prove that \( q_1 = q_2 \) and \( r_1 = r_2 \) necessarily both hold.

**Solution 3** Let \( a = bq_1 + r_1 = bq_2 + r_2 \), with \( 0 \leq r_1 < b \) and \( 0 \leq r_2 < b \), where \( a, b, q_1, r_1, q_2, r_2 \) are non-negative integers. We wish to show that \( q_1 = q_2 \) and \( r_1 = r_2 \).

Assume that \( r_1 \neq r_2 \). Then, without loss of generality, assume that \( r_2 > r_1 \). We then have that

\[ bq_1 - bq_2 = r_2 - r_1 \]

\[ \Rightarrow b(q_1 - q_2) = r_2 - r_1 \]

(8)

Since \( 0 \leq r_1 < b \) and \( 0 \leq r_2 < b \), and \( r_2 > r_1 \), it must be that

\[ 0 < (r_2 - r_1) < b, \]

(9)

since the largest difference has \( r_2 = b - 1 \) and \( r_1 = 0 \), and \( r_1 \neq r_2 \) by assumption (so \( r_2 - r_1 \neq 0 \)). But equation (8) implies that \( b \) divides \( r_2 - r_1 \), which cannot be given equation (9), because the multiples of \( b \) are \( 0, \pm b, \pm 2b, \ldots \). This is a contradiction, and we conclude that \( r_1 = r_2 \).

Since we have shown that \( r = r_1 = r_2 \), it follows that

\[ bq_1 - bq_2 = r - r \]

\[ \Rightarrow b(q_1 - q_2) = 0 \]

(10)

But equation (10) implies either that \( b = 0 \) or \( q_1 - q_2 = 0 \). Since \( b \neq 0 \) by the assumptions of the division theorem, we conclude that it must be that \( q_1 - q_2 = 0 \), meaning that \( q_1 = q_2 \), which is what we set out to prove. QED
**Problem 4** In the previous exercise, if $a, b, q_1, q_2$, are non-negative integer numbers satisfying $a = bq_1 + r_1 = bq_2 + r_2$ while $r_1, r_2$ are integers satisfying $-b < r_1 < b$ and $-b < r_2 < b$. Do we still reach the same conclusion? Justify your answer.

**Solution 4** No, we do not. Indeed, with $a = 7$ and $b = 3$, we then have two possible divisions:

\[
\begin{array}{c|c}
7 & 3 \\
1 & 3 \\
\hline
2 & 3 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
7 & 3 \\
-2 & 3 \\
\hline
& 3 \\
\end{array}
\]