Problem 1 Let $a, b, c$ be integers, where $a \neq 0$ and $b \neq 0$. Prove that if $a | b$ and $b | c$ both hold, then $a | c$ holds as well.

Solution 1 If $a | b$ and $b | c$ both hold, then there exist integers $d, e$ such that we have $b = d \cdot a$ and $c = e \cdot b$. Hence we have $c = e \cdot d \cdot a$, that is $a | c$ holds as well.

Problem 2 Let $a$ and $b$ be integers, and let $m$ be a positive integer. Let $r_a$ and $q_a$ be the remainder and the quotient in the integer division of $a$ by $m$. Similarly, let $r_b$ and $q_b$ be the remainder and the quotient in the integer division of $b$ by $m$.

1. Prove that the following two propositions are equivalent:
   (a) $a \equiv b \pmod{m}$,
   (b) $r_a = r_b$.

2. Let $r_s$ and $q_s$ be the remainder and the quotient in the integer division of $a + b$ by $m$. Prove that we have

$$r_s \equiv (r_a + r_b) \pmod{m}$$

3. Let $r_p$ and $q_p$ be the remainder and the quotient in the integer division of $ab$ by $m$. Prove that we have

$$r_p \equiv (r_a r_b) \pmod{m}$$

Solution 2

1. By definition of $q_a, r_a$, we have $a = q_a m + r_a$ and $0 \leq r_a < m$. Similarly, by definition of $q_b, r_b$, we have $b = q_b m + r_b$ and $0 \leq r_b < m$. Therefore,

$$a - b = (q_a - q_b) m + r_a - r_b. \quad (1)$$

It follows that $r_a = r_b$ implies that $a - b$ is a multiple of $m$. Now recall that $a \equiv b \pmod{m}$ means that $a - b$ is a multiple of $m$. Similarly, $r_a = r_b$. Therefore $r_a = r_b$ implies $a \equiv b \pmod{m}$. Conversely, if $a \equiv b \pmod{m}$, we deduce with Equation (1) that is,

$$r_a - r_b \equiv 0 \pmod{m}. \quad (2)$$

Since $0 \leq r_a < m$ and $0 \leq r_b < m$ both hold, we have $-m < r_a - r_b < m$. Finally, we have $r_a = r_b$, thanks to Equation (2).
2.

Problem 3  Find $s, t, \gcd(a, b)$ such that $s\, a + t\, b = \gcd(a, b)$ in the following cases:
1. $a = 2$ and $b = 3$,
2. $a = 11$ and $b = 12$,
3. $a = 12$ and $b = 15$,
4. $a = 3$ and $b = 7$.

Solution 3
1. $-1 \times 2 + 1 \times 3 = 1 = \gcd(a, b)$,
2. $-1 \times 11 + 1 \times 12 = 1 = \gcd(a, b)$,
3. $-1 \times 12 + 1 \times 15 = 3 = \gcd(a, b)$,
4. $-2 \times 3 + 1 \times 7 = 1 = \gcd(a, b)$.

Problem 4
1. Find all integers $x$ such that $0 \leq x < 21$ and $4x + 9 \equiv 13 \mod 21$. Justify your answer.
2. Find all integers $x$ and $y$ such that $0 \leq x < 21$, $0 \leq y < 21$, $x + 2y \equiv 4 \mod 21$ and $3x - y \equiv 10 \mod 21$. Justify your answer.

Solution 4
1. We have $4 \times 5 \equiv -1 \mod 21$. Thus, we have $4 \times 16 \equiv 1 \mod 21$, since $5 \equiv -16 \mod 21$. That is, 16 is the inverse of 4 modulo 21. We multiply by 16 each side of:

$$4x + 9 \equiv 13 \mod 21,$$

leading to:

$$x + 9 \times 16 \equiv 16 \times 13 \mod 21,$$

that is:

$$x \equiv 16(13 - 9) \mod 21,$$

which finally yields: $x \equiv 1 \mod 21$.
2. We eliminate $y$ in order to solve for $x$ first. Multiplying $3x - y \equiv 10 \mod 21$ by 2 yields $6x - 2y \equiv 20 \mod 21$. Adding this equation side-by-side with $x + 2y \equiv 4 \mod 21$ yields $7x \equiv 3 \mod 21$. Since $3 \times 7 \equiv 0 \mod 21$, we have $0x \equiv 9 \mod 21$, which is false. Therefore, the input problem has no solutions for $x$ and consequently no solutions for $y$. 

2