You are encouraged to type your answers to make it easier for the TAs to read your work. If you submit handwritten assignments, please try to write clearly. You must submit through OWL a file in PDF format with your answers to the following questions.

1. The following figure shows a flow network on which a flow $f$ has been computed. The value of the flow through an edge appears in the dotted arrow beside the edge. Edges without a dotted arrow carry zero flow.

   - (10 marks) Is this a maximum flow? If so, prove it. If not, give a maximum flow.
   - (10 marks) Find a minimum cut separating $s$ from $t$.

   ![Flow network diagram]

2. Let $G = (V,E)$ be a flow network with source $s$, sink $t$, and integer capacities. As always we denote with $n$ the number of vertices of $G$ and with $m$ the number of edges. A minimum cut for $G$ is a cut with minimum capacity. $G$ might have several minimum cuts, some of them having more edges than others. For example for the following flow network the following are some of its minimum cuts: $(\{s\}, \{1, 2, 3, 4, 5, 6, t\})$, $(\{s, 1\}, \{2, 3, 4, 5, 6, t\})$, $(\{s, 1, 2\}, \{3, 4, 5, 6, t\})$, $(\{s, 1, 2, 3, 4, 5, 6\}, \{t\})$. All these cuts have capacity 4, but the first cut has 2 edges, the second one has 3 edges, the third one has 4 edges, and the last one has 1 edge. A maximal minimum cut is a minimum capacity cut with the largest number of edges. For the following flow network a maximal minimum cut is $(\{s, 1, 2\}, \{3, 4, 5, 6, t\})$.

   ![Flow network diagram]

   - (10 marks) Write a polynomial time algorithm for computing a maximal minimum cut of a given flow network $G$. You can assume that all capacities are integers multiples of $m$.

   **Note.** Since the time complexity of the algorithm must be polynomial do not try to list all the minimum cuts to find one with maximum number of edges, as such an algorithm would need exponential time in the worst case.
• (15 marks) Prove that your algorithm is correct.

• (5 marks) Compute the time complexity of your algorithm in the worst case. The running time must be polynomial in the size of the input.

3. A set \( B = \{B_1, B_2, \ldots, B_r\} \) of object needs to be transported from some city \( X \) to another city \( Y \). In order to ensure that the objects are not damaged during the transportation, each object needs to be placed inside a container.

There is a set \( C = \{C_1, C_2, \ldots, C_s\} \) of containers; each container can store only one object. The container where an object \( B_i \) is placed must have specialized straps to secure the object and climate control to provide certain temperature and humidity level. Hence, for each object \( B_i \) there is only a subset of containers \( S_i \subseteq C \) that can store it.

Each container \( C_i \in C \) has size \( d_i \). Several containers can have the same size. There is also a set \( T \) of trucks that will transport the containers. Each truck has compartments of fixed dimensions, so a truck can only transport containers of one size. A truck \( T_j \) can transport up to \( t_{jd_i} \) containers of size \( d_i \), but it cannot transport containers of any other size.

The problem is to determine whether all the objects \( B \) can be stored in the containers \( C \) and transported by the trucks \( T \) to their destination subject to the above constraints.

For example consider the sets of objects \( B = \{B_1, B_2, B_3, B_4\} \) and containers \( C = \{C_1, C_2, C_3, C_4, C_5\} \). Object \( B_1 \) can be stored only in \( C_1 \) or \( C_4 \), so \( S_1 = \{C_1, C_4\} \); the other sets \( S_i \) are as follows \( S_2 = \{C_1, C_4\} \), \( S_3 = \{C_1, C_3\} \), \( S_4 = \{C_2, C_3, C_5\} \). Containers \( C_1 \) and \( C_4 \) have size 3 and containers \( C_2, C_3, C_5 \) have size 2.

There are 2 trucks \( T = \{T_1, T_2\} \), where truck \( T_1 \) can transport 2 containers of size 3, i.e., \( t_{13} = 2 \), and truck \( T_2 \) can transport 2 containers of size 2, i.e., \( t_{22} = 3 \). This problem has a solution: store \( B_1 \) in \( C_1 \) and truck \( T_1 \), store \( B_2 \) in \( C_4 \) and truck \( T_1 \), store \( B_3 \) in \( C_3 \) and truck \( T_2 \), and store \( B_4 \) in \( C_2 \) and truck \( T_2 \). Note that this solution is not unique.

However, if truck \( T_1 \) can only transport one container of capacity 3, i.e., \( t_{13} = 1 \) and truck \( T_2 \) can transport 3 containers of size 2, then there is no solution for the problem.

• (20 marks) Write an algorithm for solving this problem. If there is a way of storing the objects in the containers and the containers in the trucks, the algorithm must return the value \( true \), otherwise it must return the value \( false \).

• (25 marks) Prove that your algorithm correctly solves the above problem.

• (5 marks) Compute the worst case time complexity of your algorithm.