1. The following figure shows a flow network on which a flow $f$ has been computed. The value of the flow through an edge appears in the dotted arrow beside the edge. Edges without a dotted arrow carry zero flow.

- (10 marks) Is this a maximum flow? If so, prove it. If not, give a maximum flow.
- (5 marks) Find a minimum cut separating $s$ from $t$.

2. (10 marks) Consider the following two claims.

Let $G = (V, E)$ be a flow network with source $s$, sink $t$ and positive integer edge capacities. Let $f$ be a maximum flow for $G$.

- (10 marks) **Claim 1** For any flow network $G = (V, E)$ with integer capacities, there is at least one edge $(u, v) \in E$ such that if the capacity of $(u, v)$ is increased by 1, the value of the maximum flow increases by 1.
- (10 marks) **Claim 2** For any flow network $G = (V, E)$ with integer capacities, there is at least one edge $(u, v) \in E$ such that if the capacity of $(u, v)$ is decreased by 1, the value of the maximum flow decreases by 1.

Determine whether the above claims are true or false. If a claim is true, give a proof for it. If a claim is false, give an example proving the claim wrong.

(5 marks) **Optional question:** Write an algorithm for finding an edge $(u, v)$ as described in Claim 1, if such an edge exists. Assume that a maximum flow $f$ for $G$ is known. The algorithm must run in $O(m + n)$ time, where $n$ is the number of vertices and $m$ is the number of edges in $G$.

3. Let $G = (V, E)$ be a flow network with source $s$, sink $t$, and integer capacities. As always, let $m$ be the number of edges and $n$ be the number of vertices of $G$. A *basal minimum cut* is a minimum capacity cut with the smallest number of edges. For example, for the following flow network there are two minimum capacity cuts: $(\{s\}, \{1, 2, 3, t\})$ and $(\{s, 1, 2, 3\}, \{t\})$. Both cuts have capacity 4, but the first cut has 2 edges while the second has 1 edge; hence the second cut is a basal minimum cut.
• (10 marks) Write an efficient algorithm for computing a basal minimum cut of a given flow network \( G \). Assume that all capacities are integers multiples of \( m \).

• (10 marks) Prove that your algorithm is correct.

• (5 marks) Compute the time complexity of your algorithm. The running time must be polynomial in the size of the input.

4. Consider a group \( C = \{C_1, C_2, \ldots, C_{n_C}\} \) of companies. Each company \( C_i \) owns a set \( T_i \) of retail stores; sets \( T_i \) are disjoint. The total set of stores is \( S = \{S_1, S_2, \ldots, S_{n_S}\} \), where \( n_S = \sum_{i=1}^{n_C} |T_i| \). A group of warehouses \( W = \{W_1, W_2, \ldots, W_{n_W}\} \) can distribute computers to the stores. Warehouse \( W_i \) has \( q_i \) computers that it can distribute to the stores and a store \( S_i \) can receive up to \( r_i \) computers from warehouses. A warehouse \( W_i \) can only distribute computers to a set \( V_i \subseteq S \) of stores.

Each company \( C_i \) has a total demand of \( d_i \) computers, so the total number of computers that the warehouses must distribute to the stores owned by \( C_i \) must be exactly equal to \( d_i \). The problem is to determine whether the warehouses can distribute computers to the stores to satisfy the demands of the companies without exceeding the capacities of the stores.

For example, suppose that \( C = \{C_1, C_2\} \), \( T_1 = \{S_1, S_2\} \), \( T_2 = \{S_3, S_4, S_5\} \), \( W = \{W_1, W_2\} \), \( V_1 = \{S_1, S_2, S_3, S_5\} \), and \( V_2 = \{S_2, S_3, S_4\} \). Let \( d_1 = 6 \), \( d_2 = 10 \), \( q_1 = 8 \), \( q_2 = 9 \), \( r_1 = 3 \), \( r_2 = 4 \), \( r_3 = 5 \), \( r_4 = 5 \), and \( r_5 = 2 \). A solution is this: \( W_1 \) sends 3 computers to \( S_1 \), 3 to \( S_2 \) and 2 to \( S_5 \), while \( W_2 \) sends 5 computers to \( S_3 \) and 3 to \( S_4 \). Note that the stores owned by \( C_1 \) receive in total 6 computers and the stores owned by \( C_2 \) receive 10 computers, as required.

• (16 marks) Write an algorithm for solving this problem. If there is a way of delivering computers to stores to satisfy the demands of the companies, the algorithm must return the value true, otherwise it must return the value false.

• (16 marks) Prove that your algorithm correctly solves the above problem.

• (8 marks) Compute the time complexity of your algorithm.

(5 marks) Optional question: If the above problem is too easy for you, here is a slightly more challenging one. Assume that the computers are colored either pink or blue. Each warehouse \( W_i \) stores \( q_{pi} \) pink computers and \( q_{bi} \) blue ones. Furthermore, each store \( S_i \) has a limit \( p_i \) on the maximum number of pink computers that it can receive. The problem is to determine whether the warehouses can distribute computers to stores so as to satisfy the demands of the companies subject to these additional constraints.