1. A group $L$ of $\ell$ lecturers in some university need to teach a set $C = \{c_1, c_2, \ldots, c_n\}$ of courses. Each course is taught once per day. For each course $c_i$ we are given its starting time $t_i$ and its duration $d_i$. Two different courses can be taught by the same lecturer if they do not overlap. We want to determine whether all the courses can be taught by the $\ell$ lecturers.

   - (20 marks) Give in pseudocode an algorithm for solving this problem. If all the courses can be taught by the lecturers, the algorithm must return the value true; otherwise it must return the value false.
   - (15 marks) Prove that your algorithm correctly solves the problem. You need to show that the algorithm terminates and that it computes the correct output.
   - (5 marks) Compute the time complexity of your algorithm.

2. A set $F = \{f_1, f_2, \ldots, f_n\}$ of files with integer sizes $s_1, s_2, \ldots, s_n$ needs to be stored in a hard disk of capacity $K$. We wish to find a subset of files of maximum total size but not larger than $K$ to be stored in the disk. For example, if we have 4 files with sizes 3, 5, 8, and 6 and $K = 15$, an optimum solution is to store in the hard drive the files of size 6 and 8. The above problem is NP-hard.

   Consider the following algorithm for the problem. The following two questions refer to this algorithm.

   **Algorithm** $\text{files}(F, S, n, K)$

   **In:** Set $F$ of files, set $S = \{s_1, s_2, \ldots, s_n\}$ of $n$ file sizes, and hard disk capacity $K$

   **Out:** Set of files of total size at most $K$

   ```
   A \leftarrow \emptyset \\
   total \leftarrow 0 \\
   for i \leftarrow 1 \text{ to } n do \\
     \quad if total + s_i \leq K then \\
     \quad \quad Add file $f_i$ to $A$ \\
     \quad \quad total \leftarrow total + s_i \\
   return A
   ```

   (i) (10 marks) For the above algorithm the value $SOL$ of the solution computed by the algorithm is equal to the total size of the files in set $A$, i.e. $SOL = total$. Show that the approximation ratio, $OPT/SOL$, of this algorithm is arbitrarily large by giving an instance in which the total size of the files in the set $A$ returned by this algorithm is very small compared to the value of an optimum solution. Note that the files are not sorted in any particular manner.

   (ii) (20 marks) Assume now that the files are sorted in non-increasing order of size. Compute the approximation ratio $OPT/SOL$ of the algorithm.

3. In a graph an edge $(u, v)$ is specified by its two endpoints. A $3$-hypergraph $H = (V_H, E_H)$ consists of a set of vertices and a set of hyperedges, where a hyperedge is specified by
three endpoints \((u, w, v)\), not by two as in graphs. The minimum vertex cover problem on 3-hypergraphs is to find the smallest set \(S\) if vertices, such that every hyperedge has at least one endpoint in \(S\). For example, for the following 3-hypergraph with hyperedges \((1, 2, 3)\), \((1, 3, 4)\), \((3, 4, 5)\), and \((5, 6, 7)\) a minimum vertex cover is \(S = \{3, 5\}\).

The minimum vertex cover problem on 3-hypergraphs is NP-hard. Consider the following generalization of the algorithm that we studied in class for the minimum vertex cover problem.

**Algorithm** VertexCover\((H = (V_H, E_H))\)  
**Input:** 3-hypergraph \(H\)  
**Output:** Vertex cover of \(H\) of small size.

\[
S \leftarrow \emptyset \\
\text{while } E_H \neq \emptyset \text{ do } \\
\quad \text{Choose a hyperedge } (u, v, w) \in E_H \\
\quad S \leftarrow S \cup \{u, v, w\} \\
\quad \text{Remove from } E_H \text{ every hyperedge incident on } u, v, \text{ or } w \\
\}\n
**Output** \(S\)

- (20 marks) Compute the approximation ratio of this algorithm.
- (10 marks) Suppose that in every iteration of the **while** loop instead of adding to \(S\) the three vertices \(u, v,\) and \(w\) we only add \(u\) to \(S\). The idea, is that then we might be able to produce a smaller solution as we add only one vertex to \(S\), not three of them. Compute the approximation ratio of this modified algorithm.