1. Consider a directed graph $G = (V, E)$ representing a communication network where each node represents either a client computer, a server, or a packet switch, and each edge represents a communication link. Packet switches just re-route the information that they receive so clients and servers can communicate. Let $C$ be the set of clients and $S$ be the set of servers, where $C \cap S = \emptyset$. We wish to find a set $P$ of paths in $G$ that allow each client to connect to a server. Paths in $P$ must be edge disjoint to avoid interference. Furthermore, $P$ must connect a server to at most one client and each client must be connected to exactly one server. Paths in $P$ might go through nodes that represent clients or servers. For example, in the following graph the clients are light shaded and the servers are black. A solution $P$ for this instance of the problem consists of the paths drawn in bold.

- (10 marks) Design a polynomial time algorithm for solving the problem.
- (15 marks) Prove that your algorithm is correct.
- (5 marks) Compute the time complexity of the algorithm.
- (5 marks) Modify the above algorithm so that if $p$ is a path in $P$ from client $c_i$ to server $s_j$ then all intermediate nodes in $p$ represent packet switches.

2. A group of $k$ workers in a factory needs to perform a set $T = \{T_1, T_2, \ldots, T_n\}$ of tasks. Task $T_i$ has to be performed by one worker, it must be started at time $s_i$ and it requires time $p_i$ to be completed (so the task must be finished at time $s_i + p_i$). A worker cannot work on two tasks at the same time, but when a worker finishes a task she can work on another one. The setup time needed for a worked to go from task $T_i$ to task $T_j$ is $D_{ij}$; so if a worked completes task $T_i$ at time $s_i + p_i$ she cannot perform task $T_j$ if $s_j < s_i + p_i + D_{ij}$. We wish to determine whether the $k$ workers can perform all the tasks in $T$.

- (15 marks) Design a polynomial time algorithm for solving this problem. The algorithm must return $true$ if all the tasks can be completed by $k$ workers, and $false$ otherwise.
- (15 marks) Prove that your algorithm is correct.
- (5 marks) Compute the time complexity of the algorithm.

3. Consider the following approximation algorithm for the bin packing problem.
Algorithm LastFit(I,S)
Input: Set I of items and set S of item sizes; item I_j ∈ I has size S_j
Output: A packing of I into unit size bins
B ← ∅
for each item I_j ∈ I do {
    if I_j fits in one of the bins of B then
        Put I_j in the last bin where it fits
    else {
        Add a new bin b to B
        Put I_j in b
    }
}
Return B

• (10 points) Compute the approximation ratio of the above algorithm. You must explain how you computed the approximation ratio.

• (20 marks) Compute the approximation ratio of the First-Fit algorithm discussed in class for the case when each item has size at most 1/3. (Hint. The approximation ratio is smaller than 2.)