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Two-Dimensional Affine Transformations

Affine transformations of the plane in two dimensions include pure translations, scaling in a given direction, rotation, and shear. An affine transformation is usually and conveniently represented in matrix notation:

$$\begin{bmatrix} \vec{y} \\ 1 \end{bmatrix} = \begin{bmatrix} [A] & \vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

using homogeneous coordinates. The advantage of using homogeneous coordinates is that one can combine any number of affine transformations into one by multiplying the respective matrices. This property is used extensively in computer graphics, computer vision, and robotics.

Linear Two-Dimensional Transformations

Let's examine 2D transformations without the notion of homogeneous coordinates first. Also, let's consider the point $\vec{p}=(x, y)^T$ as that to be transformed. Translating a point (moving it somewhere else) is accomplished with a translation, and is effected in the following manner:

$$\vec{p}+\Delta\vec{p}=\vec{p}\,'$$

Fully worked out, this amounts to:

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x+\Delta x \\ y+\Delta y \end{bmatrix}$$

Scaling is also an affine transformation and can be described as an operation that performs a scalar multiplication on each component of a point:

$$S\vec{p}=\vec{p}\,'$$

and, fully worked out, amounts to:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

The ability to rotate points is also essential. To rotate a point about the origin, we use a rotation matrix:

$$R\vec{p} = \vec{p}\,'$$

which turns out to be:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

Homogeneous Coordinates

Homogeneous coordinates allow us to represent all these transformations with matrices that can be multiplied together. In this way, we can compose transformations in the order we choose to manipulate objects composed of 2D points. Translation as a matrix multiplication is expressed as:

$$\begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{bmatrix}$$

Scaling, in matrix form is:

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix}$$

Rotation, in matrix form is:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix}$$

In general, since matrix multiplication is not commutative, we must pay attention to the order in which we apply transformations. As an example of this, consider rotating point $\vec{p}$ around another point, say $\vec{c}$ instead of the origin. The correct way of performing this is to first translate the point to the origin, perform the rotation, and place the point back where it was to begin with:

$$T(\vec{c})R(\theta)T(-\vec{c})\vec{p}$$

Changing the order of the transformations yields incorrect results. We simply have to remember that $AB \neq BA$ for matrix multiplication.

Notable Two-Dimensional Transformations

Matrices can also perform reflections about an axis, and also shears, which are distortions in shape. To perform a reflection about the x axis, we apply the following matrix:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Conversely, a reflection about the y axis is performed by applying:

$$R_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Interestingly, the matrix

$$R_{xy} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

performs a reflection about the $x = y$ axis.

We can also apply transformations that distort points. These are called shears and are embodied by the following matrices:

$$S_x = \begin{bmatrix} 1 & s_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_y = \begin{bmatrix} 1 & 0 & 0 \\ s_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finding the Matrix for a Transformation

Consider $\vec{e}_1 = (1,0)^T$ and $\vec{e}_2 = (0,1)^T$, two unit vectors along the axes of the coordinate system. If we know where the transformation must send these two vectors, then we can find the transformation matrix.

Suppose we want to reflect an object about the vertical axis. We know that the transformation has the form

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

Now let's transform $\vec{e}_1$ and $\vec{e}_2$:

$$M\vec{e}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$M\vec{e}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Now, to reflect the object around the vertical axis, we observe that $M\vec{e}_1 = -\vec{e}_1$ and thus we can form the following system of equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

directly yielding $a=-1$ and $c=0$. We perform the same operations with $M\vec{e}_2=\vec{e}_2$ to find $b=0$ and $d=1$. The transformation matrix is thus

$$M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

In general, given independent unit vectors $\vec{u}_1$ and $\vec{u}_2$ we can find the transformation that sends them onto vectors $\vec{v}_1$ and $\vec{v}_2$. Let

$$M = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}$$

be the matrix that sends $\vec{e}_1$ and $\vec{e}_2$ onto $\vec{u}_1$ and $\vec{u}_2$. Therefore M^{-1} is the matrix that sends $\vec{u}_1$ and $\vec{u}_2$ onto $\vec{e}_1$ and $\vec{e}_2$. Let

$$N = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

be the transformation that sends $\vec{e}_1$ and $\vec{e}_2$ onto $\vec{v}_1$ and $\vec{v}_2$. It is now clear that the correct transformation matrix is given by NM^{-1}.

Transformations and Coordinate Systems

Suppose two coordinate systems sharing their origin but oriented differently. How can we transform points from one system to the other? Let

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\vec{u}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} \frac{-4}{5} \\ \frac{3}{5} \end{bmatrix}$$

be the orthogonal unit vectors representing the two coordinate systems. Let

$$\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

be a vector expressed in the $(\vec{e}_{1,}\vec{e}_2)$ coordinate system. Then the matrix

$$\begin{bmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \end{bmatrix}$$

transforms the point $\vec{v}$ into the $(\vec{u}_{1,}\vec{u}_2)$ coordinate system:

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Note that this transformation matrix is a rotation and in general the inverse of a rotation matrix is its transpose $M^{-1} = M^T$.

Reflections

To reflect a vector about a line that goes through the origin, let

$$\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

be a vector in the direction of the line. The matrix performing this reflection is then

$$M = \frac{1}{\|\vec{v}\|} \begin{bmatrix} v_x^2 - v_y^2 & 2\, v_x v_y \\ 2\, v_x v_y & v_y^2 - v_x^2 \end{bmatrix}$$

Orthogonal Projections

To project a vector orthogonally onto a line that goes through the origin, let

$$\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

be a vector in the direction of the line and form the following transformation matrix:

$$M = \frac{1}{\|\vec{v}\|} \begin{bmatrix} v_x^2 & v_x v_y \\ v_x v_y & v_y^2 \end{bmatrix}$$

Affine Examples

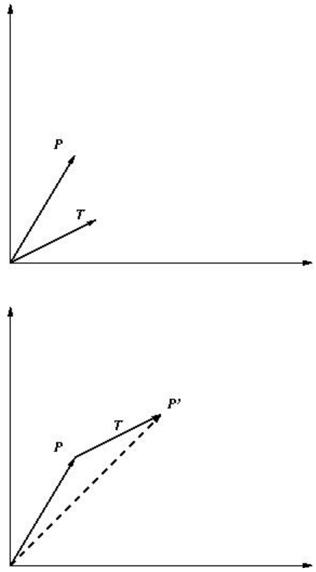

Illustration 2: Translating a point by $\vec{T}$

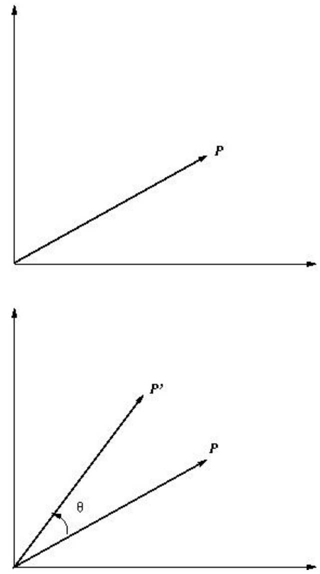

Illustration 1: Rotating a point about the origin by θ

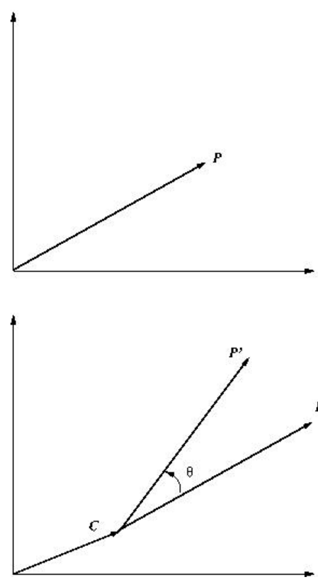

Illustration 3: Rotating a point about $\vec{C}$ *by* θ

More Examples

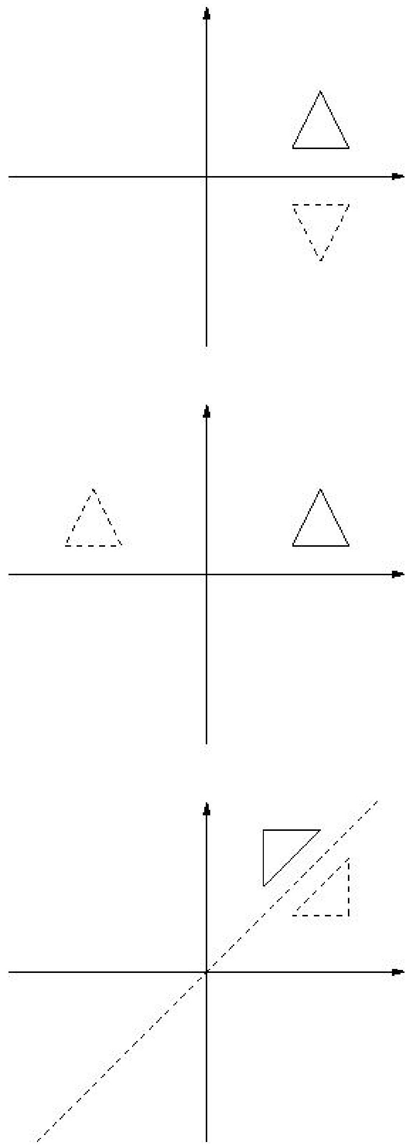

Illustration 4: Reflection transformations about various axes