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Question 1 (2 marks): Consider a 2D point in homogeneous coordinates $r=(r_1, r_2, 1)$ and a line segment formed by points $a=(a_1, a_2, 1)$ and $b=(b_1, b_2, 1)$. Give the transformation matrix that rotates the segment $a, b$ by an angle $\theta$ around point $r$.

$$M = T(r) R(\theta) T(-r) = \begin{bmatrix} \cos \theta & -\sin \theta & -r_1 \cos \theta + r_2 \sin \theta + r_1 \\ \sin \theta & \cos \theta & -r_1 \sin \theta + r_2 \cos \theta + r_2 \\ 0 & 0 & 1 \end{bmatrix}$$
Question 2 (3 marks): Suppose we have two unit vectors $\vec{e}_1=(1,0)^T$ and $\vec{e}_2=(0,1)$, representing the axes of a coordinate system $A$. We want to find a transformation that brings points from coordinate system $A$ into coordinate system $B$ with vectors $\vec{u}_1=(\sqrt{2},-\sqrt{2})^T$ and $\vec{u}_2=(\sqrt{2},\sqrt{2})^T$. Find the matrix that performs this transformation.

First, normalize $\vec{u}_1$ and $\vec{u}_2$ to obtain $\vec{u}_1=\frac{1}{2}(\sqrt{2},-\sqrt{2})^T$ and $\vec{u}_2=\frac{1}{2}(\sqrt{2},\sqrt{2})^T$. Then form the transformation matrix:

$$
\begin{bmatrix}
\vec{u}_1^T \\
\vec{u}_2^T
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\sqrt{2} & -\sqrt{2} \\
\sqrt{2} & \sqrt{2}
\end{bmatrix}
$$
Question 3 (3 marks): Given a 3D homogeneous point expressed in the coordinates of the synthetic camera $P = (10, 10, -20, 1)^T$, find its pseudo-depth $\frac{-(aZ + b)}{Z}$ knowing that the far plane is at a distance $F = 100$ and the near plane is at a distance $N = 5$ (that is to say, the intersection of the $n$ axis with the near plane is $(0, 0, -N)$ and $(0, 0, -F)$ with the far plane).

\[
a = \frac{-(F+N)}{F-N} = -1.105 \quad \quad b = \frac{-2FN}{F-N} = -10.526
\]

\[
\frac{-(aZ + b)}{Z} = 0.5787
\]
**Question 4 (2 marks):** Given a window with \( X_{\text{min}} = Y_{\text{min}} = 0 \) and \( X_{\text{max}} = Y_{\text{max}} = 10 \) and the line segment with endpoints \( P_1 = (-5, -2) \) and \( P_2 = (5, 11) \), apply Liang-Barsky's algorithm to find the coordinates of the clipped segment. Show your calculations.

\[
\begin{align*}
p_1 &= -\Delta x = -10 \\
p_2 &= \Delta x = 10 \\
p_3 &= -\Delta y = -13 \\
p_4 &= \Delta y = 13
\end{align*}
\]

\[
\begin{align*}
q_1 &= x_1 - X_{\text{min}} = -5 \\
q_2 &= X_{\text{max}} - x_1 = 15 \\
q_3 &= y_1 - Y_{\text{min}} = -2 \\
q_4 &= Y_{\text{max}} - y_1 = 12
\end{align*}
\]

\[
\begin{align*}
r_1 &= \frac{q_1}{p_1} = 0.5 \\
r_2 &= \frac{q_2}{p_2} = 1.5 \\
r_3 &= \frac{q_3}{p_3} = 0.1538 \\
r_4 &= \frac{q_4}{p_4} = 0.9231
\end{align*}
\]

\( \forall p_k < 0 \quad u_1 = \max \{0, r_k\} = 0.5 \)

\( \forall p_k > 0 \quad u_2 = \min \{1, r_k\} = 0.9231 \)

\( u_1 \leq u_2 \)  And therefore the clipped segment is

\[
P_1 = (x_1 + \Delta x u_1, y_1 + \Delta y_1 u_1) = (0, 4.5) \quad P_2 = (x_1 + \Delta x u_2, y_1 + \Delta y_1 u_2) = (4.23, 10)
\]
Question 5 (2 marks): Suppose we have a 2D composite transformation \( T = R(\theta)R^T(\theta) \) where

\[
R(\theta) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

what is the effect of performing this transformation on a point \( p = [x, y, 1]^T \)?

*Since the inverse of a rotation matrix is its transpose then \( T \) is the identity matrix and thus has no effect on point \( p \).*
Question 6 (3 marks): Given the position of a synthetic camera $\vec{e}=(5,5,5)^T$, a gaze point $\vec{g}=(0,0,0)^T$, and an up direction $\vec{p}=(0,0,1)^T$, compute matrix $M_v$.

\[
\vec{n} = \frac{\vec{e} - \vec{g}}{\|\vec{e} - \vec{g}\|} = (0.5773, 0.5773, 0.5774)^T
\]

\[
\vec{u} = \frac{\vec{p} \times \vec{n}}{\|\vec{p} \times \vec{n}\|} = (-0.7071, 0.7071, 0)^T
\]

\[
\vec{v} = \vec{n} \times \vec{u} = (-0.4082, -0.4082, 0.8165)^T
\]

\[
-e \cdot \vec{u} = 0 \quad -e \cdot \vec{v} = 0 \quad -e \cdot \vec{n} = -8.6595
\]

\[
M_v = \begin{bmatrix}
-0.7071 & 0.7071 & 0 & 0 \\
-0.4082 & -0.4082 & 0.8165 & 0 \\
0.5774 & 0.5774 & 0.5774 & -8.6603 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
**Question 7 (2 marks):** Identify the matrix or matrices in the transformation pipeline $WS_2T_2S_1T_1M_pM_v$ that have to be updated each time the camera is moved (translated or rotated). Give a short explanation for your answer.

*Only matrix $M_v$ needs to be updated, since it is the only matrix containing $\vec{u}, \vec{v}, \vec{n},$ and $e.$*
Question 8 (3 marks): Given a 3D point \( P = (x, y, z)^T \) and a 3D vector \( \vec{V} = (v_x, v_y, v_z)^T \), write them out in homogeneous coordinates and transform them with the following homogeneous transformation matrix:

\[
M = \begin{bmatrix}
1 & 0 & 0 & a \\
0 & 1 & 0 & b \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

For point \( P \):

\[
\begin{bmatrix}
1 & 0 & 0 & a \\
0 & 1 & 0 & b \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1 \\
\end{bmatrix} =
\begin{bmatrix}
x + a \\
y + b \\
z + c \\
1 \\
\end{bmatrix}
\]

For vector \( \vec{V} \):

\[
\begin{bmatrix}
1 & 0 & 0 & a \\
0 & 1 & 0 & b \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
v_x \\
v_y \\
v_z \\
0 \\
\end{bmatrix} =
\begin{bmatrix}
v_x \\
v_y \\
v_z \\
0 \\
\end{bmatrix}
\]