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Projective Geometry

The concept of homogeneous coordinates can be properly understood only within the framework of projective geometry, which defines proper invariants under projections and sections.

Definitions

Let α be a plane in $\mathbb{R}^3$. To each line in this plane $(l \in \alpha)$ we associate an ideal point (at infinity) L_∞ such that distinct intersecting lines have distinct ideal points. Thus, ideal points signify direction. When the slope of line is of interest, then its associated ideal point is denoted as P_m^∞ . The pair (l, L_∞) is called an augmented line.

For lines $r, s \in \alpha$, their intersection is an ordinary point P if the lines are not parallel, or is either R_∞ or S_∞ if they are parallel.

Given a plane π in $\mathbb{R}^3$, let $p_\infty = \{L_\infty \mid l \in \pi,\ (l, L_\infty)\}$ be the set of all ideal points (directions) on π . Then, p_∞ satisfies the axioms of a line and is called an ideal line.

The pair (π, p_∞) is called an augmented plane for it satisfies the axioms of a plane.

The set of all ideal lines for each plane $\pi \in \mathbb{R}^3$ is denoted by

$$\pi_\infty = \{p_\infty \mid \pi \in \mathbb{R}^{3,}(\pi, p_\infty)\}$$

where π_∞ is an augmented plane and $(\mathbb{R}^{3,}\pi_\infty)$ is the augmented space. Therefore, points can be ordinary or ideal, lines can be ordinary or ideal, and planes can be augmented planes or the ideal plane.

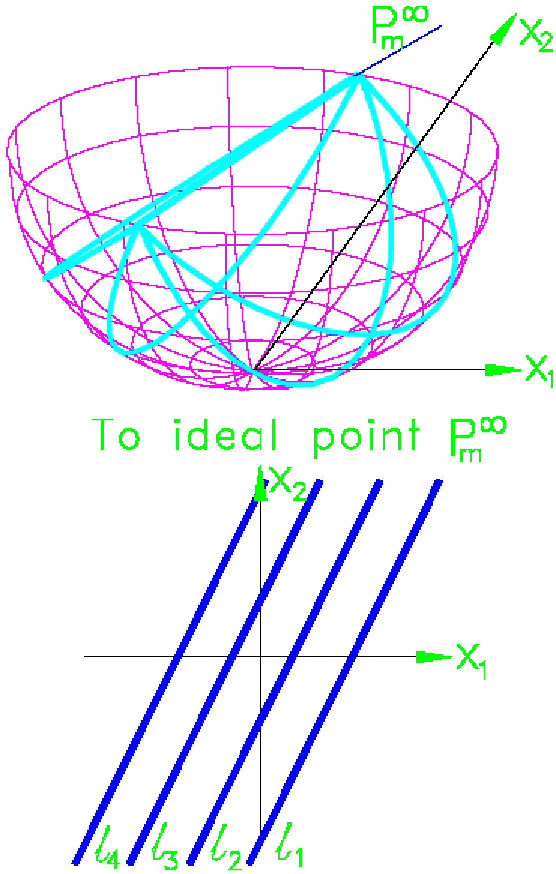

Illustration 1: Hilbert's model of the projective plane

Illustration 1 shows Hilbert's model of the projective plane. The semi-sphere sits at the origin of a plane. It is easy to see how parallel lines on this plane project onto the semi-sphere as semi-cicles, assuming that the center of projection is the center of the sphere of which the semi-sphere is a part. It can be clearly seen that these semi-circles formed by parallel lines all intersect at their common ideal point.

Axioms of Projective Geometry

- Two distinct points are incident to a unique line
- Three distinct, non-collinear points are on a unique plane
- Two distinct coplanar lines are on a unique point
- A plane and a line not contained by the plane are incident to a unique point

- Two distinct planes are incident to a unique line
- Duality expresses the fact that points and lines are equivalent in 2D, and that points an planes are equivalent in 3D.

In Euclidean geometry, infinity is a concept that does not exist and one of the purposes of homogeneous coordinates is to capture it in its essence.

Given a point $(x, y, \omega)^T$ in homogeneous coordinates, its corresponding coordinates in the xy plane is $(x/\omega, y/\omega)^T$. Hence, a point $(3,4,5)^T$ in homogeneous coordinates converts to a point $(3/5,4/5)^T$ in this plane. Similarly, a point $(x, y, z, \omega)^T$ in homogeneous coordinates converts to $(x/\omega, y/\omega, z/\omega)^T$ in $\mathbb{R}^3$ space.

Conversely, the homogeneous coordinates of $(x, y)^T$ are $(x, y, 1)^T$. But in general, the homogeneous coordinates for this point are not unique. They are represented by $(x\omega, y\omega, z\omega, \omega)^T$ for any $\omega \neq 0$.

Ideal Points

Let a point $(x, y)^T$ be fixed and converted to homogeneous coordinates by multiplying it with $1/\omega$ as we let $\omega \rightarrow 0$. Then $(x/\omega, y/\omega)^T$ moves farther away in the direction $(x, y)^T$. In the limit, $(x/\omega, y/\omega)$ is at infinity. Hence, the homogeneous coordinates $(x, y, \omega)^T = (x, y, 0)^T$ is the ideal point in the direction $(x, y)^T$, as $\omega \rightarrow 0$. In the field of computer vision, ideal points take the form of vanishing points. For example, under perspective projection, the tracks of a railroad will join at infinity while never intersecting. Similarly, the vanishing point of an optical flow field (known as the focus of expansion) specifies the 3D translation of the observer (See Illustrations 2 and 3).

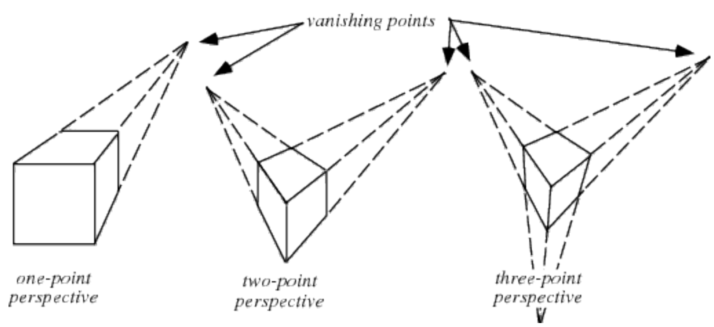

Illustration 2: Perspective projection creates vanishing points

Geometric Interpretation

The homogeneous coordinate $(x, y, \omega)^T$ intersects the plane $\omega=1$ at $(x/\omega, y/\omega, 1)^T$. This is known as a perspective projection of a 3D scene point onto a 2D image plane, parallel to the xy plane and away from the origin by a distance of 1 . One important point to remember is that the transformation from ordinary coordinates to homogeneous coordinates is not unique.

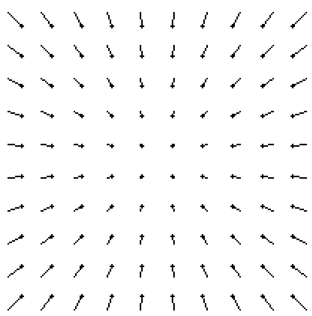

Illustration 3: The focus of expansion in an optical flow field is a vanishing point

Fundamental Transformations of Projective Geometry

Consider two directed lines p and p' and a point P not on either of these lines. Then, a line on P intersecting p at A and p' at A' establishes a one-to-one correspondence between the points of the lines, called perspectivity, denoted as:

$$p(A, B, C, D, \ldots) \overset{P}{\underset{\wedge}{=}} p'(A', B', C', \ldots)$$

and the lines are said to be in perspective with respect to point P (See illustration 4; the dual concept also exists). Given a perspectivity, a cross-ratio can be defined as:

$$(A, B; C, D) = \frac{AB/BC}{AD/BD}$$

which is invariant under perspectivity. This definition is in the Euclidean plane. However, it can be generalized to the augmented plane (π, p_∞) , the cross-ratio invariance being a projective property.

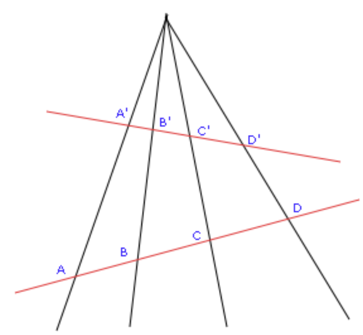

Illustration 4: Two lines in perspective of a point

The set of lines on a point is called pencil of lines (also dual), and may undergo successive perspectives:

$$p_1 \overset{p_1}{\underset{\wedge}{=}} p_2 \overset{p_1}{\underset{\wedge}{=}} \cdots \overset{p_{n-1}}{\underset{\wedge}{=}} p_n$$

A sequence of perspectivities is called a projectivity. As a consequence of the cross-ratio theorem, projective transformations are linear. An important result, for which the proof lies beyond the scope of this course.

The 2D Projective Plane

A line in the plane is represented by an equation such as $ax+by+c=0$. A line is thus naturally expressed as a vector $(a, b, c)^T$. The correspondence between lines and vectors is not a bijection (one-to-one) and the lines $ax+by+c=0$ and $kax+kby+kc=0$ are the same for any $k\neq 0$. Hence, any vector $(a, b, c)^T$ is said to be homogeneous and representative of an equivalence class

$\{k(a,b,c)^T \forall k \neq 0\}$.

A point $\boldsymbol{x}=(x,y)^T$ lies on a line $l=(a,b,c)^T$ if and only if $(x,y,1)l=0$. For any constant $k\neq 0$ and line l the equation $k(x,y,1)l=0$ if and only if $(x,y,1)l=0$. It is thus natural to consider the set of vectors $(kx,ky,k)^T$ to be a representation of the point $(x,y)^T$ in $\mathbb{R}^2$. Then, an arbitrary homogeneous vector (x_1,x_2,x_3) represents $(x_1/x_3, x_2/x_3)^T$ in $\mathbb{R}^2$.

- The point $\boldsymbol{x}$ lies on the line l if and only if $\boldsymbol{x}^T l=0$
- Given two lines l_1 and l_2 , their intersection is given by $l_1 \times l_2$.
- The line through two points $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ is $l=\boldsymbol{x_1}\times\boldsymbol{x_2}$

Ideal Points and Lines at Infinity

Consider two lines $l_1=(a,b,c)$ and $l_2=(a,b,c')$. These lines are parallel but computing their intersection as $l_1\times l_2$ is possible and yields $(c-c')(b,-a,0)^T$. Ignoring the scale factor $(c-c')$, the intersection point is thus $(b,-a,0)^T$. If we attempt to compute the inhomogeneous coordinates of this point we obtain $(b/0,-a/0)^T$ which makes no sense except to suggest that the coordinates of the intersection point are infinitely large. In general, points with homogeneous coordinates $(x,y,0)^T$ do not correspond to any finite point in $\mathbb{R}^2$.

Homogeneous vectors $(x_1,x_2,x_3)^T$ such that $x_3\neq 0$ correspond to finite points in $\mathbb{R}^2$. We can augment $\mathbb{R}^2$ by adding points with last coordinate $x_3=0$, known as ideal points. The result is the set of all homogeneous 3-vectors, called the 2D projective space. The set of all ideal points lies on a line at infinity $l_\infty=(0,0,1)^T$. This can be verified by realizing that $(x_1,x_2,0)(0,0,1)^T=0$ for all $(x_1,x_2,0)^T$.

Line $l=(a,b,c)^T$ intersects with line at infinity l_∞ at ideal point $l\times l_\infty=(b,-a,0)^T$ which is a vector tangent to l and represents the direction of the line. Hence the line at infinity can be thought of as the set of directions of lines in the plane.

We can think of the 2D projective plane as a set of rays in $\mathbb{R}^3$. The set of all vectors $k(x_1,x_2,x_3)^T$ as k varies forms a ray through the origin. Such a ray may be thought of as representing a single point in the 2D projective plane. Using this analogy, a line in the projective plane is a plane passing through the origin of $\mathbb{R}^3$.

Conics

A conic is a curve described by a second degree equation in the plane. In 2D projective geometry, all conics are equivalent under projective transformations. The equation of a conic in inhomogeneous coordinates is given by:

$$ax^2+bxy+cy^2+dx+ey+f=0$$

which is a polynomial of degree 2. Using homogeneous coordinates (that is, replacing x by x_1/x_3 and y by x_2/x_3) gives:

$$a\,x_1^2+b\,x_1x_2+c\,x_2^2+d\,x_1x_3+e\,x_2x_3+f\,x_3^2=0$$

or in matrix form:

$$\boldsymbol{x}^T C\,\boldsymbol{x}$$

where

$$C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

Note that this matrix is symmetric. A minimum number of five points define a conic. A general constraint for a point $\boldsymbol{x}_i=(x_i, y_i)^T$ can be written as:

$$(x_i^2, x_i y_i, y_i^2, x_i, y_i, 1)\,\boldsymbol{c}$$

where $\boldsymbol{c}=(a, b, c, d, e, f)^T$. Defining the system of equations for five points yields:

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\boldsymbol{c} = \vec{0}$$

where $\boldsymbol{c}$ is the solution, called the null vector, as it yields the null space. An interesting result is the form a line tangent to a conic at a point $\boldsymbol{x}$ takes: The line l tangent to a conic at $\boldsymbol{x}$ is given by $l = C\boldsymbol{x}$.

The conic C as defined above is called a point conic. However, due to duality, also five lines define a conic.

Projective Transformations

A projectivity is an invertible mapping h from the 2D projective space into itself such that three points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ lie on the same line if and only if $h(\boldsymbol{x}_1)$, $h(\boldsymbol{x}_2)$, and $h(\boldsymbol{x}_3)$ also do. A projectivity is also called a collineation, a projective transformation, or a homography. These terms are synonymous.

A planar projective transformation is a linear transformation on homogeneous 3-

vectors represented by a non-singular 3 by 3 matrix: $\mathbf{x}_2 = H\mathbf{x}_1$. Note that this matrix may be changed by a scale factor without altering the projective transformation. We consequently say that the matrix is homogeneous since only the ratio of the matrix elements is significant. There are eight independent ratios amongst the nine elements of H , and it follows that a projective transformation has eight degrees of freedom.