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Spectral Analysis

Spectral analysis of images is an important aspect of image understanding. The spectral characteristics of imagery help in determining filtering and noise-removal modalities. Further, this type of analysis is demonstrably useful in the analysis of image motion.

Special Functions

The Dirac-delta function plays an important role in spectral analysis. One operational way of defining the Dirac-delta function is to consider a Gaussian function in the limit of its variance going toward zero:

$$\delta(x-x_0)=\lim_{b\to 0}\frac{1}{|b|}G\left(\frac{x-x_0}{b}\right)$$

where

$$G\left(\frac{x-x_0}{b}\right)=\exp\left\{-\pi\left(\frac{x-x_0}{b}\right)^2\right\}$$

Properties of Dirac-delta Functions

- $\delta(x-x_0)=0 \qquad x\neq x_0$
- $\int_{x_1}^{x_2} f(\alpha)\,\delta(\alpha-x_0)\,d\alpha=f(x_0) \qquad x_1<x_0<x_2$
- $\delta\left(\frac{x-x_0}{b}\right)=|b|\,\delta(x-x_0)$
- $\delta(ax-x_o)=\frac{1}{|a|}\delta\left(x-\frac{x_0}{a}\right)$
- $\delta(-x+x_0)=\delta(x-x_0)$

- $\delta(-x)=\delta(x)$
- $f(x)\delta(x-x_0)=f(x_0)\delta(x-x_0)$
- $x\delta(x-x_0)=x_0\delta(x-x_0)$
- $\delta(x)\delta(x-x_0)=0 \quad x_0\neq 0$
- $\delta(x-x_0)\delta(x-x_0)$ is undefined
- $\int \delta(\alpha-x_0)d\alpha=1$
- $\int A\delta(\alpha-x_0)d\alpha=A$
- $\int \delta(\alpha-x_0)\delta(x-\alpha)d\alpha=\delta(x-x_0)$
- $\int e^{-ik(x-x_0)}dk=\delta(x-x_0)$

Derivatives of the Dirac-delta Function

The derivative of the Dirac-delta function is written as:

$$\delta^{(k)}=\frac{d^k\delta(x)}{dx^k}$$

If we choose a function $f(x)$ whose k^{th} derivative is known to be:

$$f^{(k)}(x)=\frac{d^k f(x)}{dx^k}$$

then we can define the derivatives of the Dirac-delta function as:

$$\delta^{(k)}(x-x_0)=0 \quad x\neq x_0$$

and

$$\int f(\alpha)\delta^{(k)}(\alpha-x_0)d\alpha=(-1)^k f^{(k)}(x_0)$$

which sifts out the derivative of $f(x)$ at x_0 . Also note that $\int \delta^{(k)}(\alpha)d\alpha=0$.

General Dirac-delta Functions

In general, we refer to the one-dimensional Dirac-delta function as a point-mass. In 2D, a point mass Dirac-delta is written as: $\delta(x-x_{0,}y-y_0)$ while a line-mass Dirac-delta function is defined as: $\delta(a_1x+b_1y+c_1)$ and is non-zero along $a_1x+b_1y+c_1=0$. Alternatively, a general point-mass Dirac-delta function can be defined as: $\delta(a_1 x+b_1 y+c_{1,} a_2+b_2 y+c_2)$ and is non-zero at the intersection of $a_1 x+b_1 y+c_1=0$ and $a_2 x+b_2 y+c_2=0$, provided that it exists.

Harmonic Analysis

A function $f(x)$ can be expressed as a Fourier series if it satisfies Dirichlet conditions, which are:

- $f(x)$ is single-valued
- it has a finite number of maxima and minima
- it has a finite number of finite discontinuities
- it is absolutely integrable: $\int |f(x)| dx<\infty$

Then we can write $f(x)$ as a Fourier series expansion

$$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{ik_0 x n}$$

where $k_0=2\pi f_o$. f_o is the fundamental frequency of $f(x)$. This Fourier series expansion may also be written as:

$$f(x)=\sum_{n=-\infty}^{\infty} C_n(\cos(k_0 x n)+i\sin(k_0 x n))$$

as per Euler's formula: $e^{i\theta}=\cos\theta+i\sin\theta$.

A Simple Fourier Integral

Let's compute the Fourier transform of a function that has the simplest frequency characteristics, such as $f(x)=A\cos(k_0 x)$:

$$\hat{f}(k)=A\int\cos(k_0 x)e^{-ikx}dx$$

From Euler's formula, we have:

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

and hence we can write:

$$\begin{aligned}\hat{f}(k) &= \frac{A}{2}\int \left[e^{ik_0x}+e^{-ik_0x}\right]e^{-ikx}\,dx \\ &= \frac{A}{2}\int e^{-ix(k-k_0)}\,dx+\frac{A}{2}\int e^{-ix(k+k_0)}\,dx \\ &= \frac{A}{2}\left[\delta(k-k_0)+\delta(k+k_0)\right]\end{aligned}$$

Consider a more complex function $f(x)$ for which our only requirement is that it satisfies Dirichlet conditions. Then, we can use a Fourier series expansion to express it as follows:

$$f(x)=\sum_{n=-\infty}^{\infty} C_n\left[\cos(k_0 x n)+i\sin(k_0 x n)\right]$$

Or, using Euler's formula:

$$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{ik_0xn}$$

Then, the Fourier transform of this Fourier series expansion is given by:

$$\begin{aligned}\hat{f}(k) &= \int\left[\sum_{n=-\infty}^{\infty} C_n e^{ik_0xn}\right]e^{-ikx}\,dx \\ &= \sum_{n=-\infty}^{\infty}\int C_n e^{-ix(k-k_0n)}\,dx \\ &= \sum_{n=-\infty}^{\infty} C_n\delta(k-k_0n)\end{aligned}$$

Hence, the Fourier transform of a function satisfying Dirichlet conditions is a train of Dirac-delta functions controlled by amplitude coefficients C_n .

Case Study: Spectral Analysis of Image Motion

Suppose we have a 1D signal translating with velocity v :

$$f(x)=I_0(x-vt)$$

where $f(x)$ satisfies Dirichlet conditions. Since it is translating in time, we can describe this signal as a 2D signal:

$$I(x,t) = I_0(x - vt)$$

Taking the Fourier transform of this signal yields:

$$\begin{aligned}
\hat{I}(k,\omega) &= \iint I_0(x - vt) e^{-ikx} e^{-i\omega t} dx\, dt \\
&= \int \left[\int I_0(x - vt) e^{-ikx} dx \right] e^{-i\omega t} dt \\
&= \int \left[\hat{I}_0(k) e^{-ikvt} \right] e^{-i\omega t} dt \\
&= \hat{I}_0(k) \int e^{-it(kv+\omega)} dt \\
&= \hat{I}_0(k) \delta(kv + \omega)
\end{aligned}$$

where $\hat{I}_0(k)$ is the Fourier transform of the signal (when not translating) and $\delta(kv+\omega)$ is a line-mass Dirac-delta function. As can be easily observed, the slope of $kv+\omega=0$ is the velocity of the translating signal.

We repeat the same exercise this time with a translating 2D image signal $I(\vec{x})$:

$$\begin{aligned}
I(\vec{x},t) &= I_0(\vec{x} - \vec{v}t) \\
\hat{I}(\vec{k},\omega) &= \iint I(\vec{x},t) e^{-i\vec{x}^T\vec{k}} e^{-i\omega t} d\vec{x}\, dt \\
&= \int \left[\int I_0(\vec{x} - \vec{v}t) e^{-i\vec{x}^T\vec{k}} d\vec{x} \right] e^{-i\omega t} dt \\
&= \hat{I}_0(\vec{k}) \int e^{-it(\vec{k}^T\vec{v}+\omega)} dt \\
&= \hat{I}_0(\vec{k}) \delta(\vec{k}^T\vec{v} + \omega)
\end{aligned}$$

where $\hat{I}_0(\vec{k})$ is the Fourier transform of the 2D signal $I_0(\vec{x})$. This result shows that for a translating signal, all the non-zero frequencies lie on a plane in the frequency domain described by $\vec{k}^T\vec{v}+\omega=0$. The orientation of this plane yields the 2D velocity of the translating signal. In addition, the Fourier transform of the optical flow equation is consonant with this result, as it is obtained in the following way:

$$F[\nabla I \vec{v} + I_t] = i\hat{I}(\vec{k},\omega)\delta(\vec{k}^T\vec{v}+\omega)$$

Gabor Filtering

It would be convenient to be able to sample the frequency spectrum of a sequence of images locally in order to estimate the amount of translation (image motion). The idea behind Gabor filters is to sample the spectrum within a spatio-temporal neighborhood without taking a Fast Fourier Transform of the entire image sequence.

A Gabor filter responds to a pre-determined frequency if it is present under the area of convolution. Gabor filters are Gaussian-windowed complex exponential functions of the form:

$$B(\vec{x}, t, \vec{k}_0, \omega_0) = e^{-i(\vec{x}^T \vec{k}_0 + \omega_0 t)} G(\vec{x}, t; \sigma)$$

In the frequency spectrum, such a Gabor filter is an un-normalized Gaussian centered at $(\vec{k}_{0,} \omega_0)$ with standard deviation $\hat{\sigma} = \frac{1}{\sigma}$. Hence, the convolution of a Gabor filter with a spatio-temporal region of an image sequence is the equivalent of multiplying the local frequency spectrum with a Gaussian centered at $(\vec{k}_{0,} \omega_0)$. It is instructive to take the Fourier transform of a Gabor filter to understand its spectral characteristics:

$$F\left[B(\vec{x}, t, \vec{k}_{0,} \omega_0)\right] = (2\pi)^{\frac{1}{3}} \sigma^3 [\int [\int [\int e^{\frac{-x^2}{2\sigma^2} - ix(k_x - k_{x0})} dx] e^{\frac{-y^2}{2\sigma^2} - iy(k_y - k_{y0})} dy] e^{\frac{-t^2}{2\sigma^2} - it(\omega - \omega_0)} dt]$$

We observe that the integral in x solves as:

$$\int \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2} - ix(k_x - k_{x0})} dx \;=\; e^{\frac{-(k_x - k_{x0})^2}{2\sigma^2}}$$

(and similarly for y and t). Hence, the Fourier transform of a Gabor filter is a Gaussian centered at $(\vec{k}_{0,} \omega_0)$ in the frequency spectrum.