A Generalization of Repetition Threshold

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Abstract

Brandenburg and (implicitly) Dejean introduced the concept of *repetition threshold*: the smallest real number α such that there exists an infinite word over a k-letter alphabet that avoids β-powers for all $\beta > \alpha$. We generalize this concept to include the lengths of the avoided words. We give some conjectures supported by numerical evidence and prove some of these conjectures. As a consequence of one of our results, we show that the pattern $ABCBABC$ is 2-avoidable. This resolves a question left open in Cassaigne's thesis.

Key words: Combinatorics on words, Repetitions

1 Introduction

In this paper we consider some variations on well-known theorems about avoiding repetitions in words.

A *square* is a repetition of the form xx, where x is a nonempty word; an example in English is `hotshots`. Let Σ_k denote the k-letter alphabet $\{0, 1, \ldots, k-1\}$. It is easy to see that every word of length ≥ 4 over Σ_2 must contain a square, so squares cannot be avoided in infinite binary words. However, Thue showed [17,18,2] that there exist infinite words over Σ_3 that avoid squares.

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Instead of avoiding *all* squares, one interesting variation is to avoid *all sufficiently large* squares. Entringer, Jackson, and Schatz [8] showed that there exist infinite binary words avoiding all squares xx with $|x| \geq 3$. Furthermore, they proved that every binary word of length ≥ 18 contains a factor of the form xx with $|x| \geq 2$, so the bound 3 is best possible. For some other papers about avoiding sufficiently large squares, see [7,14,9,15,16].

Another interesting variation is to consider avoiding fractional powers. For $\alpha \geq 1$ a rational number, we say that y is an α-power if we can write $y = x^n x'$ with x' a prefix of x and $|y| = \alpha|x|$. For example, the French word `entente` is a $\frac{7}{3}$-power and the English word `tormentor` is a $\frac{3}{2}$-power. For real $\alpha > 1$, we say a word *avoids α-powers* if it contains no factor that is a α'-power for any rational $\alpha' \geq \alpha$. Brandenburg [3] and (implicitly) Dejean [6] considered the problem of determining the *repetition threshold*; that is, the least exponent $\alpha = \alpha(k)$ such that there exist infinite words over Σ_k that avoid $(\alpha+\epsilon)$-powers for all $\epsilon > 0$. Dejean proved that $\alpha(3) = \frac{7}{4}$. She also conjectured that $\alpha(4) = \frac{7}{5}$ and $\alpha(k) = \frac{k}{k-1}$ for $k \geq 5$. In its full generality, this conjecture is still open, although Pansiot [13] proved that $\alpha(4) = \frac{7}{5}$ and Moulin-Ollagnier [11] proved that Dejean's conjecture holds for $5 \leq k \leq 11$. For more information, see [5].

In this paper we consider combining these two variations. We generalize the repetition threshold of Dejean to handle avoidance of all sufficiently large fractional powers. (Pansiot also suggested looking at this generalization at the end of his paper [13], but to the best of our knowledge no one else has pursued this question.) We give a large number of conjectures, supported by numerical evidence, about generalized repetition threshold, and prove six of them. Finally, some applications of our results to pattern avoidability are presented. In particular, we prove that the pattern $ABCBABC$ is 2-avoidable, which resolves a question left open in Cassaigne's thesis [4], and implies that every ternary pattern is either unavoidable or 3-avoidable.

2 Definitions

Let $\alpha > 1$ be a rational number, and let $\ell \geq 1$ be an integer. A word w is a *repetition of order α and length ℓ* if we can write it as $w = x^n x'$ where x' is a prefix of x, $|x| = \ell$, and $|w| = \alpha|x|$. For brevity, we also call w a (α, ℓ)-*repetition*. Notice that an α-power is an (α, ℓ)-repetition for some ℓ. We say a word is (α, ℓ)-*free* if it contains no factor that is a (α', ℓ')-repetition for $\alpha' \geq \alpha$ and $\ell' \geq \ell$. We say a word is (α^+, ℓ)-*free* if it is (α', ℓ)-free for all $\alpha' > \alpha$.

For integers $k \geq 2$ and $\ell \geq 1$, we define the *generalized repetition threshold* $R(k, \ell)$ as the real number α such that either

(a) over Σ_k there exists an (α^+, ℓ)-free infinite word, but all (α, ℓ)-free words are finite; or
(b) over Σ_k there exists a (α, ℓ)-free infinite word, but for all $\epsilon > 0$, all $(\alpha - \epsilon, \ell)$-free words are finite.

Notice that $R(k, 1)$ is essentially the repetition threshold of Dejean and Brandenburg.

Theorem 1 *The generalized repetition threshold $R(k, \ell)$ exists and is finite for all integers $k \geq 2$ and $\ell \geq 1$. Furthermore, $1 + \ell/k^\ell \leq R(k, \ell) \leq 2$.*

PROOF. Define S to be the set of all real numbers $\alpha \geq 1$ such that there exists a (α, ℓ)-free infinite word over Σ_k. Since Thue proved that there exists an infinite word over a two-letter alphabet (and hence over larger alphabets) avoiding $(2+\epsilon)$-powers for all $\epsilon > 0$, we have that $\beta = \inf S$ exists and $\beta \leq 2$. If $\beta \in S$, we are in case (b) above, and if $\beta \notin S$, we are in case (a). Thus $R(k, \ell) = \beta$.

For the lower bound, note that any word of length $\geq k^\ell + \ell$ contains $\geq k^\ell + 1$ factors of length ℓ. Since there are only k^ℓ distinct factors of length ℓ, such a word contains at least two occurrences of some word of length ℓ, and hence is not $(1 + \frac{\ell}{k^\ell}, \ell)$-free.

Remarks.

1. It may be worth noting that we know no instance where case (b) of the definition of generalized repetition threshold above actually occurs, but we have not been able to rule it out.

2. Using the Lovász local lemma, Beck [1] has proved a related result: namely, for all $\epsilon > 0$, there exists an integer n' and an infinite $(1 + n/(2-\epsilon)^n,\ n)$-free binary word for all $n \geq n'$. Thus our work can be viewed as a first attempt at an explicit version of Beck's result (although in our case the exponent does not vary with n).

3 Conjectures

In this section we give some conjectures about $R(k, \ell)$.

Figure 1 gives the established and conjectured values of $R(k, \ell)$. Entries in **bold** have been proved; the others (with question marks) are merely conjectured. However, in either case, if the entry for (k, ℓ) is α, then we have proved, using the usual tree-traversal technique discussed below, that there is no infinite (α, ℓ)-free word over Σ_k.

$R(k,\ell)$	$\ell = 1$	2	3	4	5	6	7	8
$k = 2$	$\mathbf{2}$	$\mathbf{2}$	$\boldsymbol{\frac{8}{5}}$	$\boldsymbol{\frac{3}{2}}$	$\boldsymbol{\frac{7}{5}}$	$\boldsymbol{\frac{4}{3}}$	$\frac{31}{24}$?	$\frac{24}{19}$?
3	$\boldsymbol{\frac{7}{4}}$	$\boldsymbol{\frac{3}{2}}$	$\boldsymbol{\frac{4}{3}}$	$\frac{5}{4}$?	$\frac{6}{5}$?	$\frac{7}{6}$?	$\frac{8}{7}$?	$\frac{9}{8}$?
4	$\boldsymbol{\frac{7}{5}}$	$\frac{5}{4}$?	$\frac{6}{5}$?	$\frac{7}{6}$?	$\frac{8}{7}$?	$\frac{9}{8}$?	$\frac{10}{9}$?	$\frac{11}{10}$?
5	$\boldsymbol{\frac{5}{4}}$	$\frac{6}{5}$?	$\frac{8}{7}$?	$\frac{9}{8}$?	$\frac{10}{9}$?			
6	$\boldsymbol{\frac{6}{5}}$	$\frac{36}{31}$?						
7	$\boldsymbol{\frac{7}{6}}$	$\frac{8}{7}$?						
8	$\boldsymbol{\frac{8}{7}}$							
9	$\boldsymbol{\frac{9}{8}}$							
10	$\boldsymbol{\frac{10}{9}}$							
11	$\boldsymbol{\frac{11}{10}}$							
12	$\frac{12}{11}$?							
13	$\frac{13}{12}$?							

Fig. 1. Known and conjectured values of $R(k, \ell)$.

The proved results are as follows:

- $R(2, 1) = 2$ follows from Thue's proof of the existence of overlap-free words over Σ_2 [17,18,2];
- $R(2, 2) = 2$ follows from Thue's proof together with the observation of Entringer, Jackson and Schatz [8];
- $R(3, 1) = \frac{7}{4}$ is due to Dejean [6];
- $R(4, 1) = \frac{7}{5}$ is due to Pansiot [13];
- $R(k, 1) = \frac{k}{k-1}$ for $5 \leq k \leq 11$ is due to Moulin-Ollagnier [11];
- $R(2, 3) = \frac{8}{5}$, $R(2, 4) = \frac{3}{2}$, $R(2, 5) = \frac{7}{5}$, $R(2, 6) = \frac{4}{3}$, $R(3, 2) = \frac{3}{2}$ and $R(3, 3) = \frac{4}{3}$ are new and are proved in Section 4.

We now explain how the conjectured results were obtained. We used the usual tree-traversal technique, as follows: suppose we want to determine if there are only finitely many words over the alphabet Σ_k that avoid a certain set of words S. We construct a certain tree T and traverse it using breadth-first or depth-first search. The tree T is defined as follows: the root is labelled ϵ (the empty word). If a node w has a factor contained in S, then it is a leaf. Otherwise, it has children labelled wa for all $a \in \Sigma_k$. It is easy to see that T is finite if and only if there are finitely many words avoiding S.

We can take advantage of various symmetries in S to speed traversal. For example, if S is closed under renaming of the letters (as is the case in the examples we study), we can label the root with an arbitrary single letter (instead of ϵ) and deduce the number of leaves in the full tree by multiplying by k.

Furthermore, if we use depth-first search, we can in some cases dramatically shorten the search using the following observation: if at any point some suffix of the current string strictly precedes the prefix of the same length of the same string in lexicographic order, then this suffix must have already been examined. Hence we can immediately abandon consideration of this node.

If the tree is finite, then certain parameters about the tree give useful information about the set of finite words avoiding S:

- If h is the height of the tree, then any word of length $\geq h$ over Σ_k contains a factor in S.
- If M is the length of a longest word avoiding S, then $M = h - 1$.
- If I is the number of internal nodes, then there are exactly I finite words avoiding S. Furthermore, if L is the number of leaves, then (as usual), $L = 1 + (k-1)I$.
- If I' is the number of internal nodes at depth $h-1$, then there are I' words of maximum length avoiding S.

Figure 2 gives the value of some of these parameters. Here α is the established or conjectured value of $R(k, \ell)$ from Figure 1. "NR" indicates that the value was not recorded by our program.

k	ℓ	α	L	I	h	$M{=}h{-}1$	I'
2	1	**2**	8	7	4	3	2
2	2	**2**	478	477	19	18	2
2	3	8/5	5196	5195	34	33	12
2	4	**3/2**	13680	13679	54	53	4
2	5	7/5	40642	40641	60	59	4
2	6	4/3	21476	21475	40	39	4
2	8	24/19	3480734274	3480734273	452	451	NR
3	1	**7/4**	6393	3196	39	38	18
3	2	**3/2**	11655	5827	31	30	6
3	3	**4/3**	4037361	2018680	228	227	6
3	4	5/4	188247	94123	63	62	24
3	5	6/5	493653	246826	63	62	12
3	6	7/6	782931	391465	60	59	24
3	7	8/7	2881125	1440562	68	67	24
3	8	9/8	6987903	3493951	62	61	24
4	1	**7/5**	709036	236345	122	121	48
4	2	5/4	10324	3441	17	16	24
4	3	6/5	153724	51241	24	23	96
4	4	7/6	2501620	833873	35	34	24
4	5	8/7	30669148	10223049	40	39	864
4	6	9/8	340760884	113586961	50	49	NR
5	1	**5/4**	1785	446	7	6	120
5	2	6/5	453965	113491	23	22	240
5	3	8/7	7497345	1874336	34	33	720
5	4	9/8	1521535445	380383861	52	51	NR
6	1	**6/5**	13386	2677	8	7	720
6	2	36/31	17372138466	3474427693	751	750	NR
7	1	**7/6**	112441	18740	9	8	5040
7	2	9/8	345508219	57584703	32	31	NR
8	1	**8/7**	1049448	149921	10	9	40320

Fig. 2. Tree statistics for various values of k and l

We have seen how to prove computationally that only finitely many (α, ℓ)-free words exist. But what is the evidence that suggests we have determined the smallest possible α? For this, we explore the tree corresponding to avoiding (α^+, ℓ)-repetitions using *depth-first* (and not breadth-first) search. If we are able to construct a "very long" word avoiding (α^+, ℓ)-repetitions, then we suspect we have found the optimal value of α. For each unproven α given in Figure 1, we were able to construct a word of length at least 20000 avoiding the corresponding repetitions. This constitutes weak evidence of the correctness of our conjectures, but it is evidently not conclusive.

Based on the data in Figure 1, we propose the following conjectures.

Conjecture 2 *$R(3, \ell) = 1 + \frac{1}{\ell}$ for $\ell \geq 2$.*

Conjecture 3 *$R(4, \ell) = 1 + \frac{1}{\ell+2}$ for $\ell \geq 2$.*

These conjectures are weakly supported by the numerical evidence above.

4 New Results

In this section, we prove six results of the form $R(k, l) = \alpha$. From the numerical results reported in Figure 2, we know in each case that there exist no infinite (α, l)-free words over Σ_k. It therefore suffices to exhibit an infinite (α^+, l)-free word over Σ_k. A uniform morphism $h : \Sigma_i^* \to \Sigma_k^*$ is said to be *synchronizing* if for any $a, b, c \in \Sigma_i$ and $s, r \in \Sigma_k$, if $h(ab) = rh(c)s$, then either $r = \varepsilon$ and $a = c$ or $s = \varepsilon$ and $b = c$. An α^+-repetition is an (α', l)-repetition some $\alpha' > \alpha$ and $l \geq 1$. A word is α^+-free if it contains no α^+-repetition.

Lemma 4 *Let $\alpha, \beta \in \mathbb{R}$, $1 < \alpha < \beta < 2$. Let $h : \Sigma_s^* \to \Sigma_e^*$ be a synchronizing morphism. Let $w \in \Sigma_s^*$ be an α^+-free word. Any β^+-repetition occurring in $h(w)$ is contained in the h-image of a factor t of w such that $|t| < \frac{2\beta}{\beta-\alpha}$.*

PROOF. Since h is synchronizing, it is q-uniform for some $q \geq 1$. Suppose $h(t)$ contains a β^+-repetition, that is, a factor uvu such that $\frac{|uvu|}{|uv|} > \beta$. Denote $x = |u|$ and $y = |v|$. If $x \geq 2q - 1$, then each occurrence of u contains at least one full h-image of a letter. As h is synchronizing, the two occurrences of u in uvu contain the same h-images and in the same positions. Therefore, there is a factor UVU in t such that, denoting $X = |U|$ and $Y = |V|$, we have $Yq < y + 2q$ and $Xq > x - 2q$, or equivalently $x < (X + 2)q$. (Each U is the factor of t that contains all letters whose h-images are contained in the corresponding u.) We have then $\frac{2x+y}{x+y} > \beta$, which gives $y < \frac{2-\beta}{\beta-1}x$. The fact that t is α^+-free implies that $\frac{2X+Y}{X+Y} \leq \alpha$, which gives $X \leq \frac{\alpha-1}{2-\alpha}Y$. Now we have

$$Yq < y+2q < \frac{2-\beta}{\beta-1}x+2q < \frac{2-\beta}{\beta-1}(X+2)q+2q \leq \frac{2-\beta}{\beta-1}\left(\frac{\alpha-1}{2-\alpha}Y+2\right)q+2q,$$

implying that $Y < \frac{2(2-\alpha)}{\beta-\alpha}$. By the minimality of t we get

$$|t| \leq 2+Y+2X \leq 2+Y\left(1+2\frac{\alpha-1}{2-\alpha}\right) < 2+\frac{2(2-\alpha)}{\beta-\alpha}\frac{\alpha}{2-\alpha} = \frac{2\beta}{\beta-\alpha}.$$

Consider next the case when $x \leq 2q-2$. This implies $y < \frac{2-\beta}{\beta-1}(2q-2)$ and thus $2x+y < \frac{2\beta}{\beta-1}(q-1)$. The minimality of t implies that $(|t|-2)q \leq |uvu|-2 = 2x+y-2$. By the above we get that $|t| < \frac{2(q-1)(2\beta-1)}{q(\beta-1)}$. Since $1 < \alpha < \beta < 2$ and $q \geq 1$, we can check that $\frac{2(q-1)(2\beta-1)}{q(\beta-1)} < \frac{2\beta}{\beta-\alpha}$, which completes the proof.

For convenience, let us denote the maximum in Lemma 4 by $\max_{\alpha,\beta}$. The morphisms below were found using the method described in [12].

Theorem 5 $R(2,3) = \frac{8}{5}$.

PROOF. Consider the 992-uniform morphism $h : \Sigma_4^* \longrightarrow \Sigma_2^*$ defined by

$h(0) =$ 0000101011110000111010100011110000101011110100001111000101011100
0001111010000101111000011101010001111000010111101010000111100010101110
0000111101010000101111000011101010001111000010101111010000111100010100
1110000111101000010101111000011101010001111000010111101010000111100010
0101110000111101000010111100001110101000111100001011110100001111000101
1011100001111010100001011110000111010100011110000101011110100001111000
0101011100001111010000101111000011101010001111000010111101010000111100
0010101110000111101000010101111000011101010001111000010111101000011110
0001010111000011110100001011110000111010100011110000101011110100001111
1000101011100001111010100001011110000111010100011110000101111010000111
1100010101110000111101000010101111000011101010001111000010111101010000
0111100010101110000111101010000101111000011101010001111000010101111010
0000111100010101110000111101000010101111000011101010001111000010111101
1000011110001010111000011110101000010111100001110101000111100001011110
0101000011110001010111000011110100,

$h(1) =$ 0000101011110000111010100011110000101011110100001111000101011100
0001111010000101111000011101010001111000010111101010000111100010101110
0000111101010000101111000011101010001111000010101111010000111100010100
1110000111101000010101111000011101010001111000010111101010000111100010
0101110000111101000010111100001110101000111100001011110100001111000101
1011100001111010000101011110000111010100011110000101011110100001111000
0101011100001111010100001011110000111010100011110000101111010100001110
1000101011100001111010000101011110000111010100011110000101111010000111
1100010101110000111101000010111100001110101000111100001010111101000011
1110001010111000011110101000010111100001110101000111100001011110100001
1111000101011100001111010000101011110000111010100011110000101111010101
0001111000101011100001111010000101111000011101010001111000010101111010
0000111100010101110000111101000010101111000011101010001111000010111101
1000011110001010111000011110101000010111100001110101000111100001011110
0101000011110001010111000011110100,

$h(2) =$ 0000101011110000111010100011110000101011110100001111000101011 10
000111101000010111100001110101000111100001011110100001111000101011100
001111010100001011110000111010100011110000101011110100001111000101011
100001111010000101011110000111010100011110000101111010100001111000101
011100001111010100001011110000111010100011110000101111010000111100010
101110000111101000010101111000011101010001111000010101111010000111100
010101110000111101010000101111000011101010001111000010111101010000111
100010101110000111101000010111100001110101000111100001010111101000011
110001010111000011110100001010111100001110101000111100001011110100001
111000101011100001111010100001011110000111010100011110000101011110100
001111000101011100001111010000101111000011101010001111000010111101010
000111100010101110000111101000010101111000011101010001111000010111101
000011110001010111000011110100001011110000111010100011110000101011110
100001111000101011100001111010100001011110000111010100011110000101111
0101000011110001010111000011110 1,

$h(3) =$ 000010101111000011101010001111000010101111010000111100010101110
000111101000010111100001110101000111100001011110100001111000101011100
001111010100001011110000111010100011110000101011110100001111000101011
100001111010000101011110000111010100011110000101111010100001111000101
011100001111010000101111000011101010001111000010111101000011110001010
111000011110100001010111100001110101000111100001010111101000011110001
010111000011110101000010111100001110101000111100001011110101000011110
001010111000011110100001011110000111010100011110000101011110100001111
000101011100001111010000101011110000111010100011110000101111010100001
111000101011100001111010100001011110000111010100011110000101011110100
001111000101011100001111010000101111000011101010001111000010111101010
000111100010101110000111101000010101111000011101010001111000010111101
000011110001010111000011110100001011110000111010100011110000101011110
100001111000101011100001111010100001011110000111010100011110000101111
01010000111100010101110000111101.

By a result of Pansiot [13], there exist $\frac{7}{5}^+$-free infinite words over Σ_4. Consider one such word $\mathbf{x}$. A computer check shows that h is synchronizing and that for every $\frac{7}{5}^+$-free word $t \in \Sigma_4^*$ such that $|t| < \max_{\frac{7}{5},\frac{8}{5}} = 16$, $h(t)$ is $(\frac{8}{5}^+, 3)$-free. By Lemma 4, this proves that $h(\mathbf{x})$ is an infinite binary $(\frac{8}{5}^+, 3)$-free word.

Theorem 6 $R(2,4) = \frac{3}{2}$.

PROOF. Consider the 19-uniform morphism $h : \Sigma_4^* \longrightarrow \Sigma_2^*$ defined by

$$h(0) = 0000110100100111110, \qquad h(1) = 0000011011001010111,$$
$$h(2) = 0000011010100111111, \qquad h(3) = 0000010110111110010.$$

We again consider an infinite $\frac{7}{5}^+$-free word $\mathbf{x}$ over Σ_4. A computer check shows that h is synchronizing and that for every $\frac{7}{5}^+$-free word $t \in \Sigma_4^*$ such that $|t| < \max_{\frac{7}{5},\frac{3}{2}} = 30$, $h(t)$ is $(\frac{3}{2}^+, 4)$-free. By Lemma 4, this proves that $h(\mathbf{x})$ is an infinite binary $(\frac{3}{2}^+, 4)$-free word.

Theorem 7 $R(2,5) = \frac{7}{5}$.

PROOF. Consider the 45-uniform morphism $h : \Sigma_5^* \longrightarrow \Sigma_2^*$ defined by

$$h(0) = 000000101011111001000000011010101001111111011,$$
$$h(1) = 000000101010111100010011011101000001111111011,$$
$$h(2) = 000000010101011111100100101101100010001110111,$$
$$h(3) = 000000010101011001100111111010001000101110111,$$
$$h(4) = 000000010011110101010000000110011111101010011.$$

By a result of Moulin-Ollagnier [11], there exist $\frac{5}{4}^+$-free infinite words over Σ_5. Consider one such word $\mathbf{x}$. A computer check shows that h is synchronizing and that for every $\frac{5}{4}^+$-free word $t \in \Sigma_5^*$ such that $|t| < \max_{\frac{5}{4},\frac{7}{5}} = \frac{56}{3} < 19$, $h(t)$ is $(\frac{7}{5}^+, 5)$-free. By Lemma 4, this proves that $h(\mathbf{x})$ is an infinite binary $(\frac{7}{5}^+, 5)$-free word.

Theorem 8 $R(2,6) = \frac{4}{3}$.

PROOF. Consider the 71-uniform morphism $h : \Sigma_5^* \longrightarrow \Sigma_2^*$ defined by

$$h(0) = 00000001010101111111110000100010111011110100000000110110101001001111 0111,$$
$$h(1) = 00000000101011111111000110010100101111011000010001111111101010101001100111,$$
$$h(2) = 000000001010110111111111000110011010101000000111111100100010110101100111,$$
$$h(3) = 00000000101010111111100011001010010111011110000100011011010100100 1110111,$$
$$h(4) = 000000001001010110111111111001100101000000000111111010110100001001111 0111.$$

We again consider an infinite $\frac{5}{4}^+$-free word $\mathbf{x}$ over Σ_5. A computer check shows that h is synchronizing and that for every $\frac{5}{4}^+$-free word $t \in \Sigma_5^*$ such that $|t| < \max_{\frac{5}{4},\frac{4}{3}} = 32$, $h(t)$ is $(\frac{4}{3}^+, 6)$-free. By Lemma 4, this proves that $h(\mathbf{x})$ is an infinite binary $(\frac{4}{3}^+, 6)$-free word.

Theorem 9 $R(3, 2) = \frac{3}{2}$.

PROOF. Consider the 3-uniform morphism $h : \Sigma_4^* \longrightarrow \Sigma_3^*$ defined by

$$h(0) = 021, \qquad h(1) = 100,$$
$$h(2) = 122, \qquad h(3) = 201.$$

We again consider an infinite $\frac{7}{5}^+$-free word $\mathbf{x}$ over Σ_4. A computer check shows that h is synchronizing and that for every $\frac{7}{5}^+$-free word $t \in \Sigma_4^*$ such that $|t| < \max_{\frac{7}{5},\frac{3}{2}} = 30$, $h(t)$ is $(\frac{3}{2}^+, 2)$-free. By Lemma 4, this proves that $h(\mathbf{x})$ is an infinite ternary $(\frac{3}{2}^+, 2)$-free word.

Theorem 10 $R(3, 3) = \frac{4}{3}$.

PROOF. Consider the 14-uniform morphism $h : \Sigma_5^* \longrightarrow \Sigma_3^*$ defined by

$$h(0) = 00011112122220, \qquad h(1) = 00101112202021,$$
$$h(2) = 01012111102120, \qquad h(3) = 10002212102020,$$
$$h(4) = 10100222112020.$$

We again consider an infinite $\frac{5}{4}^+$-free word $\mathbf{x}$ over Σ_5. A computer check shows that h is synchronizing and that for every $\frac{5}{4}^+$-free word $t \in \Sigma_5^*$ such that $|t| < \max_{\frac{5}{4},\frac{4}{3}} = 32$, $h(t)$ is $(\frac{4}{3}^+, 3)$-free. By Lemma 4, this proves that $h(\mathbf{x})$ is an infinite ternary $(\frac{4}{3}^+, 3)$-free word.

5 Applications to Pattern Avoidability

Our results on the repetition threshold have some interesting applications to pattern avoidability. This is due to the following observation: A word avoiding a repetition which appears in any image of a pattern, avoids the pattern itself.

For a pattern $p \in A^*$, its pattern language $p(\Sigma^+)$ is the language over Σ which contains all the words $h(p)$, where h is a non-erasing morphism from A^* to Σ^*. (For further notions and results on avoidability, we refer to Chapter 3 in [10].) We say that the pattern p has an *inherent* (α^+, ℓ)*-repetition with respect to* Σ if any word in $p(\Sigma^+)$ contains an (α', ℓ')-repetition for some $\alpha' > \alpha$ and $\ell' \geq \ell$.

We then have the following general result which can be used to prove avoidability for many patterns.

Lemma 11 *If there exists an* (α^+, ℓ)*-free infinite word over* Σ_k*, then any pattern that has an inherent* (α^+, ℓ)*-repetition is* k*-avoidable.*

According to Cassaigne [4], the pattern $ABCBABC$ was the only avoidable ternary pattern not known to be 3-avoidable. The next result solves this open problem as well as some other open ones.

Corollary 12 *The patterns* $ABCBABC$*,* $ABBCBABBC$*,* $ABCCBABC$*, and* $ABCBAABC$ *are simultaneously* 2*-avoidable.*

PROOF. Any of the patterns in the given set has an inherent $(\frac{3}{2}^+, 4)$-repetition with respect to any alphabet. Theorem 6 gives a $(\frac{3}{2}^+, 4)$-free infinite word over Σ_2 which, by Lemma 11, avoids simultaneously all patterns in the set.

Corollary 12 and the results of Cassaigne [4] give the following theorem.

Theorem 13 *Every ternary pattern is either unavoidable or* 3*-avoidable.*

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