CS840a
Learning and Computer Vision
Prof. Olga Veksler

Lecture 6

Markov Random Fields and Belief Propagation

Today

- Markov Random Fields
- Belief Propagation
- Next time paper:
  - "Object Class Recognition by Unsupervised Scale-Invariant Learning" by R. Fergus, P. Perona, A. Zisserman

Labeling Problem

Given

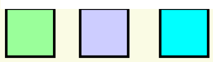
1 Image with m pixels

$S = \{1, \cdots, m\}$

1	2	...	
		...	m

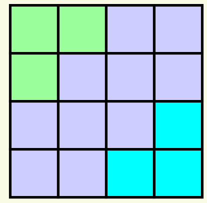
2 Set of labels

$L = \{l_1, \cdots, l_k\}$



Find

Labeling



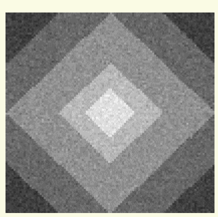
$f = \{f_1, \cdots, f_m\}, \quad f_p \in L$

pixel p is assigned l

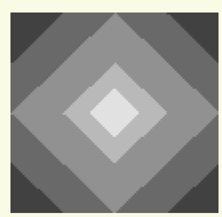
$\Updownarrow$

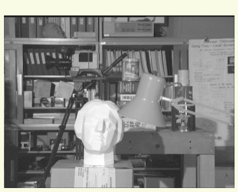
$f_p = l$

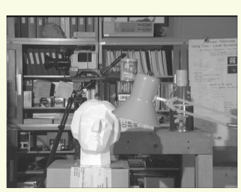
Examples



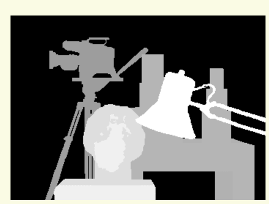
Restoration





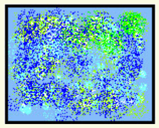


Stereo

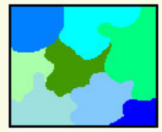


A priori knowledge is frequently available

Unlikely answer



Better answer



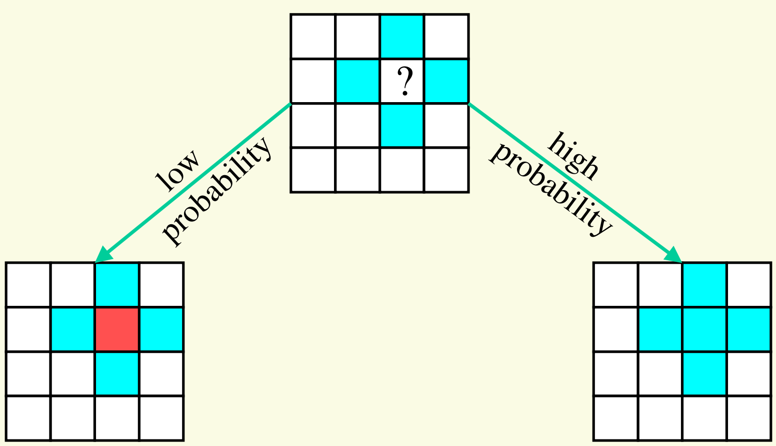
- Bayesian framework is suitable:
  assign prior probabilities *Pr(f)* to labelings

f_1	f_2	…	
		…	f_m

$$f = \{f_1, \ldots f_m\} \in L^m$$

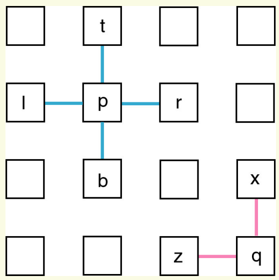
Desirable Properties of prior Pr(f)

Pixel's label depends on neighboring pixels



Markov Random Fields (MRF)

Each pixel p has a *neighborhood* $\boldsymbol{N}_p$



Examples:

$$\boldsymbol{N}_p = \{t, b, l, r\}$$
$$\boldsymbol{N}_q = \{x, z\}$$

Property: $q \in \boldsymbol{N}_p \Leftrightarrow p \in \boldsymbol{N}_q$

Markov Random Fields

- F is an MRF if:
- Markovianity:

$$Pr\left(f_p \middle| f_{S-\{p\}}\right) = Pr\left(f_p \middle| f_{N_p}\right)$$

- Positivity:

$$Pr(f) > 0, \quad \forall f \in L^m$$

Markov Random Fields

- Not obvious how to compute $Pr(f)$ from local conditional distributions $Pr\left(f_p \middle| f_{N_p}\right)$

- **Hammersley-Clifford** theorem: *f* is an MRF if and only if it has *Gibbs* distribution

$$Pr(f) \propto exp\left(-\sum_{c \in C} V_c(f)\right)$$

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- A *clique c* is a set of pixels
  $c = \{p_1, \ldots, p_k \mid p_i\text{'s are neighbors of each other}\}$
  C is the set of all cliques

- $V_c(f): L^m \to R$ is a *clique potential*

- *Clique potentials* $V_c(f)$ specify an MRF

MRF Example

- Only two-cliques are non-zero
- Two-clique potential *{p,q}* :

$$V_{pq}(f_p,f_q) = u_{pq} \cdot \delta(f_p \neq f_q)$$

$$\Downarrow$$

$$V_{pq}(f_p,f_q) = \begin{cases} 0 & \text{if } f_p = f_q \\ u_{pq} & \text{otherwise} \end{cases}$$

encourages piecewise-constant *f*

$$\boxed{Pr(f) \propto \exp\left(- \sum_{\{p,q\} \in N} u_{pq} \cdot \delta(f_p \neq f_q) \right)}$$

MAP-MRF estimation

- Given: observed data *d*, prior $Pr(f)$ and likelihood function $Pr(d|f)$
- *Maximize Posterior Probability (MAP)*

  $Pr(f|d)$ over all $f \in L^m$
- By Bayes' law, $Pr(f|d) \propto Pr(d|f)\,Pr(f)$
- MAP estimate:

$$\hat{f} = \arg\max_{f \in L^m} Pr(d|f)\,Pr(f)$$

MAP-MRF estimation

- Likelihood function:

data for pixel p

$$Pr(d|f) = \prod_p D_p(d_p|f_p)$$

- Maximization problem:

$$\arg\max_{f \in L^m} \exp\left(\sum_p \ln D_p(d_p|f_p) - \sum_{\{p,q\}} u_{pq} \cdot \delta(f_p \neq f_q) \right)$$

- Equivalent to *posterior energy* minimization:

$$\boxed{\hat{f} = \arg\min_f \left(\sum_{\{p,q\} \in N} u_{pq} \cdot \delta(f_p \neq f_q) - \sum_p \ln D_p(d_p|f_p) \right)}$$

Comments:

- MAP-MRF approach for computer vision was first advocated by *Geman and Geman, 1984*
- High optimization cost. Standard methods, like simulated annealing, work very slow because typical image size is formidable

Comments:

- MAP-MRF approach for computer vision was first advocated by *Geman and Geman, 1984*
- Computation is only tractable in some special cases
- In general, high optimization cost. Standard methods, like simulated annealing, work very slow because typical image size is formidable

Belief Propagation (BP)

- Advocated by Pearl for tree-structured graphs in the context of Bayesian Nets
- Gives exact estimates in a tree graph
- Can be applied to general graphs, however
  - No optimality guarantee
  - May not converge
- There are 2 versions of BP
  - max-product, best to use if MAP estimate is needed
    - Map estimate is $\max_f Pr(f|d)$
  - sum-product, best to use if marginals at each node are needed
    - Marginal at node ***p*** is $Pr(f_p|d)$

Max-product BP

- Iterate until convergence:
  - Each pixel ***p*** (or node) sends a message vector (in parallel) to each of its neighobrs ***q***,

$$\left[m_{p\to q}(l_1) \quad m_{p\to q}(l_2) \quad \ldots. \quad m_{p\to q}(l_k)\right]$$

where

$$m_{p\to q}(l) = min_{l'}\left[D_p(l') + V_{pq}(l,l') + \sum_{r\in N_p\backslash q} m_{r\to p}(l')\right]$$

- After "convergence" the final label at each pixel ***p*** is computed as:

$$arg\,min_l\left[D_p(l) + \sum_{q\in N_p} m_{q\to p}(l)\right]$$

Sum-product BP

- Iterate until convergence:
  - Each pixel ***p*** (or node) sends a message vector (in parallel) to each of its neighobrs ***q***,

$$\left[m_{p\to q}(l_1) \quad m_{p\to q}(l_2) \quad \ldots. \quad m_{p\to q}(l_k)\right]$$

where

$$m_{p\to q}(l) = \sum_{l'}\left[D_p(l')V_{pq}(l',l)\prod_{r\in N_p\backslash q}\sum m_{r\to p}(l')\right]$$

- After "convergence" compute the belief for each label *l* at each pixel ***p***:

$$b_l(p) = D_p(l)\prod_{q\in N_p} m_{q\to p}(l)$$

BP

- For tree-structured graph, it can be shown that
  - max-product BP gives the MAP estimate of the MRF, that is $\max_f Pr(f \mid d)$
  - sum-product BP gives the correct posterior marginals that is $Pr(f_p|d) = b(p)$
- Start passing messages at the leafs moving up to the roof
- Compute the solution label (for max-product) or the belief vector (for sum-product) from the root down to the leafs

References for BP

- Several interesting speed-up techniques for max-product BP (and code) available from
  - "Efficient Belief Propagation for Early Vision", P.F. Felzenszwalb and D.Huttenlocher, CVPR2004
- "On the optimality of solutions of the max-product belief propagation algorithm in arbitrary graphs", Y. Weiss and W.Freeman, IEEE Transactions on Information Theory, 2001
- "Generalized Belief Propagation", J.S. Yedidia , W.T. Freeman, Y.Weiss to appear in NIPS 2000

Comments about BP on General Graphs

- Pros
  - Can be applied to any energy function and any neighborhood structure
  - Very easy to implement
- Cons
  - Not guaranteed to converge on (does go into infinite loops occasionally)
  - No optimality guarantees even if converges
  - Not very fast

Better Message Passing Algorithms

- Tree-reweighed message passing algorithms
  - "MAP estimation via agreement on hyper trees: Message-passing and linear-programming approaches", by Wainwright, Jaakkola, Willsky, in IEEE trans. On Infor. Theory, 2005
  - "Convergent tree-reweighted message passing for energy minimization", V. Kolmogorov, in International workshop on Artificial Intelligence and Statistics, 2005.