CS434a/541a: Pattern Recognition
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Lecture 2

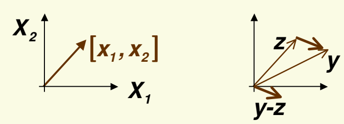
Why Linear Algebra?

- For each example (e.g. a fish image), we will extract a set of features (e.g. length, width, color)
- This set of features we will represent as a *feature vector*
  - [length, width, color,…]
- All collected examples will be represented as collection of (feature) vectors
  - $[l_1, w_1, c_1, \ldots]$, $[l_2, w_2, c_2, \ldots]$, $[l_3, w_3, c_3, \ldots]$, …
    example 1 example 2 example 3
- Besides representation, we will often use linear models since they are simple and computationally feasible

Outline

- Review of Linear Algebra
  - vectors and matrices
  - products and norms
  - vector spaces and linear transformations
  - eigenvalues and eigenvectors
- Introduction to Matlab

Vectors

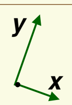


- n-dimensional row vector $x = [x_1 \; x_2 \; \ldots \; x_n]$
- Transpose of row vector is column vector $x^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
- *Vector* product (or *inner* or *dot* product)

$$\langle x, y \rangle = x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{i=1..k} x_i y_i$$

More on Vectors

- *Euclidian norm* or *length* $|\boldsymbol{x}| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\sum_{i=1\ldots n} \boldsymbol{x}_i^2}$
- If $|\boldsymbol{x}|=1$ we say $\boldsymbol{x}$ is *normalized* or *unit* length
- Angle θ between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ $\cos\theta = \dfrac{\boldsymbol{x}^T \boldsymbol{y}}{|\boldsymbol{x}||\boldsymbol{y}|}$



$\cos\theta = 0$

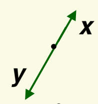
$\boldsymbol{x}^T\boldsymbol{y} = 0$

x orthogonal to y

$\boldsymbol{x} \perp \boldsymbol{y}$

$\cos\theta = 1$

$\boldsymbol{x}^T\boldsymbol{y} = |\boldsymbol{x}||\boldsymbol{y}| > 0$



$\cos\theta = -1$

$\boldsymbol{x}^T\boldsymbol{y} = -|\boldsymbol{x}||\boldsymbol{y}| < 0$

- Thus inner product captures direction relationship between $\boldsymbol{x}$ and $\boldsymbol{y}$

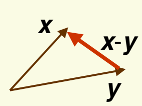
Linear Dependence and Independence

- Vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly dependent if there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ s.t.
  1. $\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \ldots + \alpha_n \boldsymbol{x}_n = \boldsymbol{0}$
  2. at least one $\alpha_i \neq 0$
- Vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly independent if $\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \ldots + \alpha_n \boldsymbol{x}_n = \boldsymbol{0} \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$

More on Vectors

- Vectors x and y are orthonormal if they are orthogonal and |x|=|y|=1
- Euclidian distance between vectors x and y

$$|\boldsymbol{x} - \boldsymbol{y}| = \sqrt{\sum_{i=1\ldots n} (\boldsymbol{x}_i - \boldsymbol{y}_i)^2}$$



Vector Spaces and Basis

- The set of all n-dimensional vectors is called a vector space $\boldsymbol{V}$
- A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ are called a basis for vector space if any $\boldsymbol{v}$ in $\boldsymbol{V}$ can be written as $\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \ldots + \alpha_n \boldsymbol{u}_n$
- $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are independent implies they form a basis, and vice versa
- $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ give an orthonormal basis if
  1. $|\boldsymbol{u}_i| = 1 \quad \forall i$
  2. $\boldsymbol{u}_i \perp \boldsymbol{u}_j \quad \forall i \neq j$

Matrices

- n by m matrix A and its m by n transpose A^T

$$A=\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{bmatrix} \qquad A^T=\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ x_{1m} & x_{2m} & \cdots & x_{nm} \end{bmatrix}$$

Matrices

- **Rank** of a matrix is the number of linearly independent rows (or equivalently columns)
- A square matrix is **non-singular** if its rank equal to the number of rows. If its rank is less than number of rows it is **singular**.
- **Identity matrix** $I=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  AI=IA=A
- Matrix ***A*** is **symmetric** if $A=A^T$ $\begin{bmatrix} 1 & 2 & 9 & 5 \\ 2 & 7 & 4 & 8 \\ 9 & 4 & 3 & 6 \\ 5 & 8 & 6 & 4 \end{bmatrix}$

Matrix Product

$$AB=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1d} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nd} \end{bmatrix}\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ b_{31} & \cdots & b_{3m} \\ \vdots & \cdots & \vdots \\ b_{d1} & \cdots & b_{dm} \end{bmatrix}=\begin{bmatrix} & & \\ & c_{ij} & \\ & & \end{bmatrix}=C$$

$$c_{ij}=\left\langle a^i, b_j \right\rangle$$

a^i is row ***i*** of ***A***
b_j is column ***j*** of ***B***

- # of columns of A = # of rows of B
- even if defined, in general $AB \neq BA$

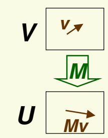
Matrices

- Matrix ***A*** is **positive definite** if
$$x^T A x = \sum_{i,j} A_{i,j} x_i x_j > 0$$
- Matrix ***A*** is **positive semi-definite** if
$$x^T A x = \sum_{i,j} A_{i,j} x_i x_j \geq 0$$
- Trace of a square matrix ***A*** is sum on the elements on the diagonal
$$tr[A]=\sum_{i=1}^{n} a_{ii}$$

Matrices

- **Inverse** of a square matrix $\boldsymbol{A}$ is matrix $\boldsymbol{A}^{-1}$ s.t. $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$
- If $\boldsymbol{A}$ is singular or not square, inverse does not exist. Pseudo-inverse $\boldsymbol{A}^{\dagger}$ is defined whenever $\boldsymbol{A}^T\boldsymbol{A}$ is not singular (it is square)
  - $\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$
  - $\boldsymbol{A}^{\dagger}\boldsymbol{A} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$

Linear Transformations



- A linear transformation from vector space $\boldsymbol{V}$ to vector space $\boldsymbol{U}$ is a mapping which can be represented by a matrix $\boldsymbol{M}$:
  - $\boldsymbol{u} = \boldsymbol{M}\boldsymbol{v}$
- If U and V have the same dimension, $\boldsymbol{M}$ is a square matrix
- In pattern recognition, often $\boldsymbol{U}$ has smaller dimensionality than $\boldsymbol{V}$, i.e. transformation $\boldsymbol{M}$ is used to reduce the number of features.

$$\boxed{\boldsymbol{M}}\ \boxed{\boldsymbol{v}} = \boxed{\boldsymbol{u}}$$

Matrices

- Determinant of n by n matrix $\boldsymbol{A}$ is

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+i} a_{ik} \det(A_{ik})$$

  - Where $\boldsymbol{A}_{ik}$ obtained from $\boldsymbol{A}$ by removing the ith row and kth column
- Absolute value of determinant gives the volume of parallelepiped spanned by the matrix rows

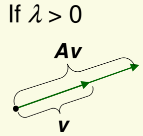
$$\{\beta_1 a^1 + \beta_2 a^2 + \ldots + \beta_n a^n\}$$

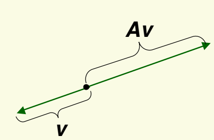
$$\beta_i \in [0,1] \quad \forall i$$

Eigenvectors and Eigenvalues

Note: A0=λ0 for any λ, not interesting

- Given n by n matrix $\boldsymbol{A}$, and nonzero vector $\boldsymbol{x}$. Suppose there is λ which satisfies $\boxed{\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}}$
  - $\boldsymbol{x}$ is called an eigenvector of $\boldsymbol{A}$
  - λ is called an eigenvalue of $\boldsymbol{A}$
- Linear transformation $\boldsymbol{A}$ maps an eigenvector $\boldsymbol{v}$ in a simple way. Magnitude changes by λ, direction
  - If $\lambda > 0$
  - If $\lambda < 0$





Eigenvectors and Eigenvalues

- If ***A*** is real and symmetric, then all eigenvalues are real (not complex)
- If ***A*** is non singular, all eigenvalues are non zero
- If ***A*** is positive definite, all eigenvalues are positive

- Starting matlab
  - xterm -fn 12X24
  - matlab
- Basic Navigation
  - quit
  - more
  - help general
- Scalars, variables, basic arithmetic
  - Clear
  - + - * / ^
  - help arith
- Relational operators
  - ==,&,|,~,xor
  - help relop
- Lists, vectors, matrices
  - A=[2 3;4 5]
  - A'
- Matrix and vector operations
  - find(A>3), colon operator
  - * / ^ .* ./ .^
  - eye(n),norm(A),det(A),eig(A)
  - max,min,std
  - help matfun
- Elementary functions
  - help elfun
- Data types
  - double
  - Char
- Programming in Matlab
  - .m files
  - scripts
  - function y=square(x)
  - help lang
- Flow control
  - if i== 1else end, if else if end
  - for i=1:0.5:2 … end
  - while i == 1 … end
  - Return
  - help lang
- Graphics
  - help graphics
  - help graph3d
- File I/O
  - load,save
  - fopen, fclose, fprintf, fscanf

MATLAB