CS434b/654b : Pattern Recognition
Prof. Olga Veksler

Lecture 9

Linear Discriminant Functions

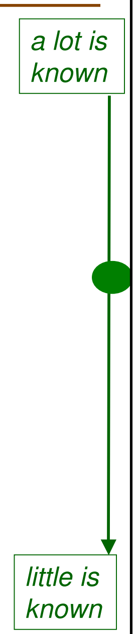
Announcements

- Final project proposal due March 8
  - 1-2 paragraph description
- Final project progress report
  - Meet with me the week of March 20-24
- Final project due April 11

Today

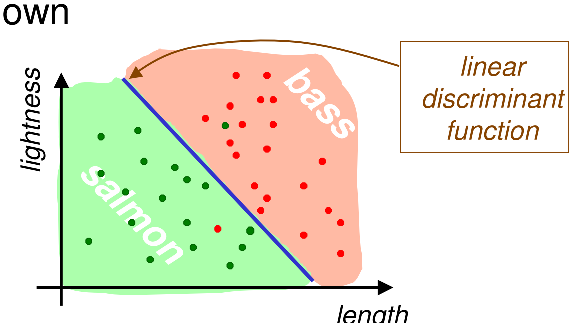
- Linear Discriminant Functions
  - Introduction
  - 2 classes
  - Multiple classes
  - Optimization with gradient descent
  - Perceptron Criterion Function
    - Batch perceptron rule
    - Single sample perceptron rule

Linear discriminant functions on Road Map



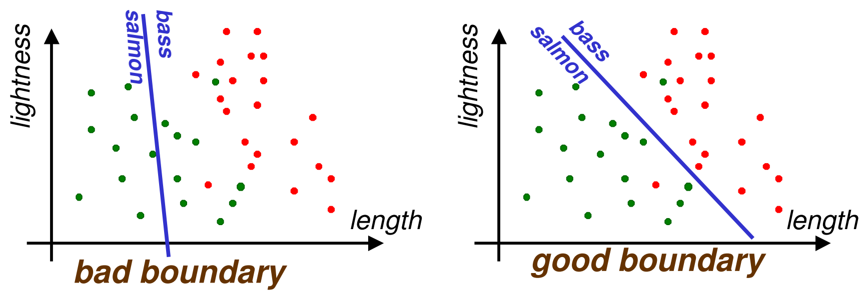
- No probability distribution (no shape or parameters are known)
- Labeled data 

- The shape of discriminant functions is known



- Need to estimate parameters of the discriminant function (parameters of the line in case of linear discriminant)

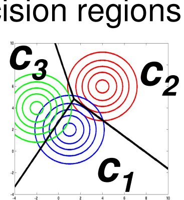
Linear Discriminant Functions: Basic Idea

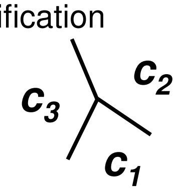


- Have samples from 2 classes $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$
- Assume 2 classes can be separated by a linear boundary $\mathbf{l}(\theta)$ with some unknown parameters θ
- Fit the "best" boundary to data by optimizing over parameters θ
- What is best?
  - Minimize classification error on training data?
    - Does not guarantee small testing error

Parametric Methods vs. Discriminant Functions

Parametric Methods	Discriminant Functions
Assume the shape of density for classes is known $p_1(\mathbf{x}\|\theta_1)$, $p_2(\mathbf{x}\|\theta_2)$,…	Assume discriminant functions are or known shape $\mathbf{l}(\theta_1)$, $\mathbf{l}(\theta_2)$, with parameters θ_1, θ_2,…
Estimate θ_1, θ_2,… from data	Estimate θ_1, θ_2,… from data
Use a Bayesian classifier to find decision regions	Use discriminant functions for classification





- In theory, Bayesian classifier minimizes the risk
  - In practice, do not have confidence in assumed model shapes
  - In practice, do not really need the actual density functions in the end
- Estimating accurate density functions is much harder than estimating accurate discriminant functions
- Some argue that estimating densities should be skipped
  - Why solve a harder problem than needed ?

LDF: Introduction

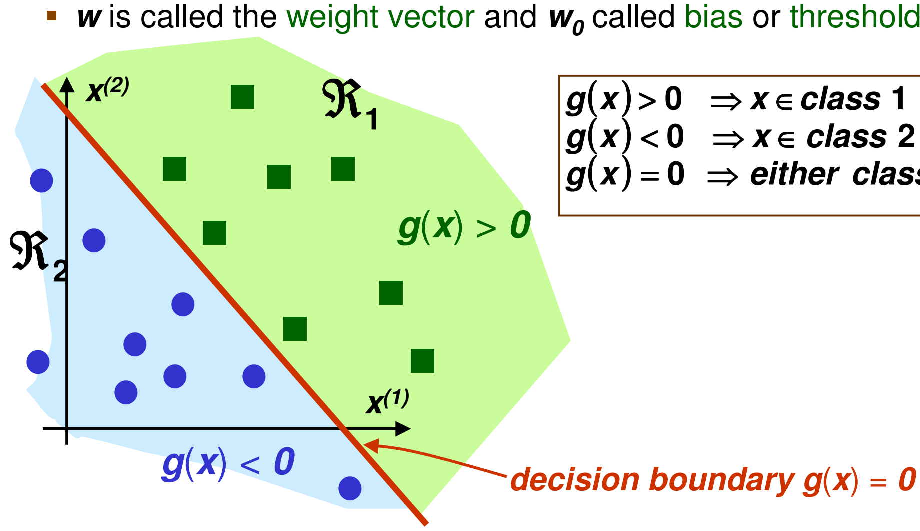
- Discriminant functions can be more general than linear
- For now, we will study linear discriminant functions
  - Simple model (should try simpler models first)
  - Analytically tractable
- Linear Discriminant functions are optimal for Gaussian distributions with equal covariance
- May not be optimal for other data distributions, but they are very simple to use
- Knowledge of class densities is not required when using linear discriminant functions
  - we can say that this is a non-parametric approach

LDF: 2 Classes

- A discriminant function is linear if it can be written as

$$g(x) = w^t x + w_0$$

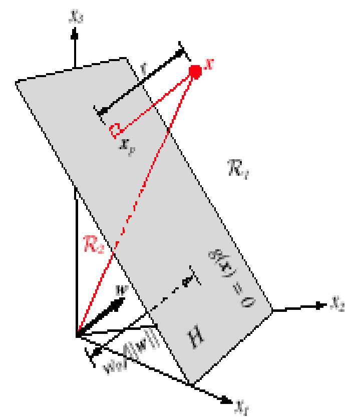
  - w is called the weight vector and w_0 called bias or threshold



$$\begin{aligned} g(x) > 0 &\Rightarrow x \in \textit{class } 1 \\ g(x) < 0 &\Rightarrow x \in \textit{class } 2 \\ g(x) = 0 &\Rightarrow \textit{either class} \end{aligned}$$

LDF: 2 Classes

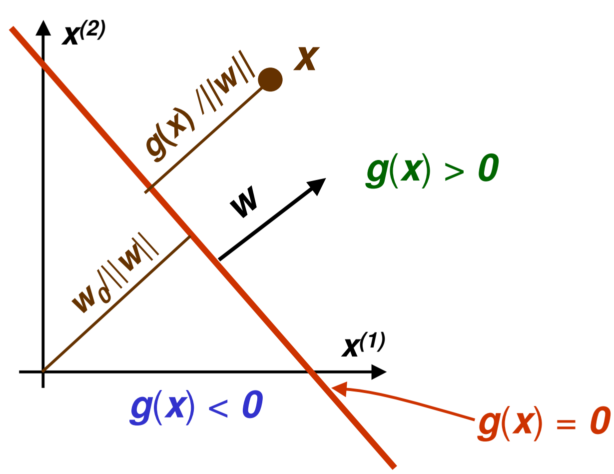
- Decision boundary $g(\mathbf{x}) = \mathbf{w}^t\mathbf{x} + w_0 = 0$ is a ***hyperplane***
  - set of vectors x which for some scalars $\alpha_0,\ldots,\alpha_d$ satisfy $\alpha_0 + \alpha_1 x^{(1)} + \ldots + \alpha_d x^{(d)} = 0$
- A hyperplane is
  - a point in 1D
  - a line in 2D
  - a plane in 3D



LDF: 2 Classes

$$g(\mathbf{x}) = \mathbf{w}^t\mathbf{x} + w_0$$

- $\mathbf{w}$ determines orientation of the decision hyperplane
- w_0 determines location of the decision surface



LDF: 2 Classes

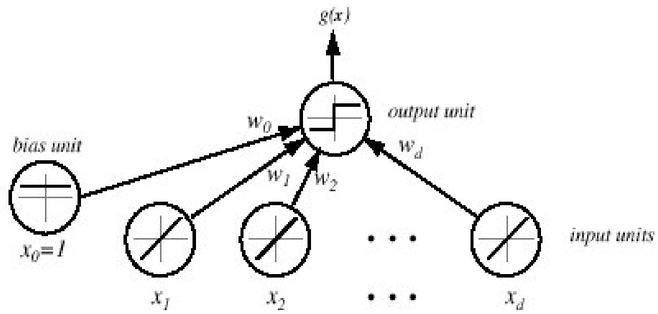


FIGURE 5.1. A simple linear classifier having d input units, each corresponding to the values of the components of an input vector. Each input feature value x_i is multiplied by its corresponding weight w_i; the effective input at the output unit is the sum all these products, $\sum w_i x_i$. We show in each unit its effective input-output function. Thus each of the d input units is linear, emitting exactly the value of its corresponding feature value. The single bias unit unit always emits the constant value 1.0. The single output unit emits a +1 if $\mathbf{w}^t\mathbf{x} + w_0 > 0$ or a −1 otherwise. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

LDF: Many Classes

- Suppose we have ***m*** classes
- Define ***m*** linear discriminant functions

$$g_i(x) = w_i^t x + w_{i0} \qquad i = 1,\ldots,m$$

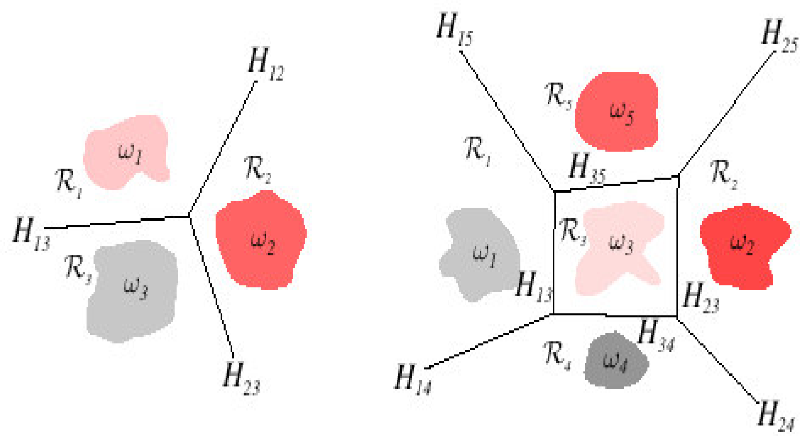
- Given ***x***, assign class c_i if

$$g_i(x) \geq g_j(x) \qquad \forall j \neq i$$

- Such classifier is called a ***linear machine***

- A linear machine divides the feature space into ***c*** decision regions, with $g_i(x)$ being the largest discriminant if ***x*** is in the region R_i

LDF: Many Classes



LDF: Many Classes

- For a two contiguous regions R_i and R_j; the boundary that separates them is a portion of hyperplane $\boldsymbol{H_{ij}}$ defined by:

$$g_i(\mathbf{x}) = g_j(\mathbf{x}) \Leftrightarrow \mathbf{w}_i^t \mathbf{x} + w_{i0} = \mathbf{w}_j^t \mathbf{x} + w_{j0}$$
$$\Leftrightarrow (\mathbf{w}_i - \mathbf{w}_j)^t \mathbf{x} + (w_{i0} - w_{j0}) = 0$$

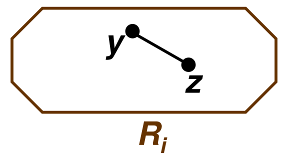
- Thus $\mathbf{w}_i - \mathbf{w}_j$ is normal to $\boldsymbol{H_{ij}}$
- And distance from $\boldsymbol{x}$ to $\boldsymbol{H_{ij}}$ is given by

$$d(\mathbf{x}, H_{ij}) = \frac{g_i(\mathbf{x}) - g_j(\mathbf{x})}{\|\mathbf{w}_i - \mathbf{w}_j\|}$$

LDF: Many Classes

- Decision regions for a linear machine are ***convex***

$$y, z \in R_i \Rightarrow \alpha y + (1-\alpha) z \in R_i$$



$$\forall j \neq i \quad g_i(y) \geq g_j(y) \text{ and } g_i(z) \geq g_j(z) \Leftrightarrow$$
$$\Leftrightarrow \forall j \neq i \quad g_i(\alpha y + (1-\alpha) z) \geq g_j(\alpha y + (1-\alpha) z)$$

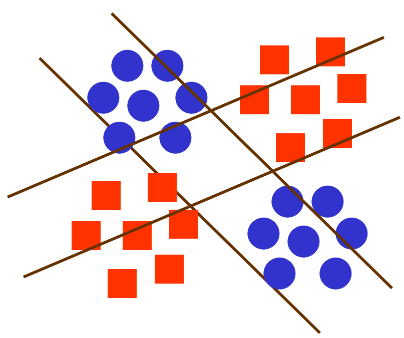
- In particular, decision regions must be spatially contiguous

R_j is a valid decision region

R_j is not a valid decision region

LDF: Many Classes

- Thus applicability of linear machine to mostly limited to unimodal conditional densities $p(x|\theta)$
  - even though we did not assume any parametric models
- Example:



  - need non-contiguous decision regions
  - thus linear machine will fail

LDF: Augmented feature vector

- Linear discriminant function: $g(x) = w^t x + w_0$
- Can rewrite it: $g(x) = \underbrace{\begin{bmatrix} w_0 & w^t \end{bmatrix}}_{\text{new weight vector } a} \underbrace{\begin{bmatrix} 1 \\ x \end{bmatrix}}_{\text{new feature vector } y} = a^t y = g(y)$
- y is called the ***augmented feature vector***
- Added a dummy dimension to get a completely equivalent new ***homogeneous*** problem

old problem

$$g(x) = w^t x + w_0$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$$

new problem

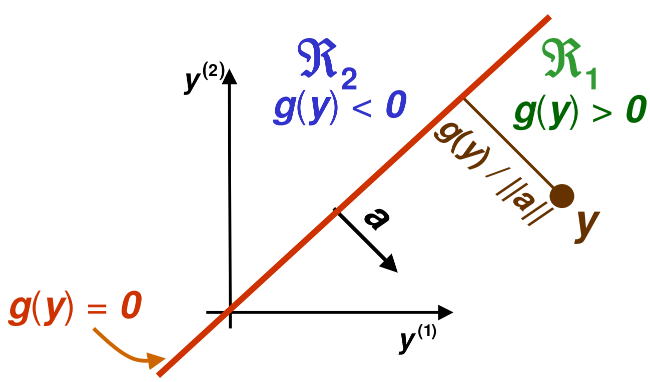
$$g(y) = a^t y$$

$$\begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_d \end{bmatrix}$$

LDF: Augmented feature vector

- Feature augmenting is done for simpler notation
- From now on we always assume that we have augmented feature vectors
  - Given samples $x_1, \ldots, x_n$ convert them to augmented samples $y_1, \ldots, y_n$ by adding a new dimension of value 1

$$y_i = \begin{bmatrix} 1 \\ x_i \end{bmatrix}$$



LDF: Training Error

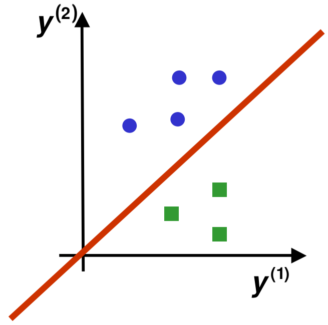
- For the rest of the lecture, assume we have 2 classes
- Samples $\boldsymbol{y_1},\ldots,\boldsymbol{y_n}$ some in class 1, some in class 2
- Use these samples to determine weights $\boldsymbol{a}$ in the discriminant function $g(\boldsymbol{y}) = \boldsymbol{a}^t\boldsymbol{y}$
- What should be our criterion for determining $\boldsymbol{a}$?
  - For now, suppose we want to minimize the training error (that is the number of misclassifed samples $\boldsymbol{y_1},\ldots,\boldsymbol{y_n}$)
- Recall that $$g(\boldsymbol{y}_i) > 0 \Rightarrow \boldsymbol{y}_i \text{ classified } c_1$$ $$g(\boldsymbol{y}_i) < 0 \Rightarrow \boldsymbol{y}_i \text{ classified } c_2$$
- Thus training error is ***0*** if $\begin{cases} g(\boldsymbol{y}_i) > 0 & \forall \boldsymbol{y}_i \in c_1 \\ g(\boldsymbol{y}_i) < 0 & \forall \boldsymbol{y}_i \in c_2 \end{cases}$

LDF: Problem "Normalization"

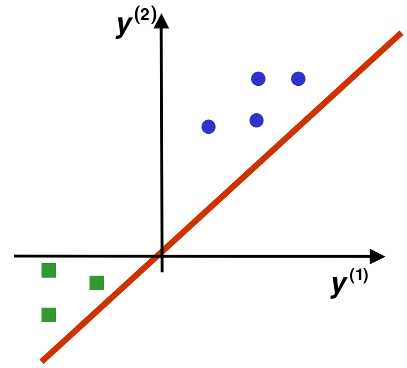
- Thus training error is ***0*** if $\begin{cases} \boldsymbol{a}^t\boldsymbol{y}_i > 0 & \forall \boldsymbol{y}_i \in c_1 \\ \boldsymbol{a}^t\boldsymbol{y}_i < 0 & \forall \boldsymbol{y}_i \in c_2 \end{cases}$
- Equivalently, training error is ***0*** if
$$\begin{cases} \boldsymbol{a}^t\boldsymbol{y}_i > 0 & \forall \boldsymbol{y}_i \in c_1 \\ \boldsymbol{a}^t(-\boldsymbol{y}_i) > 0 & \forall \boldsymbol{y}_i \in c_2 \end{cases}$$
- This suggest problem "normalization":
  1. Replace all examples from class c_2 by their negative
  $$\boldsymbol{y}_i \to -\boldsymbol{y}_i \qquad \forall \boldsymbol{y}_i \in c_2$$
  2. Seek weight vector $\boldsymbol{a}$ s.t.
  $$\boldsymbol{a}^t\boldsymbol{y}_i > 0 \qquad \forall \boldsymbol{y}_i$$
  - If such $\boldsymbol{a}$ exists, it is called a *separating* or *solution* vector
  - Original samples $\boldsymbol{x_1},\ldots,\boldsymbol{x_n}$ can indeed be separated by a line then

LDF: Problem "Normalization"

before normalization



after "normalization"

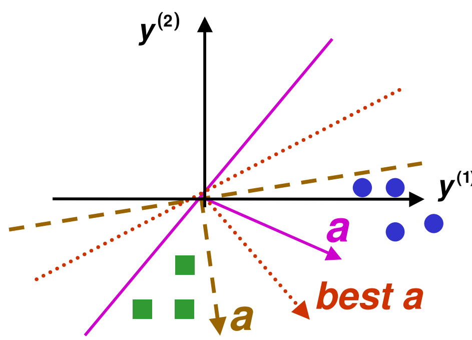


Seek a hyperplane that separates patterns from different categories ⇐ Seek hyperplane that puts *normalized* patterns on the same (positive) side

LDF: Solution Region

- Find weight vector ***a*** s.t. for all samples $\boldsymbol{y_1}, \ldots, \boldsymbol{y_n}$

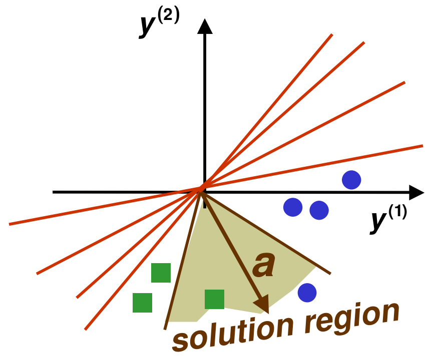
$$\boldsymbol{a}^t \boldsymbol{y}_i = \sum_{k=0}^{d} a_k y_i^{(k)} > 0$$



- In general, there are many such solutions ***a***

LDF: Solution Region

- ***Solution region*** for ***a***: set of all possible solutions
  - defined in terms of normal ***a*** to the separating hyperplane



Optimization

- Need to minimize a function of many variables

$$J(\mathbf{x}) = J(x_1, \dots, x_d)$$

- We know how to minimize $J(\mathbf{x})$
  - Take partial derivatives and set them to zero

$$\begin{bmatrix} \frac{\partial}{\partial x_1} J(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_d} J(\mathbf{x}) \end{bmatrix} = \nabla J(\mathbf{x}) = 0$$

gradient

- However solving analytically is not always easy
  - Would you like to solve this system of nonlinear equations?

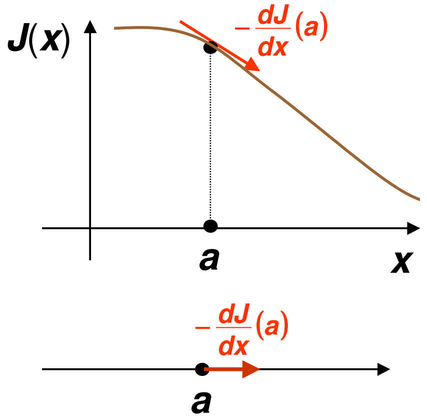
$$\begin{cases} \sin(x_1^2 + x_2^3) + e^{x_4^2} = 0 \\ \cos(x_1^2 + x_2^3) + \log(x_5^3)^{x_4^2} = 0 \end{cases}$$

  - Sometimes it is not even possible to write down an analytical expression for the derivative, we will see an example later today

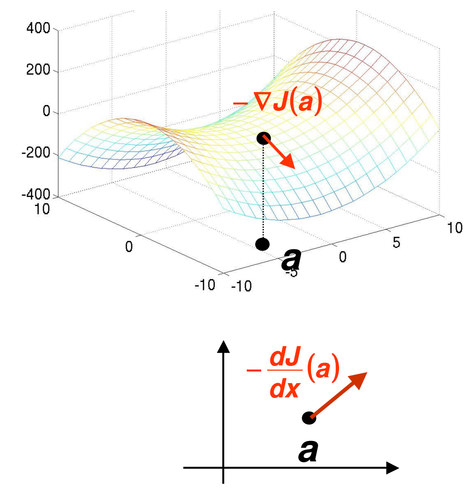
Optimization: Gradient Descent

- Gradient $\nabla J(\mathbf{x})$ points in direction of steepest increase of $J(\mathbf{x})$, and $-\nabla J(\mathbf{x})$ in direction of steepest decrease

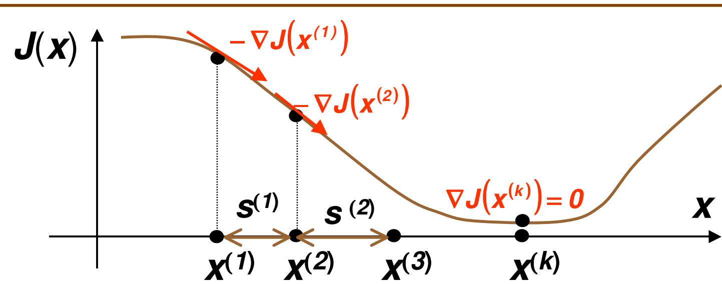
one dimension



two dimensions



Optimization: Gradient Descent



Gradient Descent for minimizing any function $J(\mathbf{x})$

set $k = 1$ and $\mathbf{x}^{(1)}$ to some initial guess for the weight vector

while $\eta^{(k)} \left| \nabla J\left(\mathbf{x}^{(k)}\right) \right| > \varepsilon$

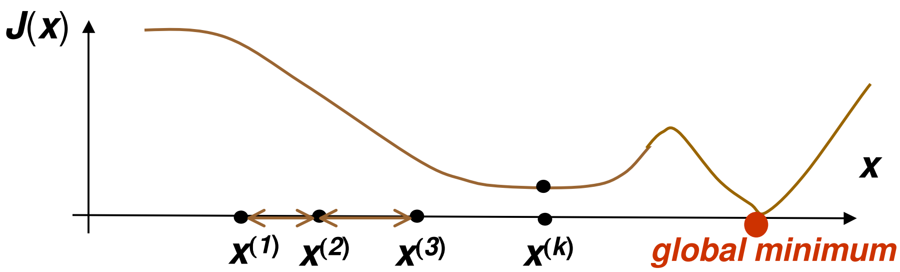
choose **learning rate** $\eta^{(k)}$

$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \eta^{(k)} \nabla J(\mathbf{x})$ (*update rule*)

$k = k + 1$

Optimization: Gradient Descent

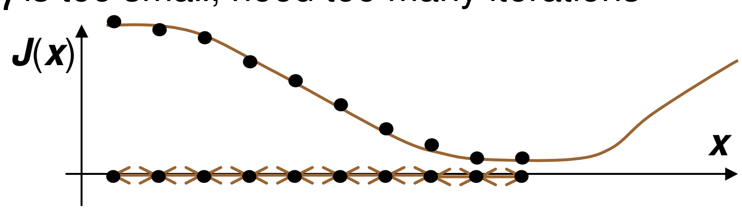
- Gradient descent is guaranteed to find only a local minimum



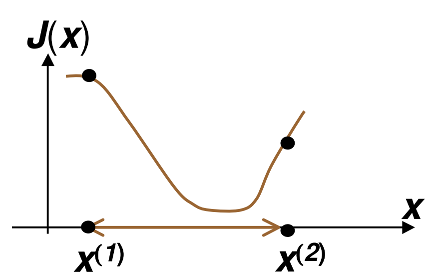
- Nevertheless gradient descent is very popular because it is simple and applicable to any function

Optimization: Gradient Descent

- Main issue: how to set parameter η (***learning rate***)
- If η is too small, need too many iterations



- If η is too large may overshoot the minimum and possibly never find it (if we keep overshooting)



Today

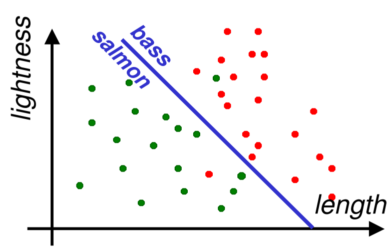
- Continue Linear Discriminant Functions
  - Perceptron Criterion Function
    - Batch perceptron rule
    - Single sample perceptron rule

LDF: Augmented feature vector

- Linear discriminant function:

$$g(x) = w^t x + w_0$$

  - need to estimate parameters w and w_0 from data



- Augment samples x to get equivalent homogeneous problem in terms of samples y:

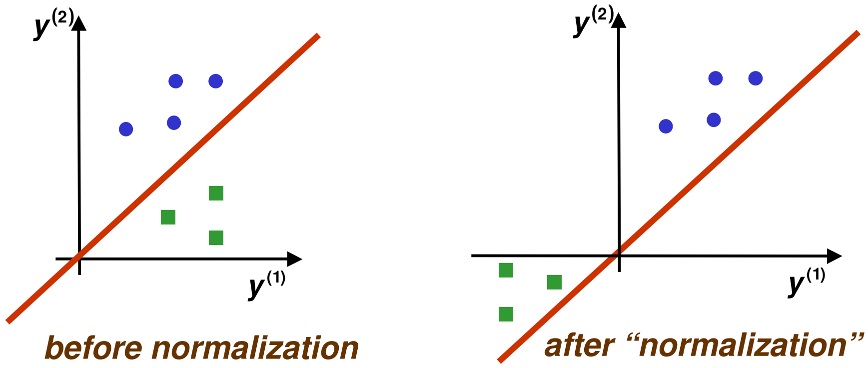
$$g(x) = \begin{bmatrix} w_0 & w^t \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = a^t y = g(y)$$

- "normalize" by replacing all examples from class c_2 by their negative

$$y_i \rightarrow -y_i \qquad \forall y_i \in c_2$$

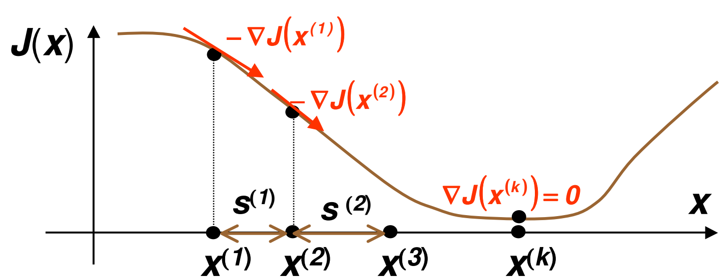
LDF

- Augmented and "normalized" samples $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$
- Seek weight vector $\boldsymbol{a}$ s.t. $\boldsymbol{a}^t \boldsymbol{y}_i > 0 \quad \forall \boldsymbol{y}_i$



- If such $\boldsymbol{a}$ exists, it is called a *separating* or *solution* vector
- original samples $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ can indeed be separated by a line then

Optimization: Gradient Descent



$$s^{(k+1)} = x^{(k+1)} - x^{(k)} = \eta^{(k)}\left(-\nabla J\left(x^{(k)}\right)\right)$$

Gradient Descent for minimizing any function $J(x)$

set $k = 1$ *and* $x^{(1)}$ to some initial guess for the weight vector

while $\eta^{(k)}\left|\nabla J\left(x^{(k)}\right)\right| > \varepsilon$

choose **learning rate** $\eta^{(k)}$

$x^{(k+1)} = x^{(k)} - \eta^{(k)} \nabla J(x)$ **(update rule)**

$k = k + 1$

LDF: Criterion Function

- Find weight vector $\boldsymbol{a}$ s.t. for all samples $\boldsymbol{y_1}, \ldots, \boldsymbol{y_n}$

$$a^t y_i = \sum_{k=0}^{d} a_k y_i^{(k)} > 0$$

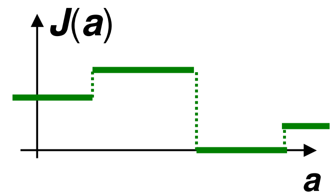
- Need criterion function $\boldsymbol{J}(\boldsymbol{a})$ which is minimized when $\boldsymbol{a}$ is a solution vector
- Let $\boldsymbol{Y_M}$ be the set of examples misclassified by $\boldsymbol{a}$

$$Y_M(a) = \{ \text{sample } y_i \text{ s.t. } a^t y_i < 0 \}$$

- First natural choice: number of misclassified examples

$$J(a) = |Y_M(a)|$$

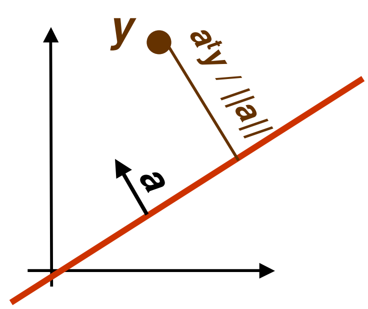
  - piecewise constant, gradient descent is useless



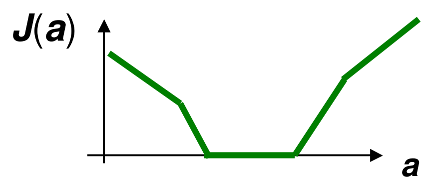
LDF: Perceptron Criterion Function

- Better choice: ***Perceptron*** criterion function

$$J_p(a) = \sum_{y \in Y_M} (-a^t y)$$



- If $\boldsymbol{y}$ is misclassified, $a^t y \le 0$
- Thus $J_p(a) \ge 0$
- $\boldsymbol{J_p}(\boldsymbol{a})$ is -||$\boldsymbol{a}$|| times sum of distances of misclassified examples to decision boundary
- $\boldsymbol{J_p}(\boldsymbol{a})$ is piecewise linear and thus suitable for gradient descent



LDF: Perceptron Batch Rule

$$J_p(\mathbf{a}) = \sum_{y \in Y_M} \left(-\mathbf{a}^t \mathbf{y}\right)$$

- Gradient of $J_p(\mathbf{a})$ is $\nabla J_p(\mathbf{a}) = \sum_{y \in Y_M} (-\mathbf{y})$
  - Y_M are samples misclassified by $\mathbf{a}^{(k)}$
  - It is not possible to solve $\nabla J_p(\mathbf{a}) = 0$ analytically because of Y_M
- Update rule for gradient descent: $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \eta^{(k)} \nabla J(\mathbf{x})$
- Thus *gradient decent batch update rule* for $J_p(\mathbf{a})$ is:

$$\mathbf{a}^{(k+1)} = \mathbf{a}^{(k)} + \eta^{(k)} \sum_{y \in Y_M} \mathbf{y}$$

- It is called batch rule because it is based on all misclassified examples

LDF: Perceptron Single Sample Rule

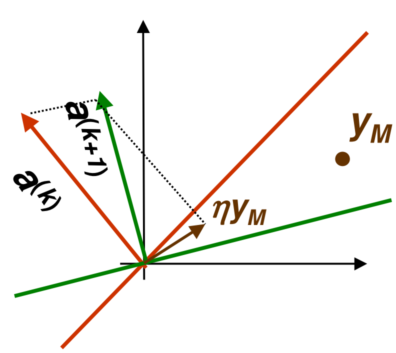
- Thus *gradient decent single sample rule* for $J_p(\mathbf{a})$ is:

$$\mathbf{a}^{(k+1)} = \mathbf{a}^{(k)} + \eta^{(k)} \mathbf{y}_M$$

  - note that y_M is one sample misclassified by $\mathbf{a}^{(k)}$
  - must have a consistent way of visiting samples
- Geometric Interpretation:
  - y_M misclassified by $\mathbf{a}^{(k)}$

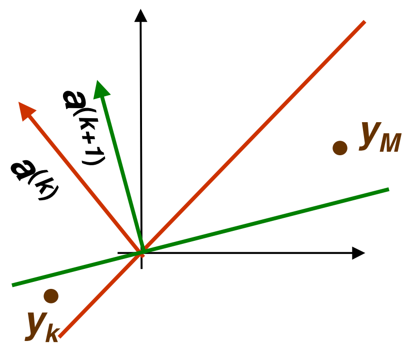
    $$\left(\mathbf{a}^{(k)}\right)^t \mathbf{y}_M \le 0$$

  - y_M is on the wrong side of decision hyperplane
  - adding $\eta \mathbf{y}_M$ to $\mathbf{a}$ moves new decision hyperplane in the right direction with respect to $\mathbf{y}_M$

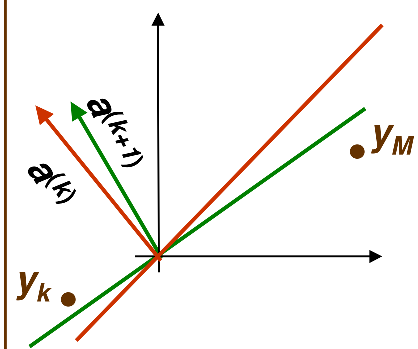


LDF: Perceptron Single Sample Rule

$$a^{(k+1)} = a^{(k)} + \eta^{(k)} y_M$$



η is too large, previously correctly classified sample y_k is now misclassified



η is too small, y_M is still misclassified

LDF: Perceptron Example

	features				grade
name	***good attendance?***	***tall?***	***sleeps in class?***	***chews gum?***	
Jane	*yes (1)*	*yes (1)*	*no (-1)*	*no (-1)*	*A*
Steve	*yes (1)*	*yes (1)*	*yes (1)*	*yes (1)*	*F*
Mary	*no (-1)*	*no (-1)*	*no (-1)*	*yes (1)*	*F*
Peter	*yes (1)*	*no (-1)*	*no (-1)*	*yes (1)*	*A*

- ***class 1***: students who get grade *A*
- ***class 2***: students who get grade *F*

LDF Example: Augment feature vector

	features					grade
name	***extra***	***good attendance?***	***tall?***	***sleeps in class?***	***chews gum?***	
Jane	*1*	*yes (1)*	*yes (1)*	*no (-1)*	*no (-1)*	*A*
Steve	*1*	*yes (1)*	*yes (1)*	*yes (1)*	*yes (1)*	*F*
Mary	*1*	*no (-1)*	*no (-1)*	*no (-1)*	*yes (1)*	*F*
Peter	*1*	*yes (1)*	*no (-1)*	*no (-1)*	*yes (1)*	*A*

- convert samples $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to augmented samples $\mathbf{y}_1, \ldots, \mathbf{y}_n$ by adding a new dimension of value 1

LDF: Perform "Normalization"

	features					grade
name	***extra***	***good attendance?***	***tall?***	***sleeps in class?***	***chews gum?***	
Jane	*1*	*yes (1)*	*yes (1)*	*no (-1)*	*no (-1)*	*A*
Steve	*-1*	*yes (-1)*	*yes (-1)*	*yes (-1)*	*yes (-1)*	*F*
Mary	*-1*	*no (1)*	*no (1)*	*no (1)*	*yes (-1)*	*F*
Peter	*1*	*yes (1)*	*no (-1)*	*no (-1)*	*yes (1)*	*A*

- Replace all examples from class c_2 by their negative

$$\mathbf{y}_i \rightarrow -\mathbf{y}_i \qquad \forall \mathbf{y}_i \in c_2$$

- Seek weight vector $\mathbf{a}$ s.t. $\mathbf{a}^t \mathbf{y}_i > 0 \qquad \forall \mathbf{y}_i$

LDF: Use Single Sample Rule

	features					grade
name	***extra***	***good attendance?***	***tall?***	***sleeps in class?***	***chews gum?***	
Jane	*1*	*yes (1)*	*yes (1)*	*no (-1)*	*no (-1)*	*A*
Steve	*-1*	*yes (-1)*	*yes (-1)*	*yes (-1)*	*yes (-1)*	*F*
Mary	*-1*	*no (1)*	*no (1)*	*no (1)*	*yes (-1)*	*F*
Peter	*1*	*yes (1)*	*no (-1)*	*no (-1)*	*yes (1)*	*A*

- Sample is misclassified if $a^t y_i = \sum_{k=0}^{4} a_k y_i^{(k)} < 0$
- gradient descent single sample rule: $a^{(k+1)} = a^{(k)} + \eta^{(k)} y_M$
- Set ***fixed*** learning rate to $\eta^{(k)} = 1$: $a^{(k+1)} = a^{(k)} + y_M$

LDF: Gradient decent Example

- set equal initial weights $a^{(1)}$=[0.25, 0.25, 0.25, 0.25]
- visit all samples sequentially, modifying the weights for after finding a misclassified example

name	$a^t y$	***misclassified?***
Jane	*0.25*1+0.25*1+0.25*1+0.25*(-1)+0.25*(-1) >0*	*no*
Steve	*0.25*(-1)+0.25*(-1)+0.25*(-1)+0.25*(-1)+0.25*(-1)<0*	*yes*

- new weights

$$a^{(2)} = a^{(1)} + y_M = [0.25\ \ 0.25\ \ 0.25\ \ 0.25\ \ 0.25] + \\ + [-1\ \ -1\ \ -1\ \ -1\ \ -1] = \\ = [-0.75\ \ -0.75\ \ -0.75\ \ -0.75\ \ -0.75]$$

LDF: Gradient decent Example

$$a^{(2)} = [-0.75 \ \ -0.75 \ \ -0.75 \ \ -0.75 \ \ -0.75]$$

name	$a^t y$	misclassified?
Mary	-0.75*(-1)-0.75*1 -0.75 *1 -0.75 *1 -0.75*(-1) <0	yes

- new weights

$$a^{(3)} = a^{(2)} + y_M = [-0.75 \ \ -0.75 \ \ -0.75 \ \ -0.75 \ \ -0.75] +$$
$$+ [-1 \ \ 1 \ \ 1 \ \ 1 \ \ -1] =$$
$$= [-1.75 \ \ 0.25 \ \ 0.25 \ \ 0.25 \ \ -1.75]$$

LDF: Gradient decent Example

$$a^{(3)} = [-1.75 \ \ 0.25 \ \ 0.25 \ \ 0.25 \ \ -1.75]$$

name	$a^t y$	misclassified?
Peter	-1.75 *1 +0.25* 1+0.25* (-1) +0.25 *(-1)-1.75*1 <0	yes

- new weights

$$a^{(4)} = a^{(3)} + y_M = [-1.75 \ \ 0.25 \ \ 0.25 \ \ 0.25 \ \ -1.75] +$$
$$+ [1 \ \ 1 \ \ -1 \ \ -1 \ \ 1] =$$
$$= [-0.75 \ \ 1.25 \ \ -0.75 \ \ -0.75 \ \ -0.75]$$

LDF: Gradient decent Example

$$a^{(4)} = [-0.75 \;\; 1.25 \;\; -0.75 \;\; -0.75 \;\; -0.75]$$

name	$a^t y$	*misclassified?*
Jane	*-0.75 *1 +1.25*1 -0.75*1 -0.75 *(-1) -0.75 *(-1)+0*	*no*
Steve	*-0.75*(-1)+1.25*(-1) -0.75*(-1) -0.75*(-1)-0.75*(-1)>0*	*no*
Mary	*-0.75 *(-1)+1.25*1-0.75*1 -0.75 *1 –0.75*(-1) >0*	*no*
Peter	*-0.75 *1+ 1.25*1-0.75* (-1)-0.75* (-1) -0.75 *1 >0*	*no*

- Thus the discriminant function is

$$g(y) = -0.75 * y^{(0)} + 1.25 * y^{(1)} - 0.75 * y^{(2)} - 0.75 * y^{(3)} - 0.75 * y^{(4)}$$

- Converting back to the original features ***x***:

$$g(x) = 1.25 * x^{(1)} - 0.75 * x^{(2)} - 0.75 * x^{(3)} - 0.75 * x^{(4)} - 0.75$$

LDF: Gradient decent Example

- Converting back to the original features ***x***:

$$1.25 * x^{(1)} - 0.75 * x^{(2)} - 0.75 * x^{(3)} - 0.75 * x^{(4)} > 0.75 \Rightarrow grade\ A$$
$$1.25 * x^{(1)} - 0.75 * x^{(2)} - 0.75 * x^{(3)} - 0.75 * x^{(4)} < 0.75 \Rightarrow grade\ F$$

↙ good attendance ↙ tall ↘ sleeps in class ↘ chews gum

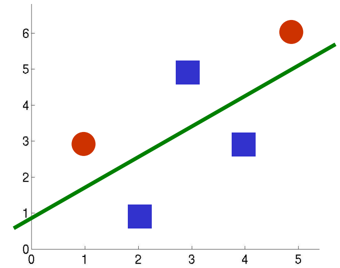
- This is just one possible solution vector
- If we started with weights $a^{(1)}$=[0,0.5, 0.5, 0, 0], solution would be [-1,1.5, -0.5, -1, -1]

$$1.5 * x^{(1)} - 0.5 * x^{(2)} - x^{(3)} - x^{(4)} > 1 \Rightarrow grade\ A$$
$$1.5 * x^{(1)} - 0.5 * x^{(2)} - x^{(3)} - x^{(4)} < 1 \Rightarrow grade\ F$$

  - In this solution, being tall is the least important feature

LDF: Nonseparable Example

- Suppose we have 2 features and samples are:
  - Class 1: [2,1], [4,3], [3,5]
  - Class 2: [1,3] and [5,6]
- These samples are not separable by a line

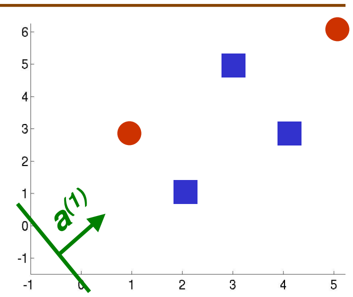


- Still would like to get approximate separation by a line, good choice is shown in green
  - some samples may be "noisy", and it's ok if they are on the wrong side of the line
- Get y_1, y_2, y_3, y_4 by adding extra feature and "normalizing"

$$y_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad y_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad y_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad y_4 = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} \quad y_5 = \begin{bmatrix} -1 \\ -5 \\ -6 \end{bmatrix}$$

LDF: Nonseparable Example

- Let's apply Perceptron single sample algorithm
- initial equal weights $a^{(1)} = [1\ 1\ 1]$
  - this is line $x^{(1)}+x^{(2)}+1=0$
- fixed learning rate $\eta = 1$



$$a^{(k+1)} = a^{(k)} + y_M$$

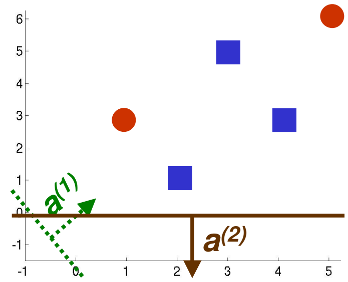
$$y_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad y_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad y_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad y_4 = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} \quad y_5 = \begin{bmatrix} -1 \\ -5 \\ -6 \end{bmatrix}$$

- $y^t_1 a^{(1)} = [1\ 1\ 1]*[1\ 2\ 1]^t > 0$ ✔
- $y^t_2 a^{(1)} = [1\ 1\ 1]*[1\ 4\ 3]^t > 0$ ✔
- $y^t_3 a^{(1)} = [1\ 1\ 1]*[1\ 3\ 5]^t > 0$ ✔

LDF: Nonseparable Example

$a^{(1)} = [1\ 1\ 1]$ $\quad a^{(k+1)} = a^{(k)} + y_M$

$y_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$ $y_2 = \begin{bmatrix}1\\4\\3\end{bmatrix}$ $y_3 = \begin{bmatrix}1\\3\\5\end{bmatrix}$ $y_4 = \begin{bmatrix}-1\\-1\\-3\end{bmatrix}$ $y_5 = \begin{bmatrix}-1\\-5\\-6\end{bmatrix}$



- $y^t_4 a^{(1)}$=[1 1 1]*[-1 -1 -3]^t = -5< 0

$$a^{(2)} = a^{(1)} + y_M = [1\ 1\ 1] + [-1\ -1\ -3] = [0\ 0\ -2]$$

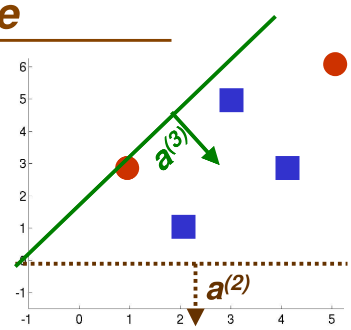
- $y^t_5\, a^{(2)}$=[0 0 -2]*[-1 -5 -6]^t = 12 > 0 ✔
- $y^t_1\, a^{(2)}$=[0 0 -2]*[1 2 1]^t < 0

$$a^{(3)} = a^{(2)} + y_M = [0\ 0\ -2] + [1\ 2\ 1] = [1\ 2\ -1]$$

LDF: Nonseparable Example

$a^{(3)} = [1\ 2\ -1]$ $\quad a^{(k+1)} = a^{(k)} + y_M$

$y_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$ $y_2 = \begin{bmatrix}1\\4\\3\end{bmatrix}$ $y_3 = \begin{bmatrix}1\\3\\5\end{bmatrix}$ $y_4 = \begin{bmatrix}-1\\-1\\-3\end{bmatrix}$ $y_5 = \begin{bmatrix}-1\\-5\\-6\end{bmatrix}$



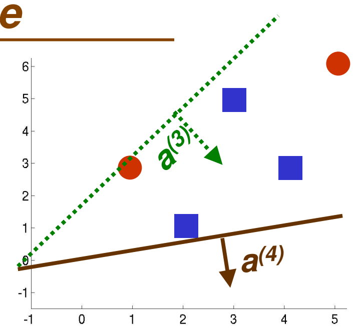
- $y^t_2\, a^{(3)}$=[1 4 3]*[1 2 -1]^t=6 > 0 ✔
- $y^t_3\, a^{(3)}$=[1 3 5]*[1 2 -1]^t > 0 ✔
- $y^t_4\, a^{(3)}$=[-1 -1 -3]*[1 2 -1]^t = 0

$$a^{(4)} = a^{(3)} + y_M = [1\ 2\ -1] + [-1\ -1\ -3] = [0\ 1\ -4]$$

LDF: Nonseparable Example

$a^{(4)} = [0 \;\; 1 \;\; -4]$ $\qquad a^{(k+1)} = a^{(k)} + y_M$

$y_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad y_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad y_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad y_4 = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} \quad y_5 = \begin{bmatrix} -1 \\ -5 \\ -6 \end{bmatrix}$



- $y^t_2\, a^{(3)}$=[1 4 3]*[1 2 -1]^t=6 > 0 ✔
- $y^t_3\, a^{(3)}$=[1 3 5]*[1 2 -1]^t > 0 ✔
- $y^t_4\, a^{(3)}$=[-1 -1 -3]*[1 2 -1]^t = 0

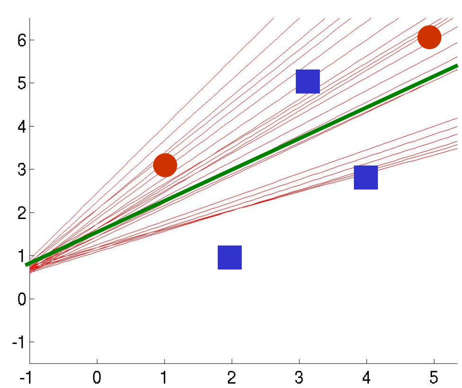
$$a^{(4)} = a^{(3)} + y_M = [1 \;\; 2 \;\; -1] + [-1 \;\; -1 \;\; -3] = [0 \;\; 1 \;\; -4]$$

LDF: Nonseparable Example

- we can continue this forever
  - there is no solution vector ***a*** satisfying for all ***i***

$$a^t y_i = \sum_{k=0}^{5} a_k y_i^{(k)} > 0$$

- need to stop but at a good point:



- solutions at iterations 900 through 915. Some are good some are not.
- How do we stop at a good solution?

LDF: Convergence of Perceptron rules

- If classes are linearly separable, and use fixed learning rate, that is for some constant ***c***, $\eta^{(k)} = c$
  - *both single sample and batch perceptron rules converge to a correct solution* (could be any ***a*** in the solution space)
- If classes are not linearly separable:
  - algorithm does not stop, it keeps looking for solution which does not exist
  - by choosing appropriate learning rate, can always ensure convergence: $\eta^{(k)} \to 0 \text{ as } k \to \infty$
  - for example inverse linear learning rate: $\eta^{(k)} = \frac{\eta^{(1)}}{k}$
  - for inverse linear learning rate convergence in the linearly separable case can also be proven
  - no guarantee that we stopped at a good point, but there are good reasons to choose inverse linear learning rate

LDF: Perceptron Rule and Gradient decent

- Linearly separable data
  - perceptron rule with gradient decent works well
- Linearly non-separable data
  - need to stop perceptron rule algorithm at a good point, this maybe tricky

Batch Rule	***Single Sample Rule***
▪ Smoother gradient because all samples are used	▪ easier to analyze ▪ Concentrates more than necessary on any isolated "noisy" training examples