On the Complexity and Parallel Implementation of Hensel's Lemma and Weierstrass Preparation

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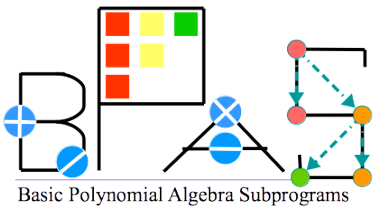

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The Problem

Given a polynomial $f \in \mathbb{K}[X_1, \ldots, X_n, Y]$ compute the roots of f in Y.

- → The classical Newton–Puiseux method is known to compute the roots of $f \in \mathbb{K}[X_1, Y]$ in Y as Puiseux series in X
- → The **Hensel–Sasaki Construction** or **Extended Hensel Construction** supports multivariate polynomials, computing the roots of f in Y as multivariate Puiseux series [12]
- → If f is monic, roots in Y can be computed as multivariate *power series*; we call such a method **Hensel Factorization**

Applications:

- → Computing limits of multivariate rational functions [2]
- → Computing topological closures, limit points of quasi-components [1]
- → An effective variant of the Abhyankar-Jung Theorem

Goals and Previous Work

Our goal is to improve the practical performance of the Hensel–Sasaki Construction and its applications.

→ First step is improving the monic case of Hensel factorization.

Practical efficiency:

→ High-performance and memory-efficient underlying polynomial arithmetic [4]

→ Lazy power series supporting lazy Weierstrass preparation and lazy Hensel factorization [6]

→ Now: **parallel and concurrent processing**

→ Now: dynamic load-balancing using **complexity estimates**

In This Work

- → Examine the complexity of a lazy scheme for Weierstrass Preparation Theorem and Hensel factorization
- → Present parallel algorithms for WPT and Hensel factorization
- → Composition of parallel schemes, WPT within Hensel factorization
- → Application of complexity estimates to dynamically load-balance parallel algorithm
- → Implementation of parallel algorithms supporting arbitrary number of variables in the BPAS library [3]
- → Experimental evaluation of parallel performance

Outline

Notations

$\mathbb{A} = \mathbb{K}[[X_1, \ldots, X_n]]$ is the ring of multivariate power series over an algebraically closed field. Its maximal ideal is $\mathcal{M} = \langle X_1, \ldots, X_n \rangle$.

- $f = \sum_{e \in \mathbb{N}^n} a_e X^e \in \mathbb{K}[[X_1, \ldots, X_n]]$
- $X^e = X_1^{e_1} \cdots X_n^{e_n}, \quad |e| = e_1 + \cdots + e_n$
- $f_{(k)} = \sum_{|e|=k} a_e X^e$ is the **homogeneous part** of f of degree k
- $f_{(k)} \in \mathcal{M}^k \setminus \mathcal{M}^{k+1}$
- The units of $\mathbb{A}$ are $\{u \,|\, u \notin \mathcal{M}\}$

$\mathbb{A}[Y]$ is the ring of Univariate Polynomials over Power Series (UPoPS)

- $f = \sum_{i=0}^{d} a_i Y^i, \; a_i \in \mathbb{A}, \; a_d \neq 0$ is a UPoPS of degree d
- Denote degree of f in Y by $\deg(f) = d$

Lazy Power Series

Power series implemented using a *lazy evaluation* scheme allow for terms to be computed on demand, increasing **precision** as needed.

Our lazy power series require:

1. an **update function** to compute homogeneous parts of a given degree
2. capturing parameters required for the update function
3. storing previously computed homogeneous parts

Where update parameters are power series, they are called **ancestors**.

Addition, $f = g + h$

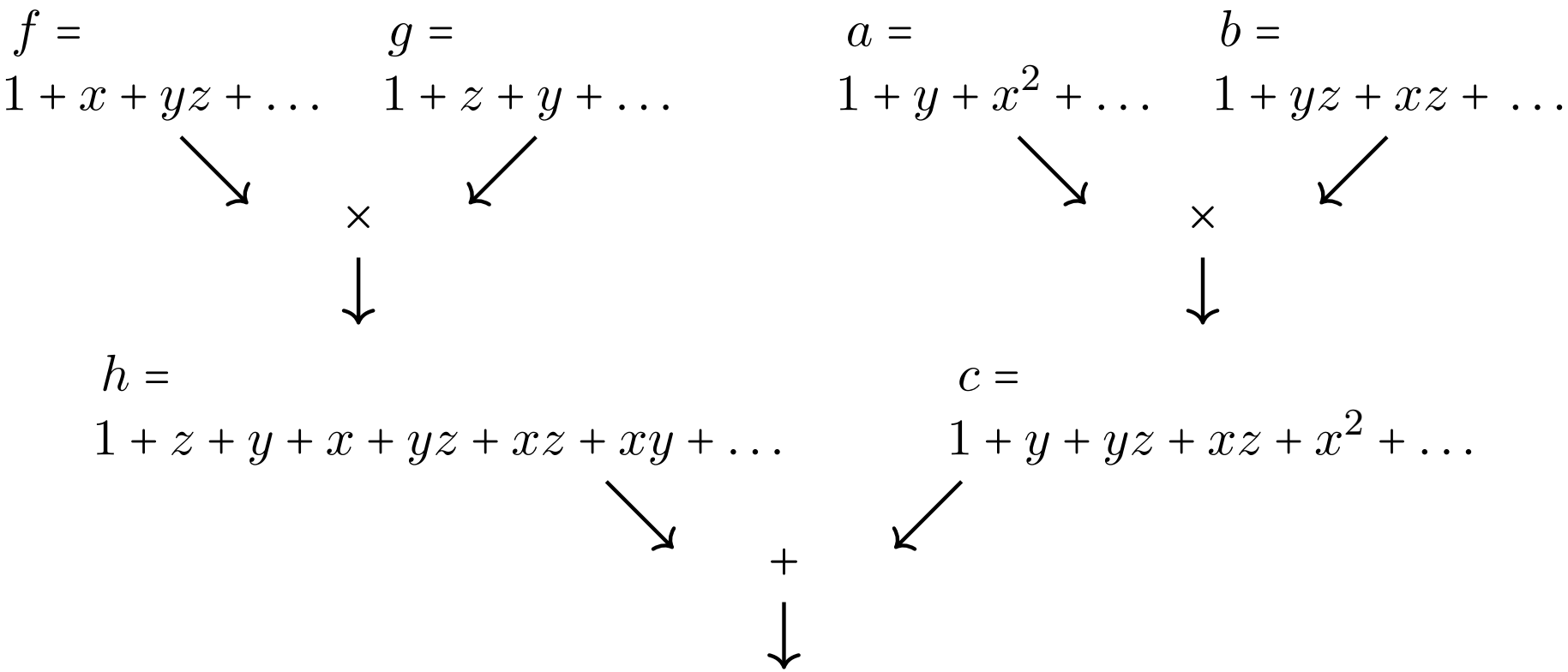
$\rightarrow \; f_{(k)} = g_{(k)} + h_{(k)}$

Multiplication $f = gh$

$\rightarrow \; f_{(k)} = \sum_{i=0}^{k} g_{(i)} h_{(k-i)}$

Ancestry Example

$$p = fg + ab$$



Outline

Weierstrass Preparation Theorem for UPoPS

Theorem 1 (Weierstrass Preparation)

Let $f = \sum_{i=0}^{d+m} a_i Y^i \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$ *where* $d \geq 0$ *be the smallest integer such that* $a_d \notin \mathcal{M}$ *and* $m \in \mathbb{Z}^+$. *Assume* $f \not\equiv 0 \mod \mathcal{M}[Y]$. *Then, there exists a unique pair* p, α *satisfying the following:*

1. $f = p\,\alpha$,
2. α *is an invertible element of* $\mathbb{K}[[X_1, \ldots, X_n]][[Y]]$,
3. p *is a monic polynomial of degree* d,
4. *writing* $p = Y^d + b_{d-1}Y^{d-1} + \cdots b_1 Y + b_0$, *we have* $b_{d-1}, \ldots, b_0 \in \mathcal{M}$.

Theorem 2 (Lazy Weierstrass Complexity)

For $f = p\,\alpha \in \mathbb{K}[[X_1]][Y], \deg(p) = d, \deg(\alpha) = m$, *computing* p *and* α *to precision* k *requires* $d(m+1)k^2 + dmk$ *operations in* $\mathbb{K}$.

Lazy Weierstrass Preparation

Let $f = \sum_{\ell=0}^{d+m} a_\ell Y^\ell$, $p = Y^d + \sum_{j=0}^{d-1} b_j Y^j$, $\alpha = \sum_{i=0}^{m} c_i Y^i$ be UPoPS.

↳ a_ℓ, b_j, c_i are power series ↳ $b_j \in \mathcal{M}$ for $j = 0, \ldots, d-1$

$$
f = p\,\alpha \implies
\begin{aligned}
a_0 &= b_0 c_0 \\
a_1 &= b_0 c_1 + b_1 c_0 \\
&\vdots \\
a_{d-1} &= b_0 c_{d-1} + b_1 c_{d-2} + \cdots + b_{d-2} c_1 + b_{d-1} c_0 \\
a_d &= b_0 c_d + b_1 c_{d-1} + \cdots + b_{d-1} c_1 + c_0 \\
&\vdots \\
a_{d+m-1} &= b_{d-1} c_m + c_{m-1} \\
a_{d+m} &= c_m
\end{aligned}
$$

We update p and α by solving these equations modulo $\mathcal{M}^k$, $k = 1, 2, \ldots$

modulo $\mathcal{M}$ we have:

(1) $b_j \equiv 0 \mod \mathcal{M}$, $j = 0, \ldots, d-1$ (2) $c_i \equiv a_{d+i} \mod \mathcal{M}$ $i = 0, \ldots, m$

Lazy Weierstrass Phase 1: update p

Let $f = \sum_{\ell=0}^{d+m} a_\ell Y^\ell$, $p = Y^d + \sum_{j=0}^{d-1} b_j Y^j$, $\alpha = \sum_{i=0}^{m} c_i Y^i$ be UPoPS.

↳ a_ℓ, b_j, c_i are power series

↳ $b_j \in \mathcal{M}$ for $j = 0, \ldots, d-1$

$$\begin{aligned} a_0 &= b_0 c_0 \\ a_1 - b_0 c_1 &= b_1 c_0 \\ a_2 - b_0 c_2 - b_1 c_1 &= b_2 c_0 \\ &\vdots \\ a_{d-1} - b_0 c_{d-1} - b_1 c_{d-2} + \cdots - b_{d-2} c_1 &= b_{d-1} c_0 \end{aligned}$$

→ let $F_j = a_j - \sum_{i=0}^{j-1} b_i c_{j-i}$

→ with power series division we can solve $F_j = b_j c_0$ from $j = 0$ to $d-1$

↳ $b_{j\,(k)} = {}^{1}\!/_{c_{0\,(0)}} \left(F_{j\,(k)} - \sum_{i=1}^{k-1} b_{j\,(i)} c_{0\,(k-i)} \right)$

→ each F_j lazily updated through lazy power series arithmetic

Lazy Weierstrass Phase 2: update α

Let $f = \sum_{\ell=0}^{d+m} a_\ell Y^\ell$, $p = Y^d + \sum_{j=0}^{d-1} b_j Y^j$, $\alpha = \sum_{i=0}^{m} c_i Y^i$ be UPoPS.

↳ a_ℓ, b_j, c_i are power series

↳ $b_j \in \mathcal{M}$ for $j = 0, \ldots, d-1$

$$\begin{aligned} c_m &= a_{d+m} \\ c_{m-1} &= a_{d+m-1} - b_{d-1}c_m \\ c_{m-2} &= a_{d+m-2} - b_{d-2}c_m - b_{d-1}c_{m-1} \\ &\vdots \\ c_0 &= a_d - b_0 c_d - b_1 c_{d-1} - \cdots - b_{d-1}c_1 \end{aligned}$$

→ $c_{m-i(k)} = a_{d+m-i(k)} - \sum_{j=1}^{i} \left(b_{d-j}c_{m-i+j}\right)_{(k)}$, for $i \leq d$

→ each c_{m-i} lazily updated through lazy power series arithmetic

→ $\left(b_{d-j}c_{m-i+j}\right)_{(k)}$ only requires up to $c_{m-i+j\,(k-1)}$ since $b_{d-j\,(0)} = 0$

→ each c_{m-i} can thus be updated simultaneously

Parallel Opportunities in Weierstrass

parallel map-reduce or **parallel for** loops

In phase 1: $b_{j(k)} = {}^{1}/_{c_{0(0)}} \left(F_{j(k)} - \sum_{i=1}^{k-1} b_{j(i)} c_{0(k-i)} \right)$

→ compute summations using map-reduce:

$$\sum_{i=1}^{k} b_{j(i)} c_{0(k-i)}$$

→ notice in multivariate case, e.g., $b_{j(1)} c_{0(k-1)}$ is less work than $b_{j(\frac{k}{2})} c_{0(k-\frac{k}{2})}$

In phase 2: $c_{m-i(k)} = a_{d+m-i(k)} - \sum_{j=1}^{i} \left(b_{d-j} c_{m-i+j} \right)_{(k)}$

→ compute each $c_{m-i(k)}$, $0 \leq i \leq m$ simultaneously, and/or

→ compute product homogeneous parts using a map-reduce:

$$\left(b_{d-j} c_{m-i+j} \right)_{(k)} = \sum_{\ell=1}^{k} b_{d-j(\ell)} c_{m-i+j(k-\ell)}$$

→ load-balance issue again in the multivariate case

Update To Deg: Map-Reduce

Algorithm 1 UpdateToDegParallel$(k,\ f,\ t)$

Input: $k \in \mathbb{Z}^+$, $f \in \mathbb{K}[[X_1, \ldots, X_n]]$ known to at least precision $k-1$. If f has ancestors, it is the result of a binary operation. $t \in \mathbb{Z}^+$ for the number of threads to use.

Output: f is updated to precision k, in place.

1: $g,\ h :=$ FirstAncestor(f), SecondAncestor(f)
2: UpdateToDegParallel$(k,\ g,\ t)$;
3: UpdateToDegParallel$(k,\ h,\ t)$;

4: **if** f is a product **then** ▷ compute $f_{(k)}$ by map-reduce
5: | $\mathcal{V} := [0, \ldots, 0]$ ▷ 0-indexed list of size t
6: | **parallel_for** $j := 0$ to $t-1$
7: | | **for** $i := {}^{jk}/_{t}$ to ${}^{(j+1)k}/_{t} - 1$ **while** $i \leq k$ **do**
8: | | | $\mathcal{V}[j] := \mathcal{V}[j] + g_{(i)} h_{(k-i)}$
9: | $f_{(k)} := \sum_{j=0}^{t-1} \mathcal{V}[j]$ ▷ reduce

...

Parallel Challenges

$$
\begin{aligned}
c_m &= a_{d+m} \\
c_{m-1} &= a_{d+m-1} - b_{d-1}c_m \\
c_{m-2} &= a_{d+m-2} - b_{d-2}c_m - b_{d-1}c_{m-1} \\
c_{m-3} &= a_{d+m-3} - b_{d-3}c_m - b_{d-2}c_{m-1} - b_{d-1}c_{m-2} \\
&\vdots \\
c_0 &= a_d - b_0c_d - b_1c_{d-1} - \cdots - b_{d-1}c_1
\end{aligned}
$$

- → Computing $c_{m-i(k)}$ for different i requires a different amount of work
- → c_{m-i} requires i products
- → For load-balance, must compute a different number of c_{m-i} with each thread; the total number of products computed per thread should be same

Outline

Hensel's Lemma

Theorem 3 (Hensel's Lemma)

Let $f = Y^d + \sum_{i=0}^{d-1} a_i Y^i$ *be a monic polynomial in* $\mathbb{K}[[X_1, \ldots, X_n]][Y]$. *Let* $\bar{f} = f(0, \ldots, 0, Y) = (Y - c_1)^{d_1}(Y - c_2)^{d_2} \cdots (Y - c_r)^{d_r}$ *for* $c_1, \ldots, c_r \in \mathbb{K}$ *and positive integers* $d_1, \ldots, d_r$. *Then, there exists* $f_1, \ldots, f_r \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$, *all monic in Y, such that:*

1. $f = f_1 \cdots f_r$,
2. $\deg(f_i, Y) = d_i$ *for* $1 \leq i \leq r$, *and*
3. $\bar{f}_i = (Y - c_i)^{d_i}$ *for* $1 \leq i \leq r$.

Proof:

Let $g = f(X_1, \ldots, X_n, Y + c_r) = Y^d + \sum_{i=0}^{d-1} b_i Y^i$, sending c_r to the origin. By construction, $b_0, \ldots, b_{d_r - 1} \in \mathcal{M}$ and Weierstrass preparation can be applied to produce $g = p\,\alpha$ with $\deg(p) = d_r, \deg(\alpha) = d - d_r$.

Reversing the shift, $f_r = p(Y - c_r)$.
Induction on $\widehat{f} = \alpha(Y - c_r)$ completes the proof. □

Hensel Factorization

Algorithm 2 HenselFactorization(f)

Input: $f = Y^d + \sum_{i=0}^{d-1} a_i Y^i, a_i \in \mathbb{K}[[X_1, \ldots, X_n]]$.
Output: $f_1, \ldots, f_r$ satisfying Theorem 3.

1: $\bar{f} = f(0, \ldots, 0, Y)$
2: $(c_1, \ldots, c_r), (d_1, \ldots, d_r) :=$ roots and their multiplicities of $\bar{f}$
3: $c_1, \ldots, c_r :=$ SORT$([c_1, \ldots, c_r])$ by increasing multiplicity
4: $\widehat{f}_1 := f$
5: **for** $i := 1$ to $r - 1$ **do**
6: $\quad g_i := \widehat{f}_i(Y + c_i)$
7: $\quad p_i, \alpha_i :=$ WeierstrassPreparation(g_i)
8: $\quad f_i := p_i(Y - c_i)$
9: $\quad \widehat{f}_{i+1} := \alpha_i(Y - c_i)$
10: $f_r := \widehat{f}_r$
11: **return** $f_1, \ldots, f_r$

Hensel Complexity (1/2)

Let $\widehat{d_i} = \deg(\widehat{f_i}) = \sum_{j=i}^{r} d_i$

At each iteration but the last:

1. a shift of degree $\widehat{d_i}$ for $\widehat{f_i}(Y + c_i)$
2. a Weierstrass preparation producing $\deg(p) = d_i, \deg(\alpha) = \widehat{d_{i+1}}$
3. a shift of degree d_i for $p(Y - c_i)$
4. a shift of degree $\widehat{d_{i+1}}$ for $\alpha(Y - c_i)$

Theorem 4 (Hensel Factorization Complexity)

For $f = f_1 \cdots f_r \in \mathbb{K}[[X_1]][Y], \deg(f) = d$, updating all factors to precision k can be done within $\mathcal{O}(d^3k + d^2k^2)$ operations in $\mathbb{K}$.

Conjecture: using relaxed algorithms [8] in Weierstrass may yield $\mathcal{O}(d^3k + d^2k\log^2 k)$

Hensel Complexity (2/2)

Let $f \in \mathbb{K}[[X_1]][Y]$ have degree d_Y in Y and total degree d.
Let $M(n)$ be a polynomial multiplication time.

Factorizing f requires...

- → Our method: $\mathcal{O}(d_Y^3 k + d_Y^2 k^2)$
- → Hensel–Sasaki Construction (EHC): $\mathcal{O}(d^3 M(d) + k^2 d M(d))$ [2]
- → Over $\mathbb{C}$, Newton–Puiseux can be done in:
  - ↳ $\mathcal{O}(d^2 M(k))$; Kung and Traub [9]
  - ↳ $\mathcal{O}(d^2 k)$; Chudnovsky and Chudnovsky [7]
- → Berthomieu, Lecerf, Quintin: $\mathcal{O}(M(d_Y)\log(d_Y)\, k M(k))$ [5]

Including the initial root-finding cost as $R(d_Y)$:

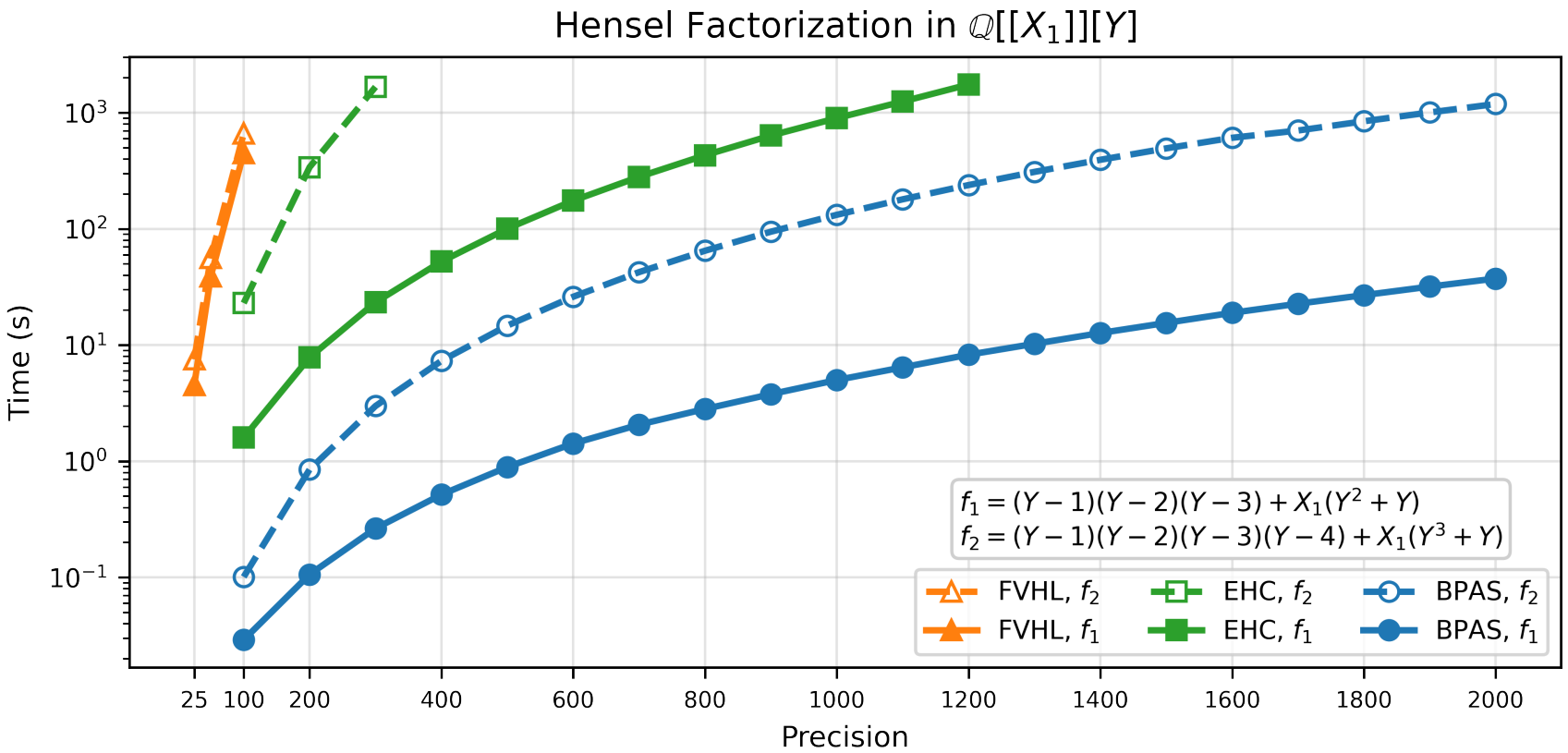
- → Our method: $\mathcal{O}(d_Y^3 k + d_Y^2 k^2 + R(d_Y))$
- → Neiger, Rosenkilde, Schost: $\mathcal{O}^{\sim}(d_Y k + k R(d_Y))$ [11]

Note $M(n) \in \mathcal{O}(n^2)$ or $\mathcal{O}(n \log n)$; $R(n) \in \mathcal{O}^{\sim}(n^2)$

Serial Performance

EHC outperforms Kung and Traub's method by orders of magnitude [2]

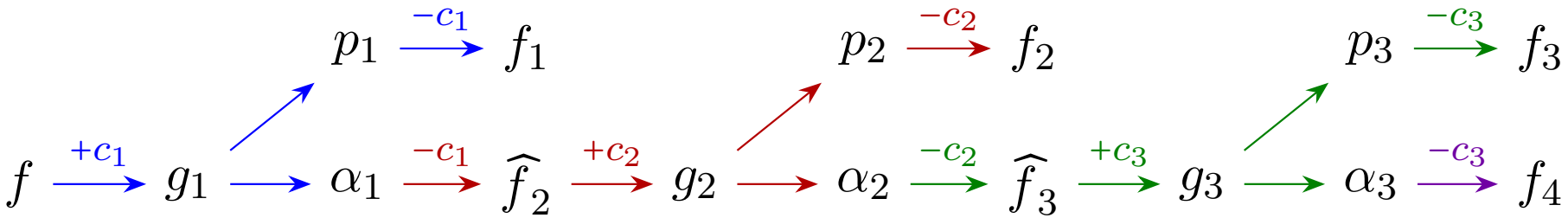
From CASC 2020, our serial method outperforms EHC and Factorization via Hensel's Lemma of `PowerSeries` sub-package of `RegularChains` [10]



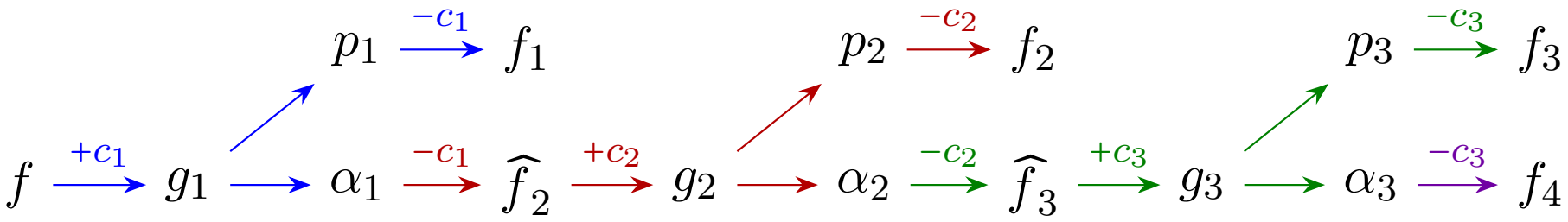
Parallel Opportunities in Hensel

- → The output of one Weierstrass becomes input to another
- → $f_{i+i_{(k)}}$ relies on $f_{i_{(k)}}$
- → Can compute $f_{i_{(k+1)}}$ and $f_{i+i_{(k)}}$ concurrently in a **pipeline**

	Stage 1 (f_1)	Stage 2 (f_2)	Stage 3 (f_3)	Stage 4 (f_4)
Time 1	$f_{1_{(1)}}$			
Time 2	$f_{1_{(2)}}$	$f_{2_{(1)}}$		
Time 3	$f_{1_{(3)}}$	$f_{2_{(2)}}$	$f_{3_{(1)}}$	
Time 4	$f_{1_{(4)}}$	$f_{2_{(3)}}$	$f_{3_{(2)}}$	$f_{4_{(1)}}$
Time 5	$f_{1_{(5)}}$	$f_{2_{(4)}}$	$f_{3_{(3)}}$	$f_{4_{(2)}}$
Time 6	$f_{1_{(6)}}$	$f_{2_{(5)}}$	$f_{3_{(4)}}$	$f_{4_{(3)}}$



Parallel Challenges and Composition



→ Degrees and computational work diminish with each stage
  ↳ $\deg(g_1) = d,\ \deg(g_2) = d - d_1,\ \deg(g_3) = d - d_1 - d_2,\ \ldots$

→ To load-balance each stage, give a decreasing number of threads to each stage to be used within Weierstrass preparation.

→ From Theorem 1: updating f_i requires $\mathcal{O}(d_i \widehat{d}_{i+1} k^2)$ operations

→ Assign t_i threads to stage i so that $d_i \widehat{d}_{i+1}/t_i$ is equal for each stage.

→ Better still, update a *group of successive factors* per stage.
  ↳ To each stage s assign factors $f_{s_1}, \ldots, f_{s_2}$ and t_s threads so that $\sum_{i=s_1}^{s_2} d_i \widehat{d}_{i+1}/t_s$ is roughly equal for each stage.

Hensel Pipeline Example

$$f = (Y-1)(Y-2)(Y-3)(Y-4) + X_1(Y^3+Y)$$
$$\bar{f}_1 = Y-1, \quad \bar{f}_2 = Y-2, \quad \bar{f}_3 = Y-3, \quad \bar{f}_4 = Y-4$$

$k = 600$

factor	serial time (s)	shift time (s)	$d_i \widehat{d}_{i+1}$	Complexity-est. threads	parallel time (s)	wait time (s)	Time-est. threads	parallel time (s)	wait time (s)
f_1	18.1989	0.0012	$1 \cdot 3$	6	4.5380	0.0000	7	3.5941	0.0000
f_2	6.6681	0.0666	$1 \cdot 2$	4	4.5566	0.8530	3	3.6105	0.6163
f_3	3.4335	0.0274	$1 \cdot 1$	1	4.5748	1.0855	0	-	-
f_4	0.0009	0.0009	$1 \cdot 0$	1	4.5750	4.5707	2	3.6257	1.4170

- → f_4 requires at least one thread so that it (the last factor) gets updated
- → work estimates based on complexity results okay, but does not account for, e.g., coefficient size or data locality.
- → can also use serial time to suggest thread assignments

Hensel Pipeline Algorithm

Algorithm 3 HenselPipeStage(k, f_i, t, gen)

Input: $k \in \mathbb{Z}^+$, f_i a UPoPS, $t \in \mathbb{Z}^+$ the number of threads to use within this stage. gen a generator for the previous pipeline stage.

Output: a sequence of integers j signalling f_i is known to precision j; the sequence ends at k.

```
p := Precision(f_i)  ▷ current precision of f_i
do
    ▷ A blocking function call until gen yields
    k' := gen()
    for j := p to k' do
        UpdateToDegParallel(j, f_i, t)
        yield j
    p := k'
while k' < k
```

Algorithm 4 HenselFactorizationPipeline(k, $\mathcal{F}$, $\mathcal{T}$)

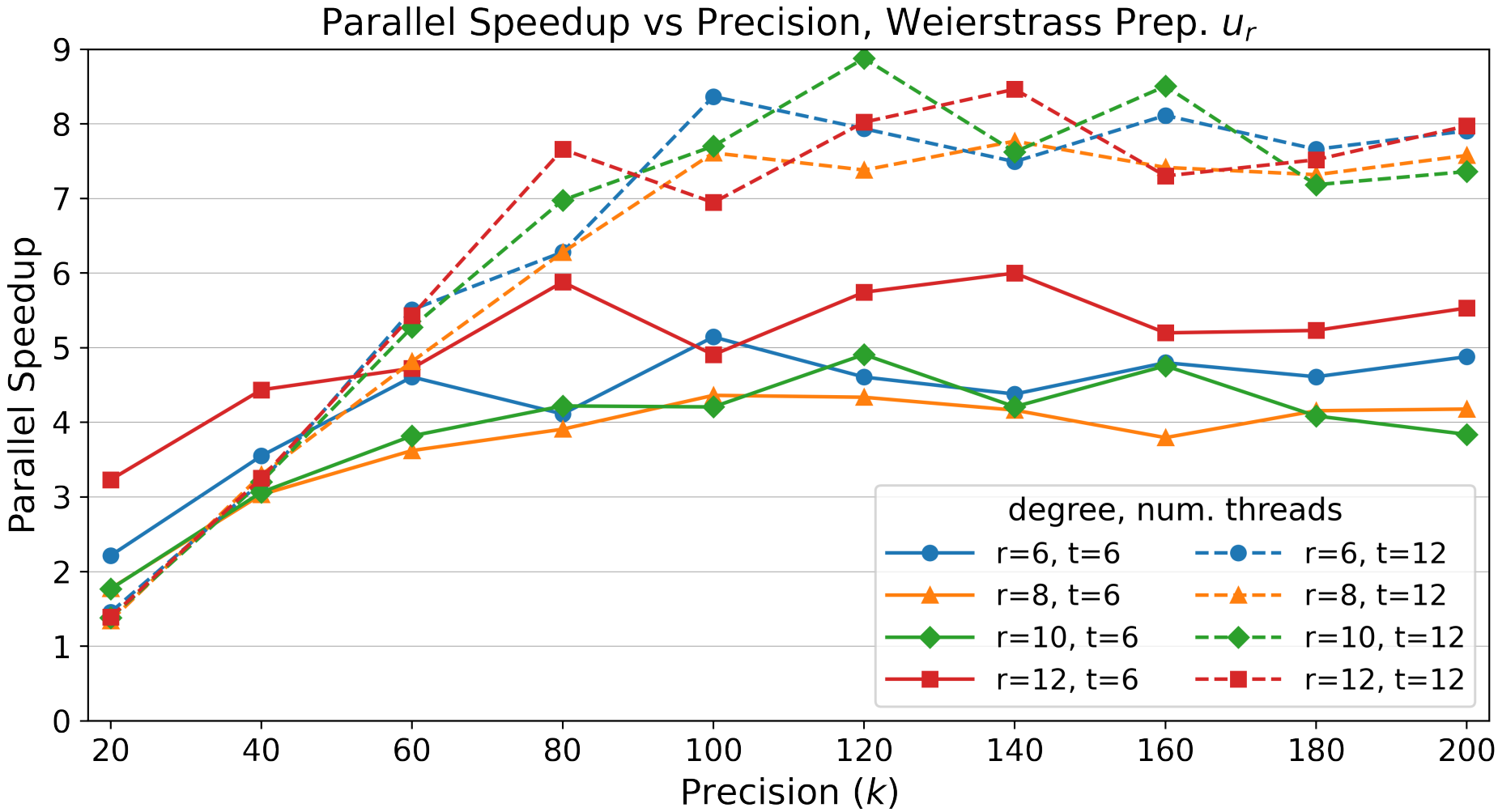
Input: $k \in \mathbb{Z}^+$, $\mathcal{F} = \{f_1, \ldots, f_r\}$ the output of HenselFactorization. $\mathcal{T} \in \mathbb{Z}^r$ a 0-indexed list of the number of threads to use in each stage, $\mathcal{T}[r-1] > 0$.

Output: $f_1, \ldots, f_r$ updated in-place to precision k.

```
▷ an anonymous function asynchronous generator
gen := ( ) → {yield k}
for i := 0 to r − 1 do
    if T[i] > 0 then
        ▷ Capture function as a function object,
          passing the previous gen as input
        gen := AsyncGenerator(
          HenselPipeStage, k, f_{i+1}, T[i], gen)
▷ ensure last stage completes before returning
do
    k' := gen()
while k' < k
```

Outline

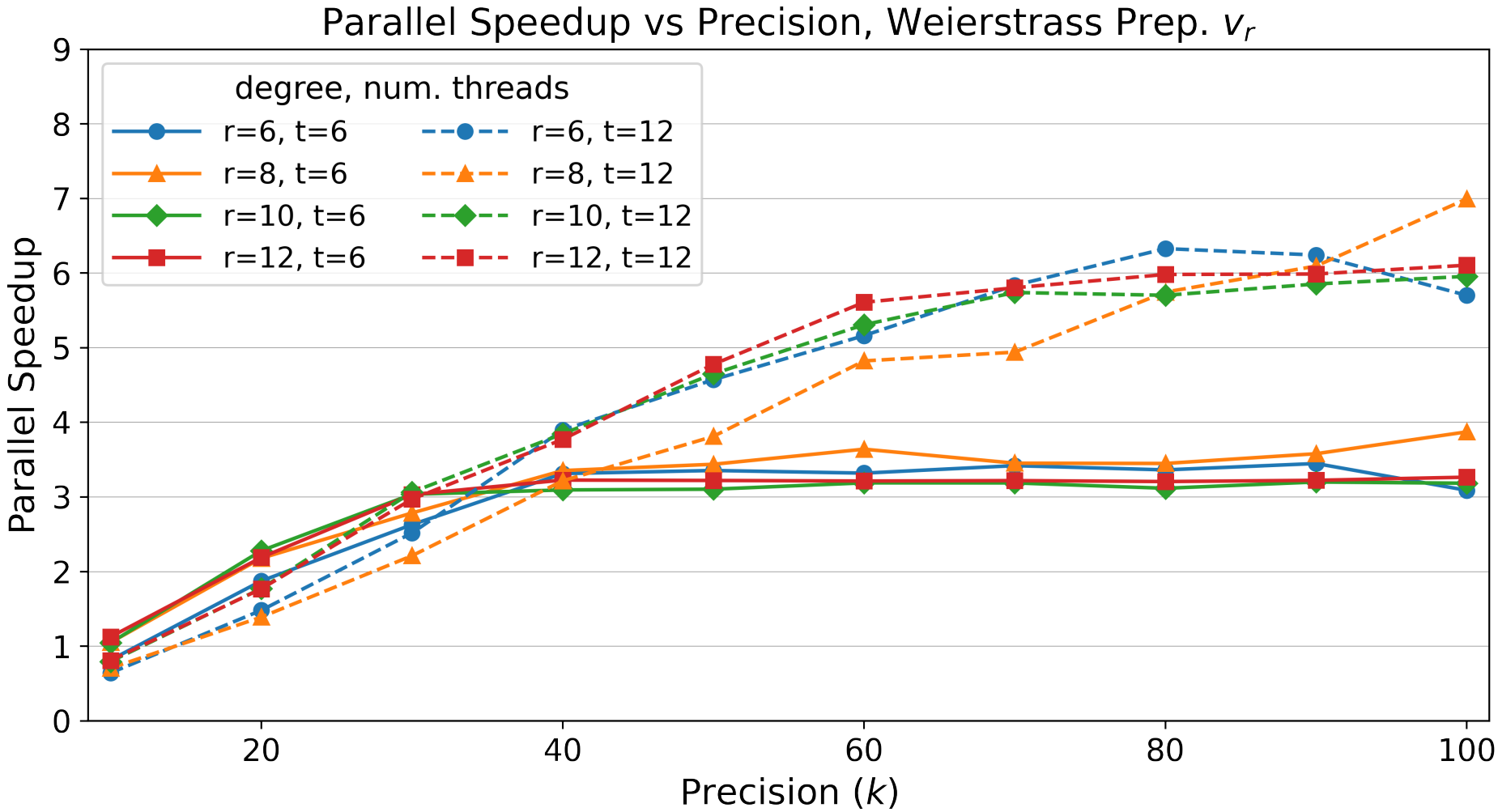
Parallel Speed-up WPT 1



$$u_r = \sum_{i=2}^{r} (X_1^2 + X_2 + i)Y^i + (X_1^2 + X_2)Y + X_1^2 + X_1X_2 + X_2^2$$

$$\implies \deg(p) = 2 \quad \deg(\alpha) = r - 2$$

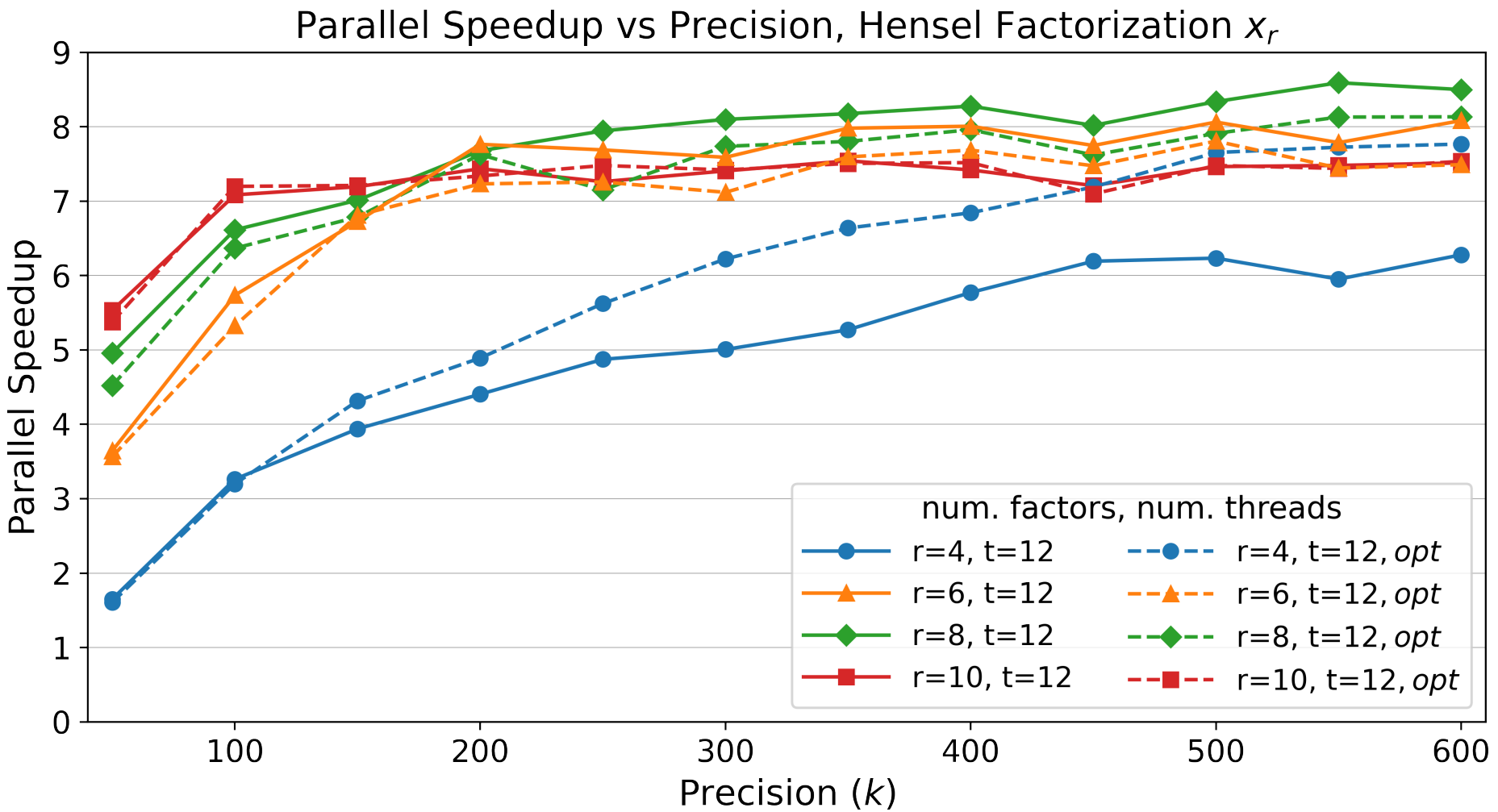
Parallel Speed-up WPT 2



$$v_r = \sum_{i=r/2}^{r} (X_1^2 + X_2 + i)Y^i + \sum_{i=1}^{r/2-1} (X_1^2 + X_2)Y^i + X_1^2 + X_1X_2 + X_2^2$$

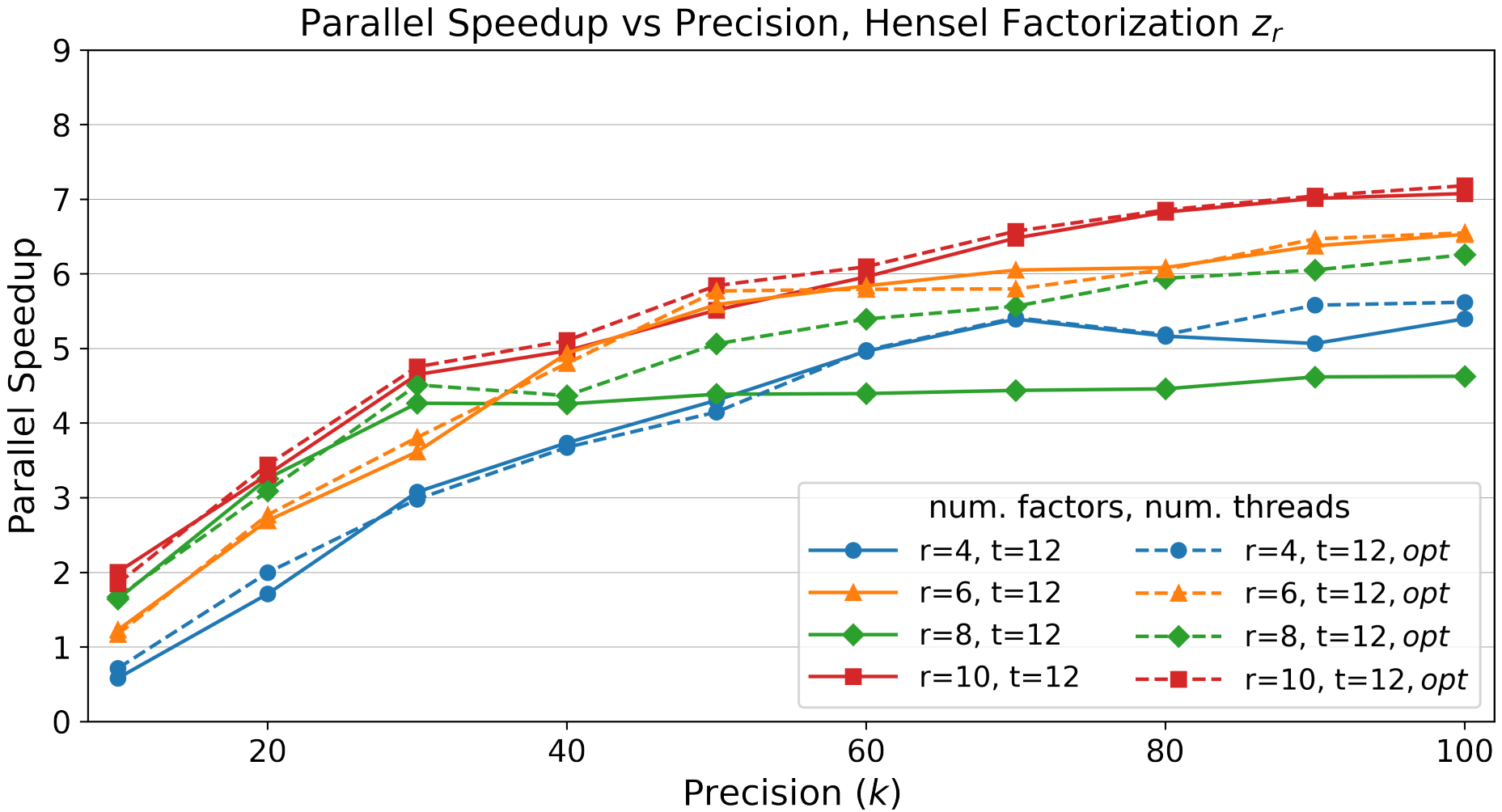
$$\Longrightarrow \deg(p) = \deg(\alpha) = r/2$$

Parallel Speed-up Hensel Factorization 1



$$x_r = \prod_{i=1}^{r} (Y - i) + X_1(Y^3 + Y)$$

Parallel Speed-up Hensel Factorization 2



$$z_r = \prod_{i=1}^{r} (Y + X_1 + X_2 - i) + X_1 X_2 (Y^3 + Y)$$

Conclusions and Future Work

→ We have seen complexity estimates and parallel algorithms for lazy Weierstrass preparation and lazy Hensel factorization.

→ The composition of parallel patterns (map-reduce within pipeline) provides good performance.

In future:

→ In multivariate case, must improve loop partitioning to better compute homogeneous parts of products: $f_{(k)} = \sum_{i=0}^{k} g_{(i)} h_{(k-i)}$

→ Improved thread distribution between Hensel pipeline stages to consider practical performance: coefficient size, locality

→ Improved thread distribution between Hensel pipeline stages for multivariate case

→ Relaxed algorithms within Weierstrass preparation

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