Classification

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Linear models in general (HTF Ch. 2.8.3)

- By linear models, we mean that the hypothesis function $h_{\mathbf{w}}(\mathbf{x})$ is a (transformed) *linear function of the parameters* $\mathbf{w}$.
- Predictions are a (transformed) *linear combination of feature values*

$$h_{\mathbf{w}}(\mathbf{x}) = g\left(\sum_{k=0}^{p} w_k \phi_k(\mathbf{x})\right) = g(\boldsymbol{\phi}(\mathbf{x})^{\mathsf{T}}\mathbf{w})$$

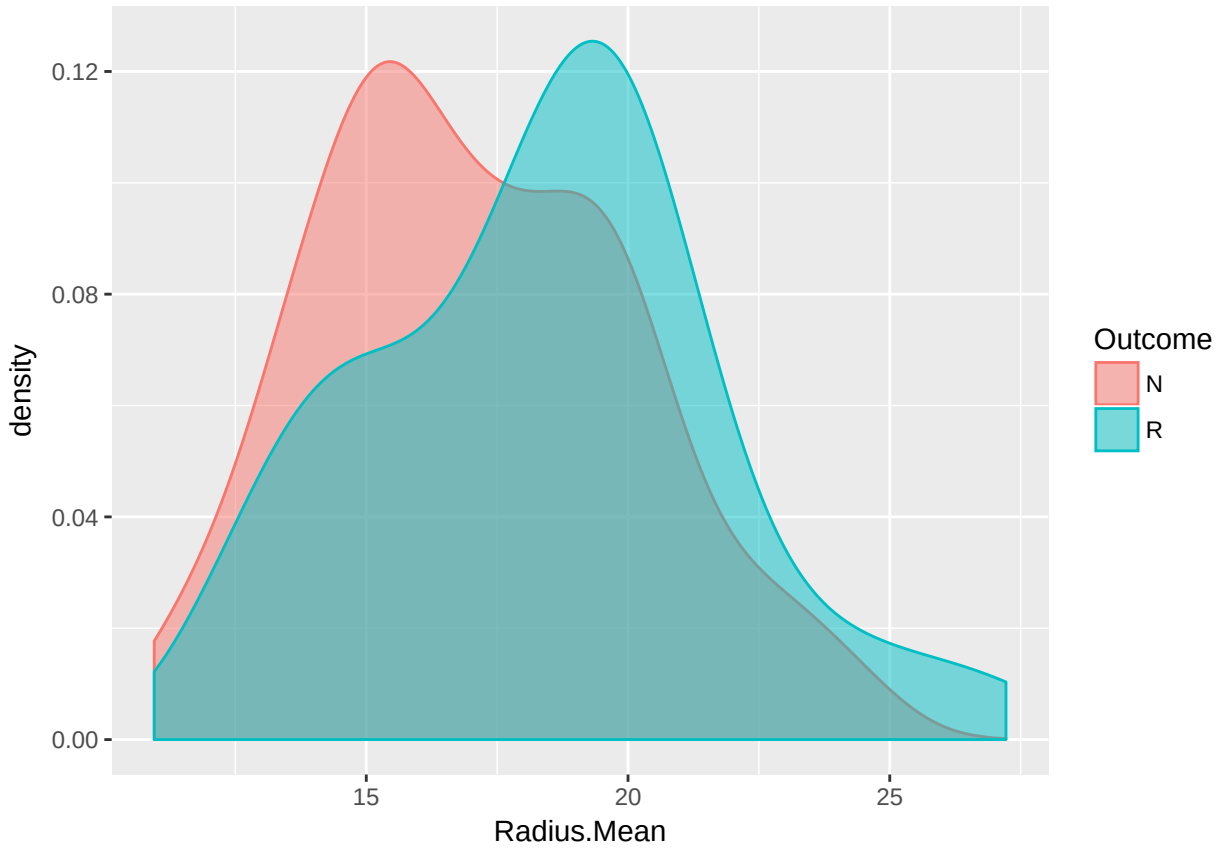
- where ϕ_k are called *basis functions* As usual, we let $\phi_0(\mathbf{x}) = 1, \forall \mathbf{x}$, so that we don't force $h_{\mathbf{w}}(0) = 0$
- Polynomial regression: set $\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2, ..., \phi_d(x) = x^d$ and set $g(x) = x$.
- Basis functions are *fixed* for training (but can be chosen through model selection)

Linear Methods for Classification

- Classification tasks
- Loss functions for classification
- Logistic Regression
- Support Vector Machines

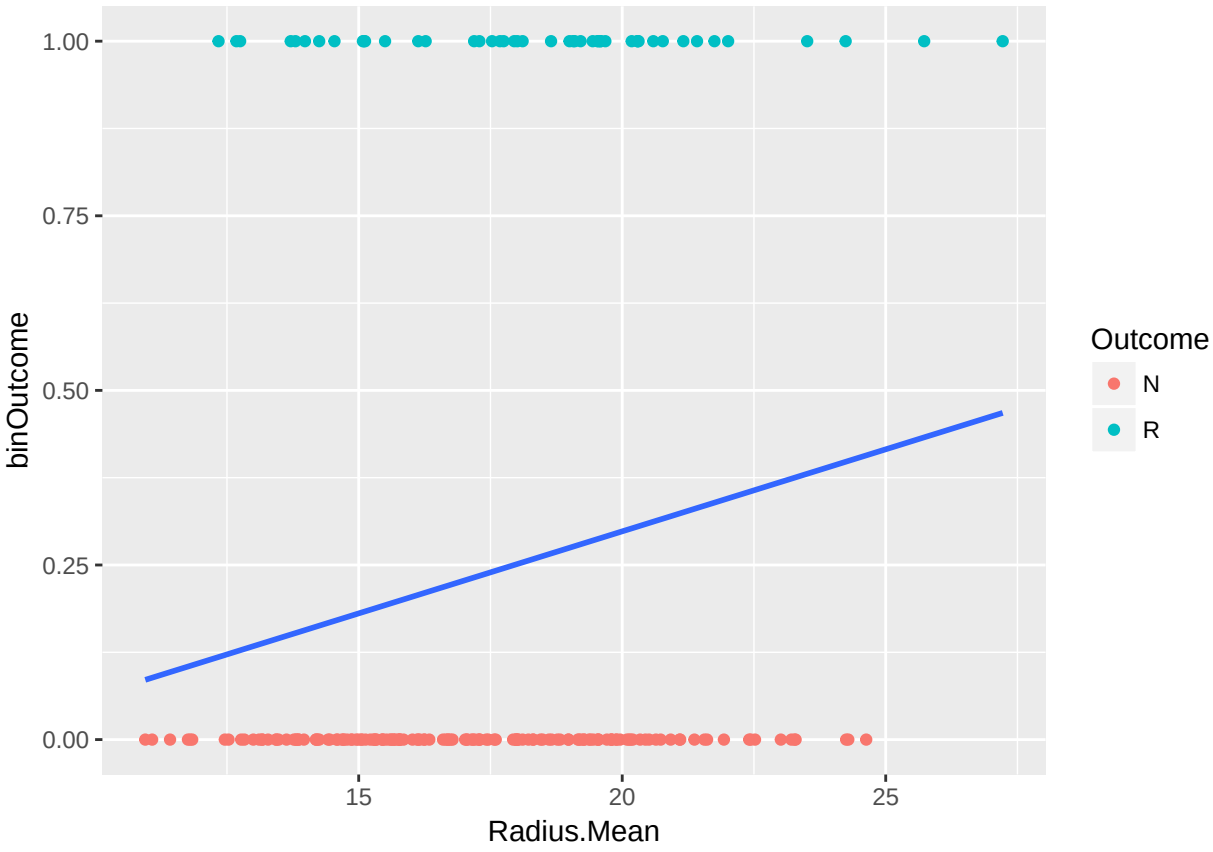
Example: Given nucleus radius, predict cancer recurrence

```
ggplot(bc,aes(Radius.Mean,fill=Outcome,color=Outcome)) + geom_density(alpha=I(1/2))
```



Example: Solution by linear regression

- Univariate real input: nucleus size
- Output coding: non-recurrence = 0, recurrence = 1
- **Sum squared error** minimized by the blue line



Linear regression for classification

- The predictor shows an increasing trend towards recurrence with larger nucleus size, as expected.
- Output *cannot be directly interpreted* as a class prediction.
- Thresholding output (e.g., at 0.5) could be used to predict 0 or 1.
  (In this case, prediction would be 0 except for extremely large nucleus size.)

Probabilistic view

- Suppose we have two possible classes: $y \in \{0, 1\}$.
- The symbols "0" and "1" are unimportant. Could have been $\{a, b\}$, $\{up, down\}$, whatever.
- Rather than try to predict the class label directly, ask:
  What is the *probability* that a given input $\mathbf{x}$ to has class $y = 1$?
- Conditional Probability:

$$P(y = 1|\mathbf{X} = \mathbf{x}) = \frac{P(\mathbf{X} = \mathbf{x}, y = 1)}{P(\mathbf{X} = \mathbf{x})}$$

Probabilistic models for binary classification

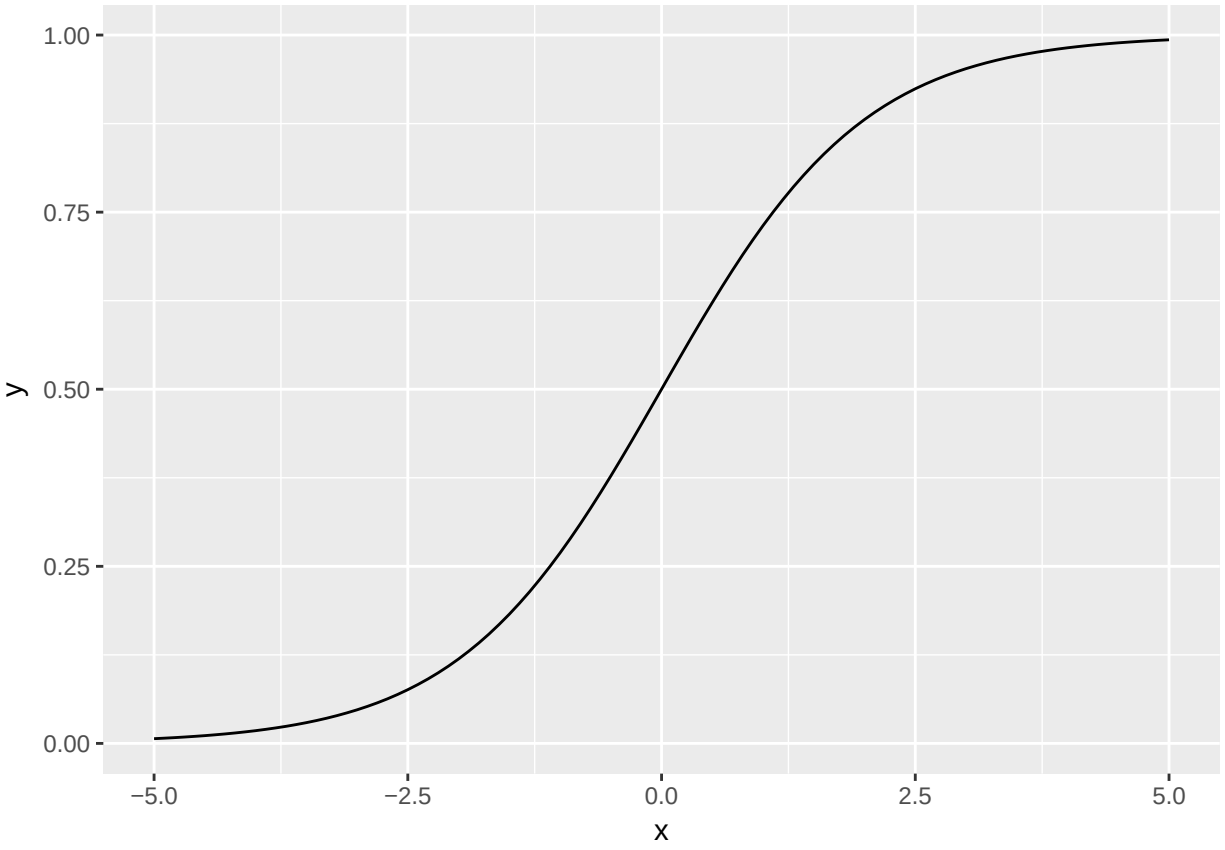
What kind of function do we use for $P(y = 1|\mathbf{X} = \mathbf{x})$?

Idea: $h_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}^\mathsf{T}\mathbf{x}$

Why? Why not?

Sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



Logistic Regression HTF (Ch. 4.4)

- Represent the hypothesis as a logistic function of a linear combination of inputs, interpret $h(\mathbf{x})$ as $P(y = 1|\mathbf{X} = \mathbf{x})$:

$$h_{\mathbf{w}}(\mathbf{x}) = \sigma(\mathbf{x}^\mathsf{T}\mathbf{w})$$

- $\sigma(a) = \frac{1}{1+\exp(-a)}$ is the *sigmoid* or *logistic function*
- With a little algebra, we can write:

$$P(y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\log \frac{P(y = 1|\mathbf{X} = \mathbf{x})}{P(y = 0|\mathbf{X} = \mathbf{x})}\right)$$

  - Interpret $\mathbf{x}^\mathsf{T}\mathbf{w}$ as the log-odds

Logistic regression training HTF (Ch. 4.4)

- How do we choose $\mathbf{w}$?
- In the probabilistic framework, observing $\langle \mathbf{x}_i, 1 \rangle$ does not mean $h_{\mathbf{w}}(\mathbf{x}_i)$ should be as close to 1 as possible.
- **Maximize probability the model assigns to the y_i, *given the* $\mathbf{x}_i$** by adjusting $\mathbf{w}$.

Reminder: Independence

- Two random variables X and Y that are part of a random vector are **independent** iff:

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

If they have a joint density or joint PMF, then

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

Max Conditional Likelihood

- **Maximize probability the model assigns to the y_i, *given the* $\mathbf{x}_i$** by adjusting $\mathbf{w}$.
- Assumption 1: Examples are i.i.d. Probability of observing all ys is product

$$\begin{aligned} P(\text{all y}|\text{all x}) &= P(Y_1 = y_1, Y_2 = y_2, ..., Y_n = y_n | X_1 = \mathbf{x}_1, X_2 = \mathbf{x}_2, ..., X_n = \mathbf{x}_n) \\ &= \prod_{i=1}^{n} P(Y_i = y_i | X_1 = \mathbf{x}_1, X_2 = \mathbf{x}_2, ..., X_n = \mathbf{x}_n) \\ &= \prod_{i=1}^{n} P(Y_i = y_i | X_i = \mathbf{x}_i) \end{aligned}$$

- Assumption 2: $\begin{array}{l} P(y = 1 | \mathbf{X} = \mathbf{x}) = h_{\mathbf{w}}(\mathbf{x}) = 1/(1 + \exp(-\mathbf{x}^{\mathsf{T}}\mathbf{w})) \\ P(y = 0 | \mathbf{X} = \mathbf{x}) = (1 - h_{\mathbf{w}}(\mathbf{x})) \end{array}$

Max Conditional Likelihood

- **Maximize probability the model assigns to the y_i, *given the* $\mathbf{x}_i$** by adjusting $\mathbf{w}$.
- More stable to maximize log probability. Note

$$\log P(Y_i = y_i | X_i = \mathbf{x}_i) = \begin{cases} \log h_{\mathbf{w}}(\mathbf{x}_i) & \text{if } y_i = 1 \\ \log(1 - h_{\mathbf{w}}(\mathbf{x}_i)) & \text{if } y_i = 0 \end{cases}$$

- Therefore,

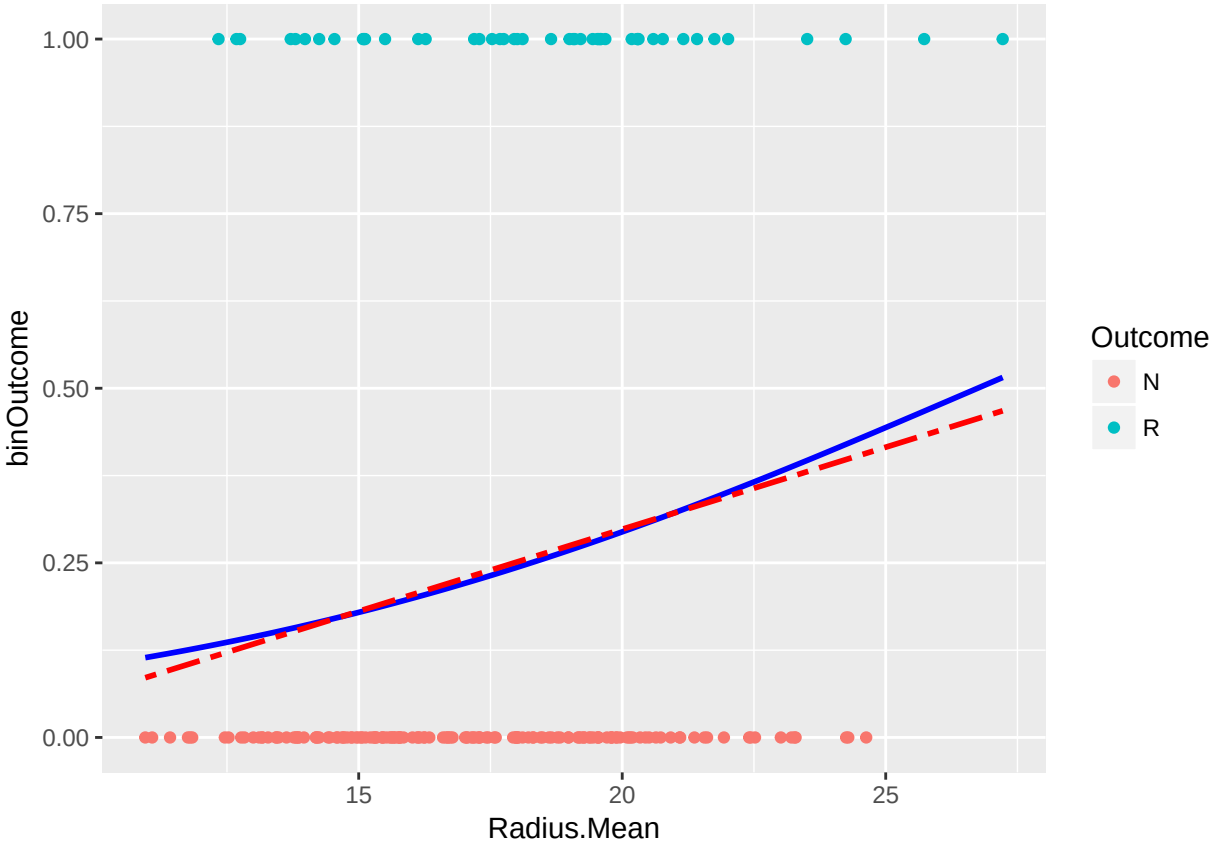
$$\log \prod_{i=1}^{n} P(Y_i = y_i | X_i = \mathbf{x}_i) = \sum_{i=1}^{n} [y_i \log(h_{\mathbf{w}}(\mathbf{x}_i)) + (1 - y_i) \log(1 - h_{\mathbf{w}}(\mathbf{x}_i))]$$

- Suggests an error

$$J(h_{\mathbf{w}}) = -\sum_{i=1}^{n} [y_i \log(h_{\mathbf{w}}(\mathbf{x}_i)) + (1 - y_i) \log(1 - h_{\mathbf{w}}(\mathbf{x}_i))]$$

- This is the *cross entropy*. Number of bits to transmit y_i if both parties know $h_{\mathbf{w}}$ and $\mathbf{x}_i$.

Back to the breast cancer problem



Logistic Regression:

```
## (Intercept) Radius.Mean
##  -3.4671348   0.1296493
```

Least Squares:

```
## (Intercept) Radius.Mean
## -0.17166939  0.02349159
```

Probability and Expectation

- Why are these so close?
- The *expected value* of a discrete random variable Y is denoted

$$\mathrm{E}[Y] = \sum_{y \in \mathcal{Y}} y \cdot p_Y(Y = y)$$

- Consider a random variable $Y \in \{0, 1\}$

$$\begin{aligned}\mathrm{E}[Y] &= \sum_{y \in \{0,1\}} y \cdot p_Y(Y = y) \\ &= 0 \cdot p_Y(Y = 0) + 1 \cdot p_Y(Y = 1) \\ &= p_Y(Y = 1)\end{aligned}$$

- Though we did not discuss in this way, linear regression tries to estimate $\mathrm{E}[Y = y|X = x]$. So, makes sense that the OLS and logistic regression answers can be close.

Supervised Learning Methods: "Objective-driven"

Mthd.	Form	Objective
OLS	$h_w(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{w}$	$\sum_{i=1}^n (h_\mathbf{w}(\mathbf{x}_i) - y_i)^2$
	$\approx E[Y = y\|\mathbf{X} = \mathbf{x}]$...	...using a linear function
LR	$h_w(\mathbf{x}) = \frac{1}{1+\mathrm{e}^{-\mathbf{x}^\mathsf{T}\mathbf{w}}}$	$-\sum_{i=1}^n y_i \log h_\mathbf{w}(\mathbf{x}_i) + (1 - y_i) \log(1 - h_\mathbf{w}(\mathbf{x}_i))$
	$\approx P(Y = y\|\mathbf{X} = \mathbf{x})$...	...using sigmoid of a linear function

- Both model the ***conditional mean of*** y using a (transformed) ***linear function***
- Both use ***maximum conditional likelihood*** to estimate

Decision boundary HTF Ch. 2.3.1,2.3.2

- How complicated is a classifier?
- One way to think about it is in terms of its *decision boundary*, i.e. the line it defines for separating examples
- *Linear classifiers* draw a hyperplane between examples of the different classes. *Non-linear classifiers* draw more complicated surfaces between the different classes.
- For a probabilistic classifier with a cutoff of 0.5,
  the decision boundary is the curve on which:

$$\frac{P(y = 1|\mathbf{X} = \mathbf{x})}{P(y = 0|\mathbf{X} = \mathbf{x})} = 1, \text{i.e., where } \log \frac{P(y = 1|\mathbf{X} = \mathbf{x})}{P(y = 0|\mathbf{X} = \mathbf{x})} = 0$$

- For logistic regression, this this is where $\mathbf{x}^\mathsf{T}\mathbf{w} = 0$.

Decision boundaries of linear classifiers

- Recall: predictions are a (transformed) *linear combination of feature values*

$$h_\mathbf{w}(\mathbf{x}) = g(\mathbf{x}^\mathsf{T}\mathbf{w})$$

- Suppose our decision boundary is

$$h_\mathbf{w}(\mathbf{x}) = c$$

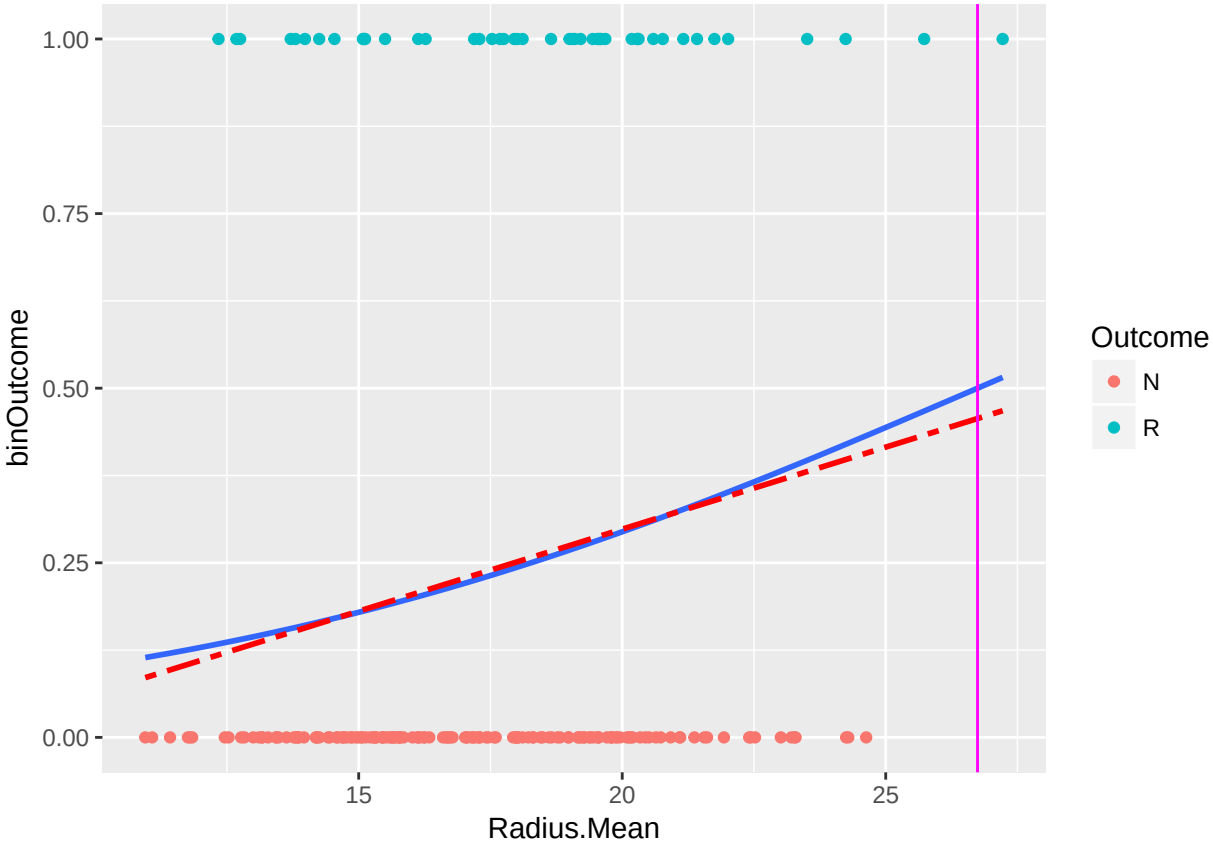
  .
- This is equivalent to

$$\mathbf{x}^\mathsf{T}\mathbf{w} = c'$$

where $c' = g^{-1}(c)$.

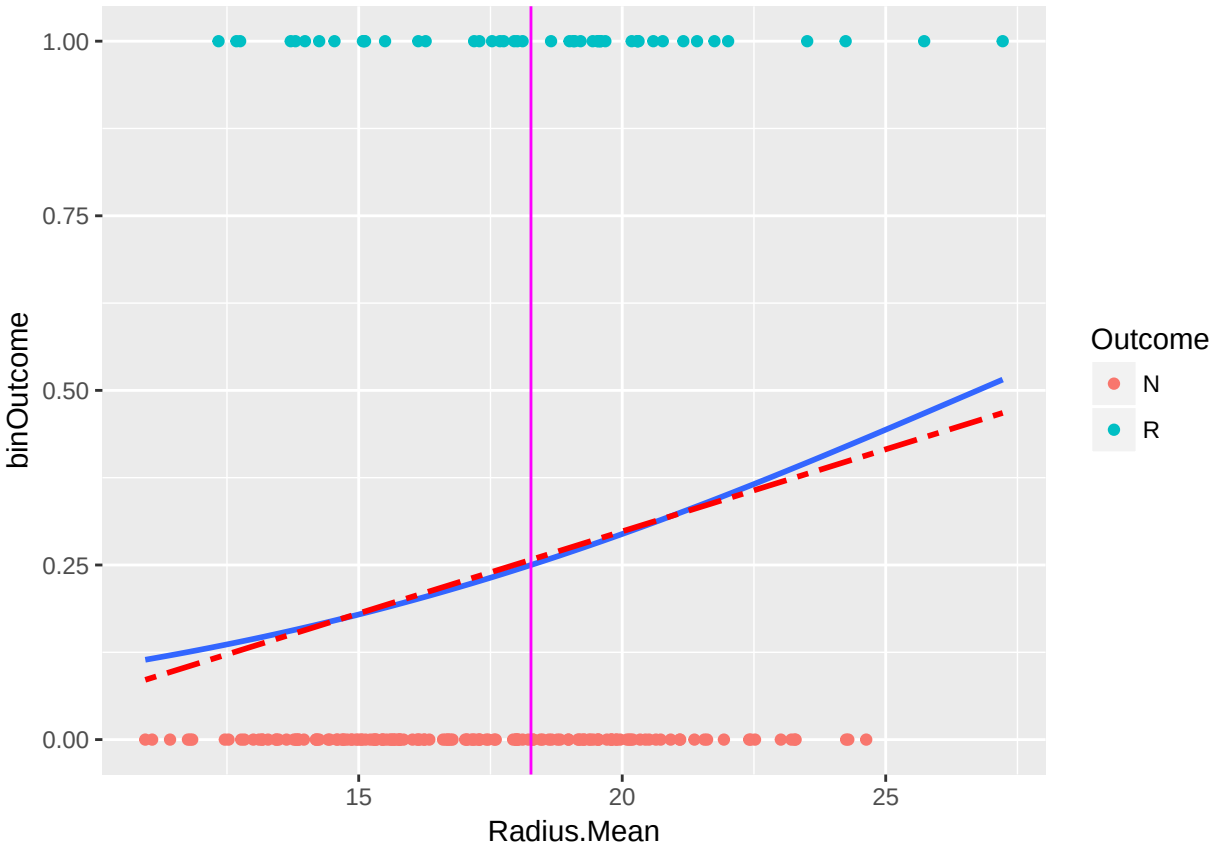
Decision boundary

Class = R if $\Pr(Y = 1|X = x) > 0.5$



Decision boundary

Class = R if $\Pr(Y = 1|X = x) > 0.25$



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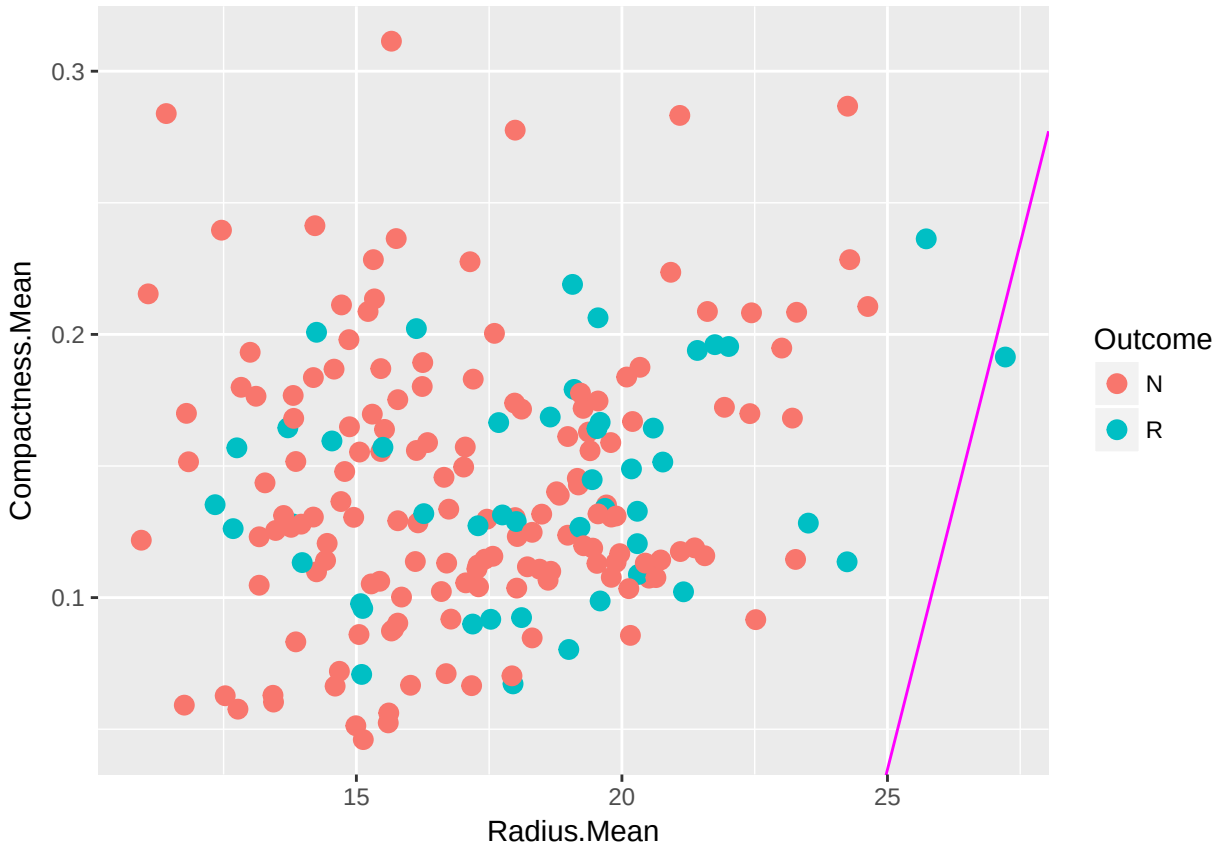

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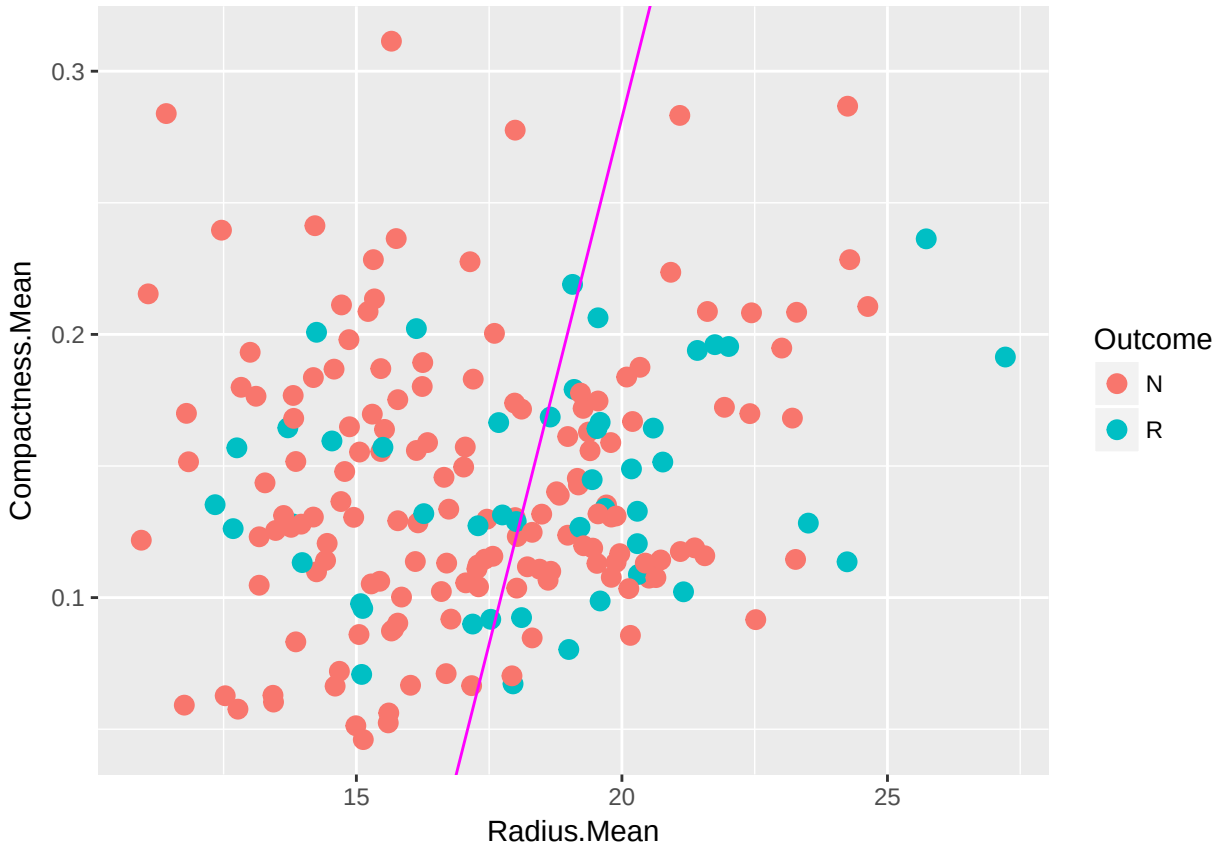

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	$\approx E[Y = y\|\mathbf{X} = \mathbf{x}]$...	...using a linear function
LR	$h_\mathbf{w}(\mathbf{x}) = \frac{1}{1+\mathrm{e}^{-\mathbf{x}^\mathsf{T}\mathbf{w}}}$	$-\sum_{i=1}^n y_i \log h_\mathbf{w}(\mathbf{x}_i) + (1 - y_i) \log(1 - h_\mathbf{w}(\mathbf{x}_i))$
	$\approx P(Y = y\|\mathbf{X} = \mathbf{x})$...	...using sigmoid of a linear function
SVM	$h_\mathbf{w}(\mathbf{x}) = \mathrm{sgn}(\mathbf{x}^\mathsf{T}\mathbf{w})$	

Large Margin Classifiers: Linear Support Vector Machines

- Linear classifiers that focus on learning the *decision boundary* rather than the conditional distribution $P(Y = y|\mathbf{X} = \mathbf{x})$
  - Perceptrons
    - Definition
    - Perceptron learning rule
    - Convergence
  - "Margin" idea and max margin classifiers

- (Linear) support vector machines
  - Formulation as optimization problem

Marvin Minsky, 1927-2016

Perceptrons HTF Ch. 4.5

- Consider a binary classification problem with data $\{\mathbf{x}_i, y_i\}_{i=1}^n$, $y_i \in \{-1, +1\}$. **Note coding of** y_i.
- A *perceptron* (Rosenblatt, 1957) is a classifier of the form:

$$h_{\mathbf{w},w_0}(\mathbf{x}) = \text{sign}(\mathbf{x}^\mathsf{T}\mathbf{w} + w_0) = \begin{cases} +1 & \text{if } \mathbf{x}^\mathsf{T}\mathbf{w} + w_0 \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

  Here, $\mathbf{w}$ is a vector of weights, and w_0 is a constant offset. (**Note** $x_0 = 1$ **is omitted.**)
- The decision boundary is $\mathbf{x}^\mathsf{T}\mathbf{w} + w_0 = 0$.
- Perceptrons output a **class**, not a probability
- An example $(\mathbf{x}, y)$ is classified correctly if:

$$y \cdot (\mathbf{x}^\mathsf{T}\mathbf{w} + w_0) > 0$$

Linear separability

- The data set is *linearly separable* if and only if there exists $\mathbf{w}$, w_0 such that:
  - For all i, $y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) > 0$.
  - Or equivalently, the 0-1 loss $\sum_i \mathbf{1}_{y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w}+w_0)<0}$ is zero for some set of parameters $(\mathbf{w}, w_0)$.

Linear Separability

The Perceptron Learning Rule

- Consider the following procedure:
  1. Initialize $\mathbf{w}$ and w_0 randomly
  2. While any training examples remain incorrecty classified
     1. Loop through all misclassified examples
     2. For misclassified example i, perform the updates:

$$\mathbf{w} \leftarrow \mathbf{w} + \delta y_i \mathbf{x}_i, \qquad w_0 \leftarrow w_0 + \delta y_i$$

        where δ is a step-size parameter.
- The update equation, or sometimes the whole procedure, is called the *perceptron learning rule*.
- Intuition: For positive examples misclassified as negative, change $\mathbf{w}$ to increase $\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0$, and vice versa

Error Minimization Interpretation

- PLR can be interpreted as a gradient descent on the following function:

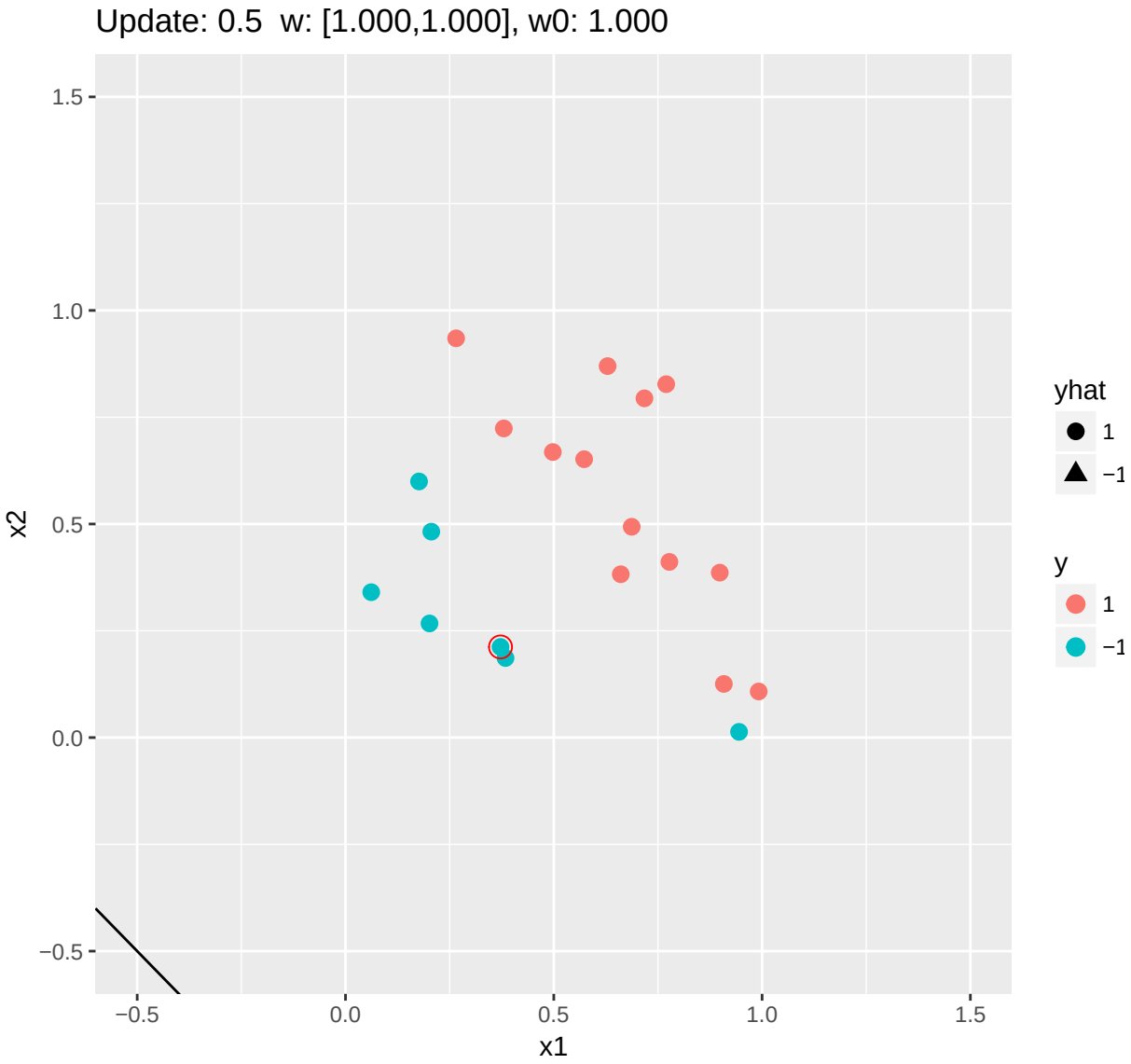
$$J(\mathbf{w}, w_0) = \sum_{i=1}^{n} \begin{cases} 0 & \text{if } y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 0 \\ -y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) & \text{if } y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) < 0 \end{cases}$$

- For correctly classified examples, the error is zero.
- For incorrectly classified examples, the error is by how much $\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0$ is on the wrong side of the decision boundary.
- J is piecewise linear, so it has a gradient almost everywhere; stochastic gradient descent gives the perceptron learning rule.
- J is zero if and only if all examples are classified correctly – just like the 0-1 loss function.

Perceptron convergence theorem

- ***If*** classes are linearly separable ***then*** the perceptron learning rule will find a separater after some finite number of updates.
- The number of updates depends on the data set, and also on the step size parameter.
- If the classes are not linearly separable, there will be oscillation (which can be detected automatically).

Perceptron Learning Example



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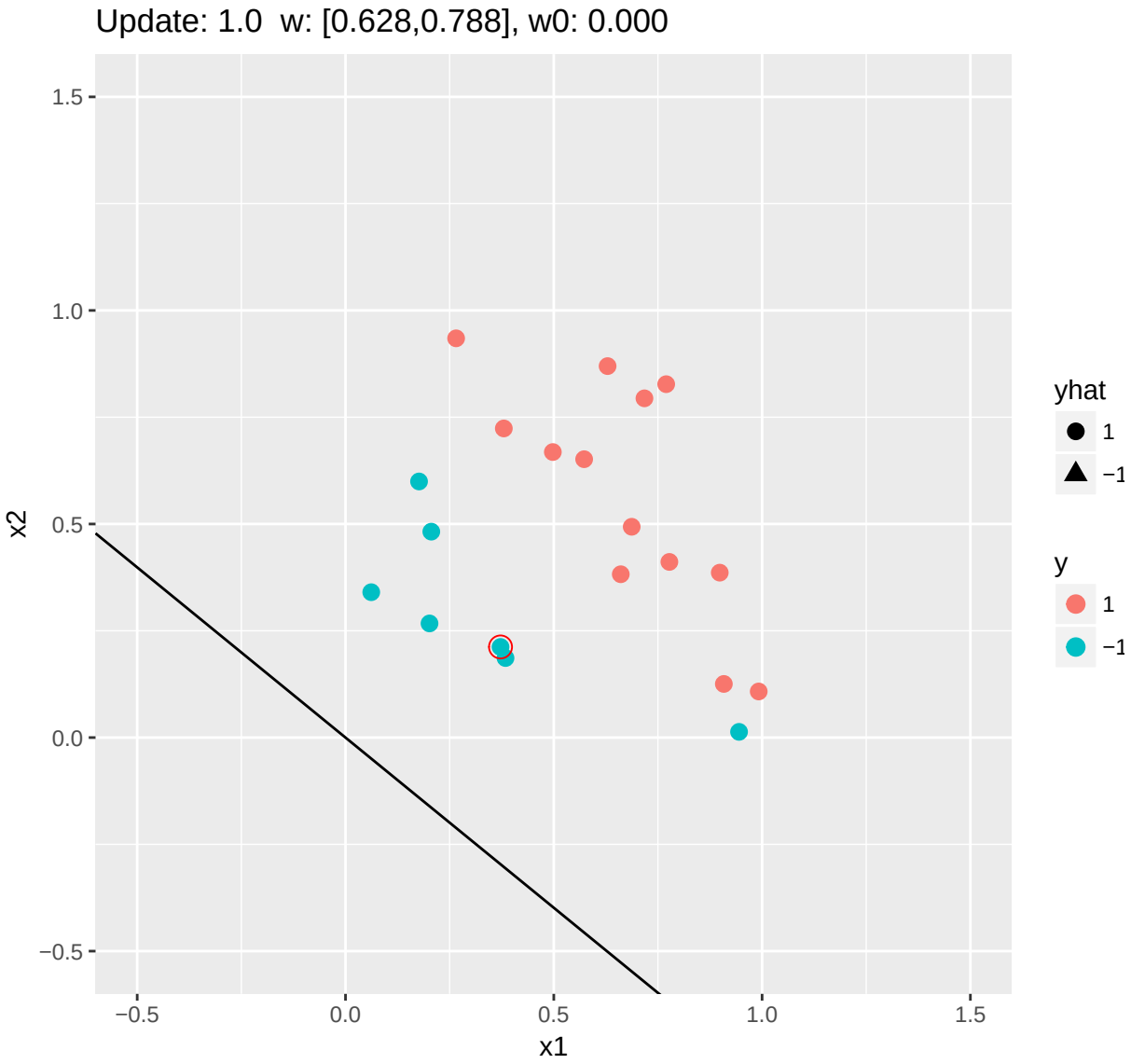

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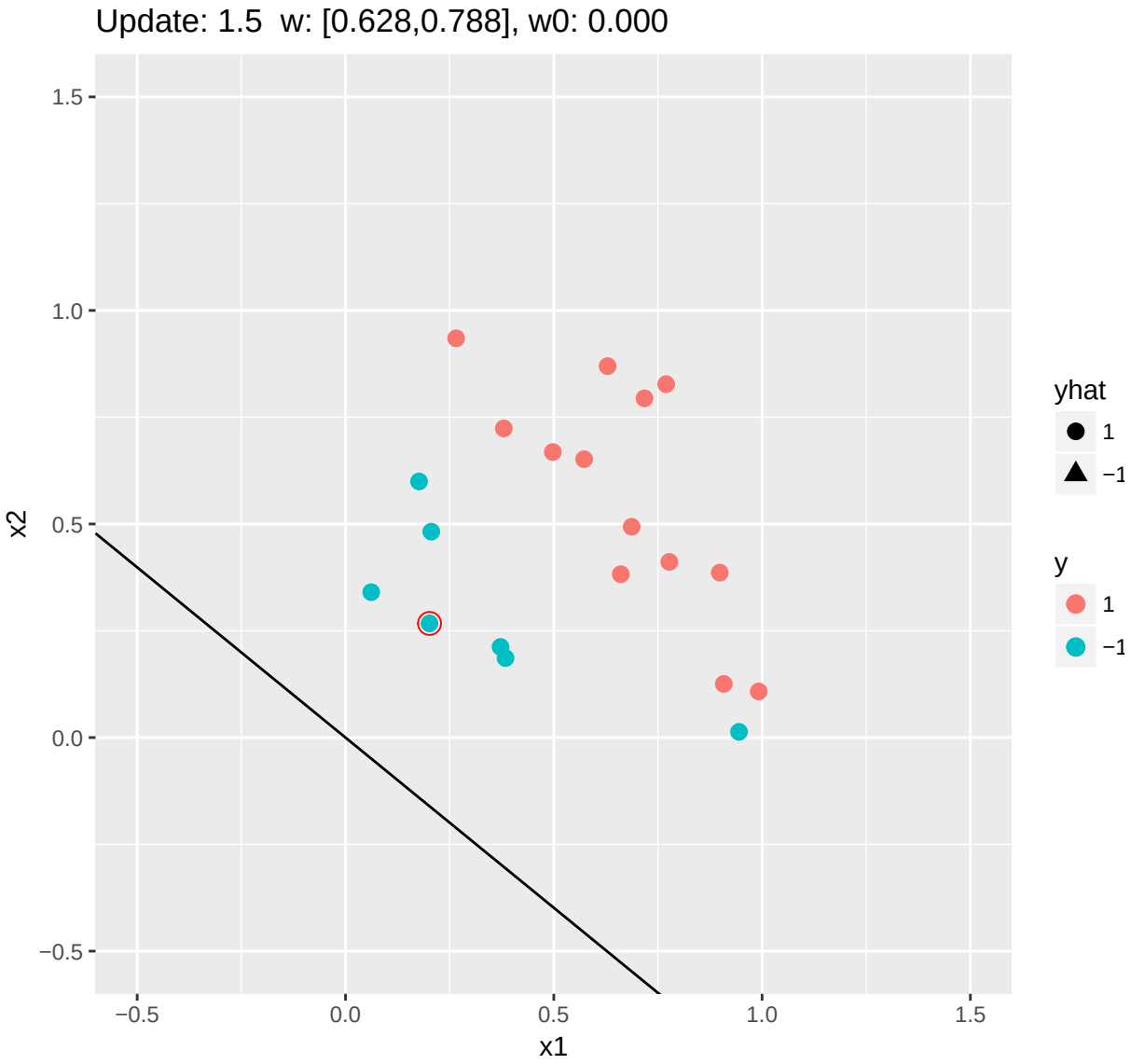

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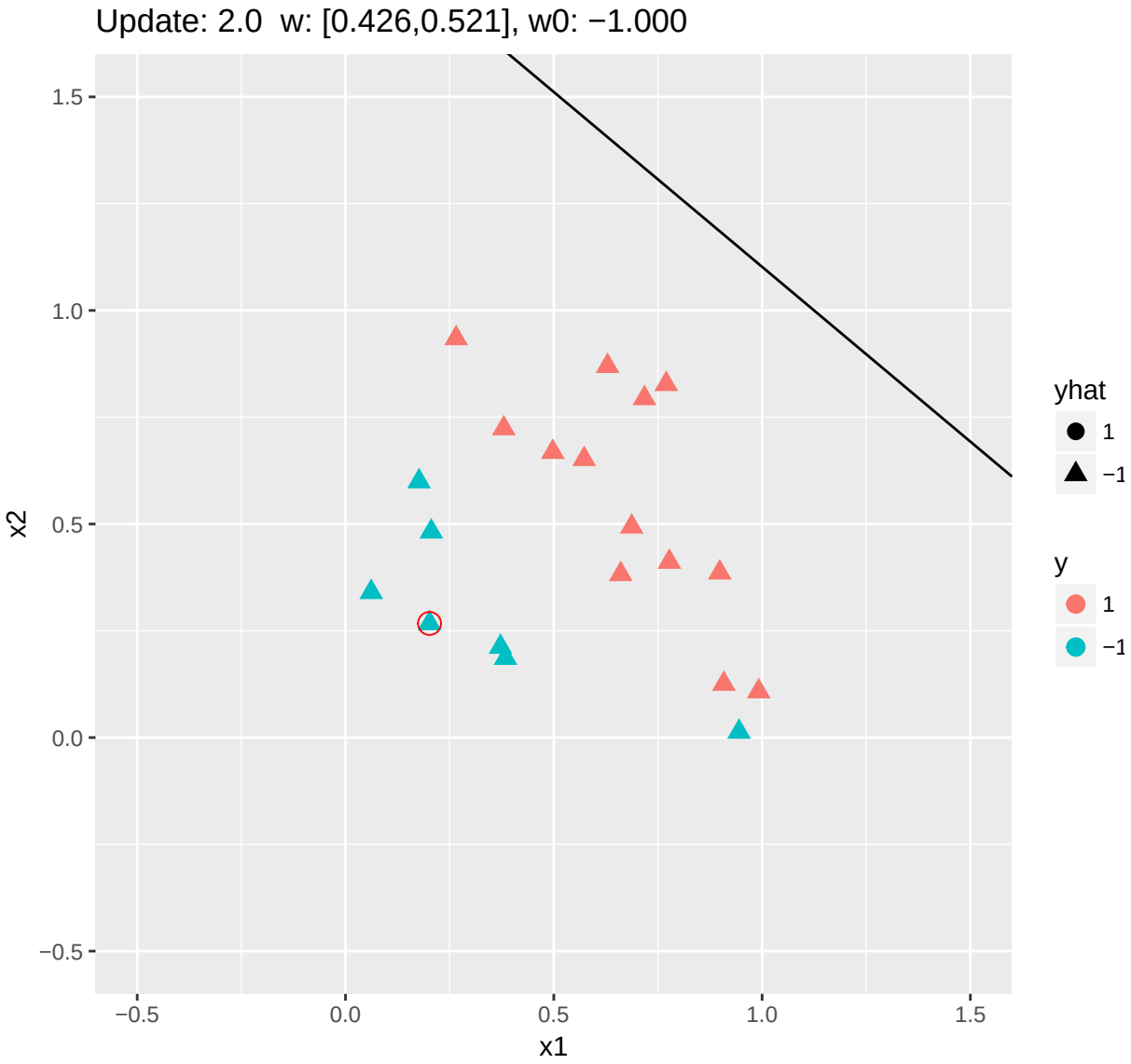

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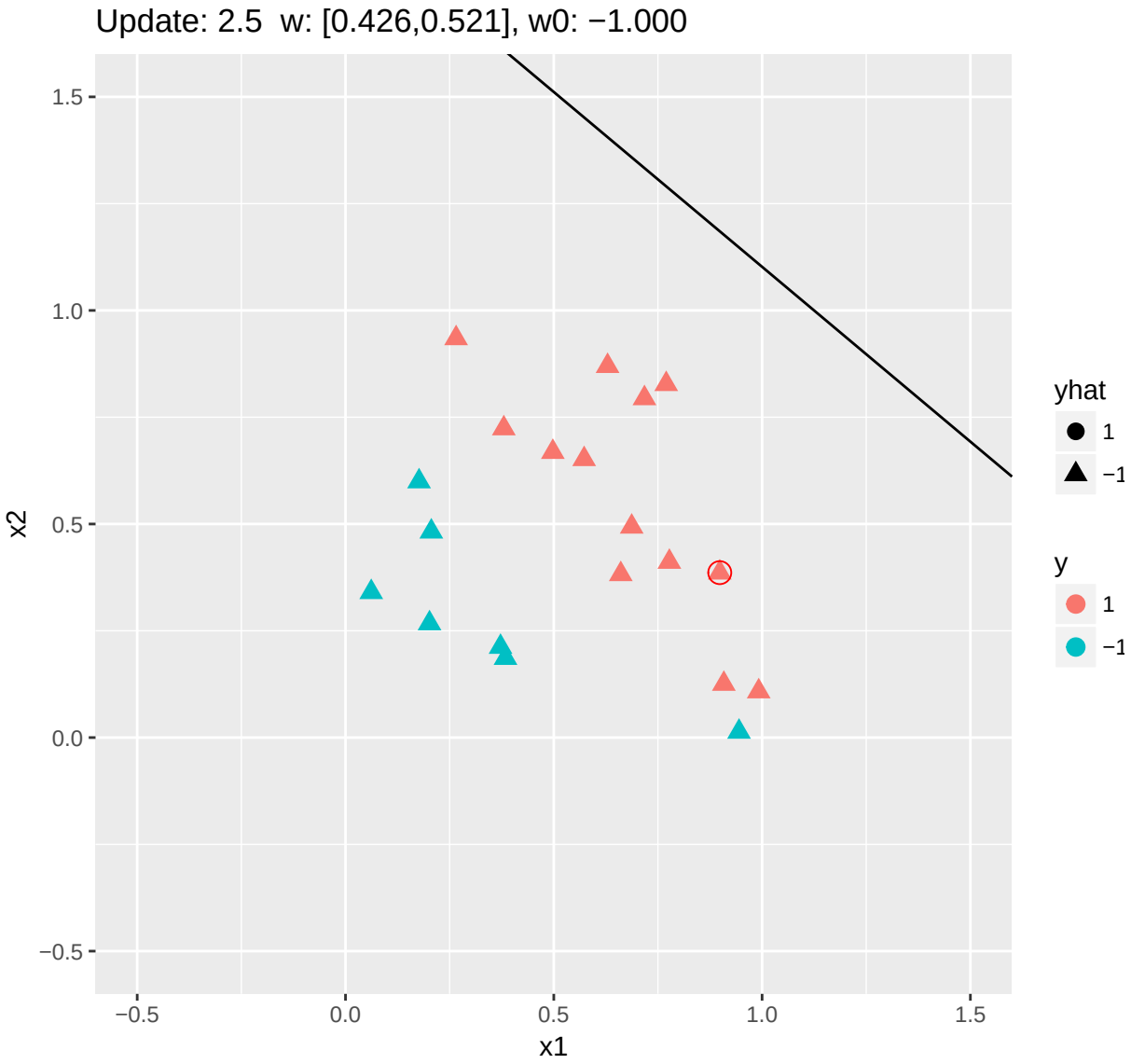

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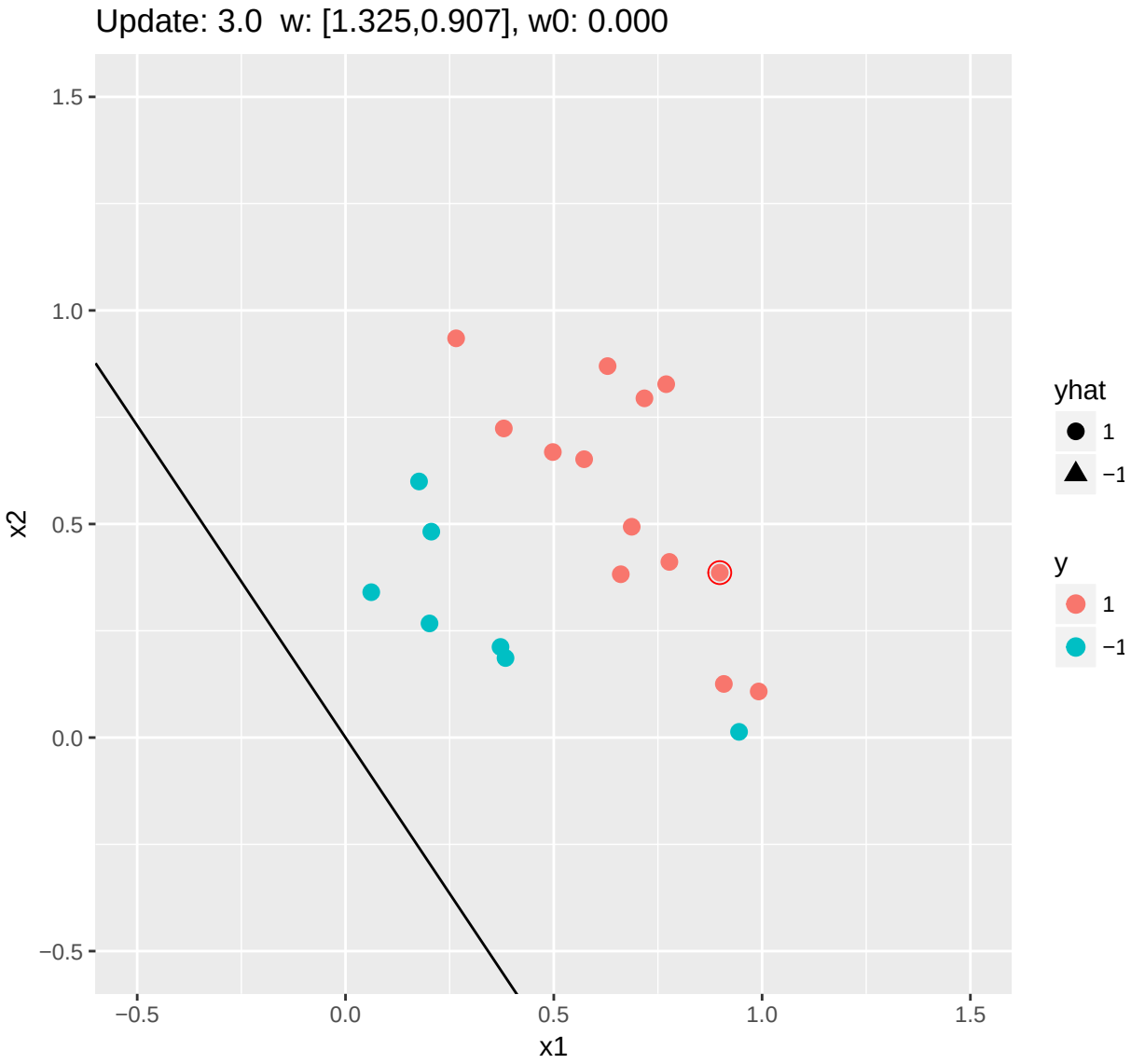

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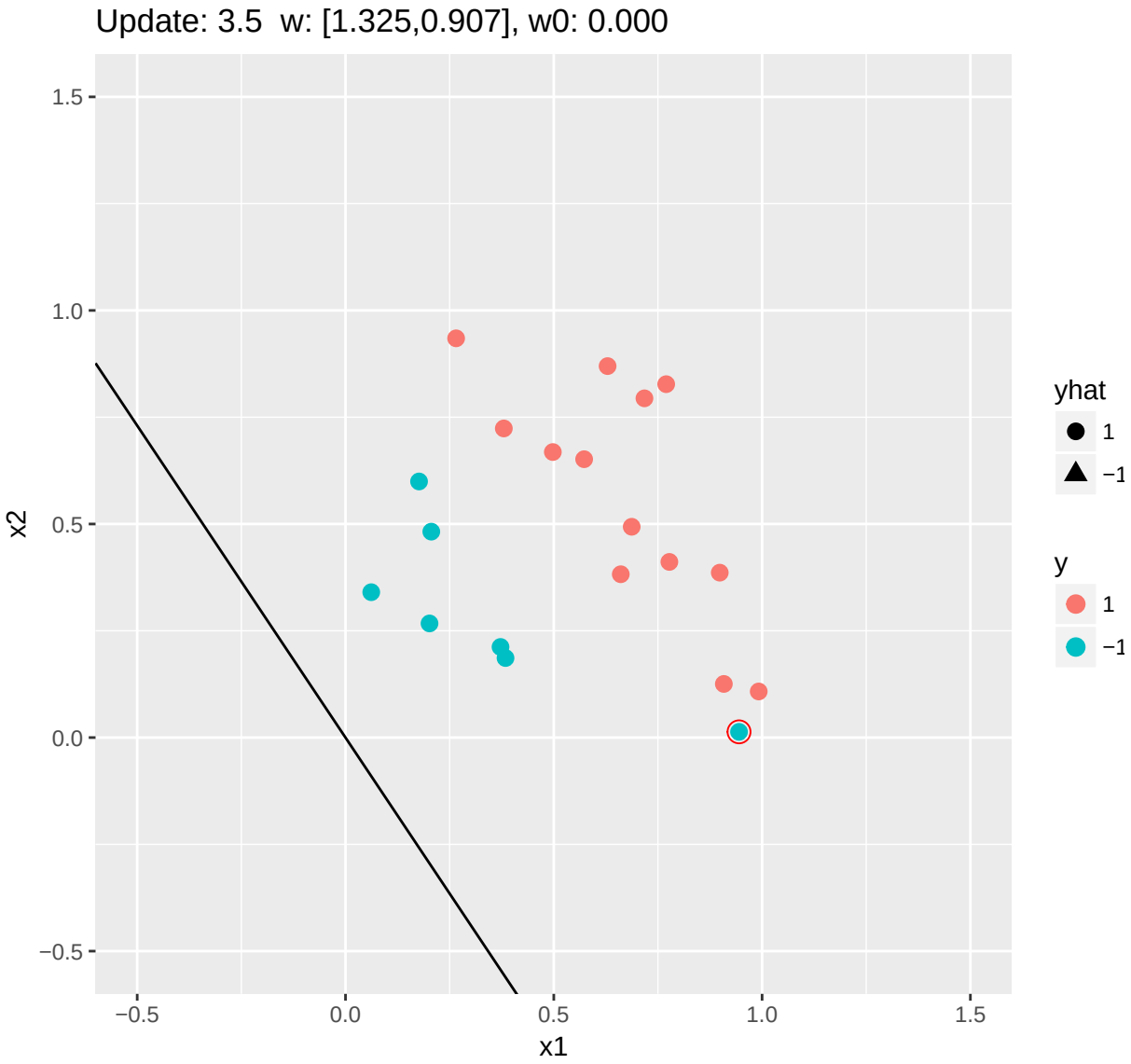

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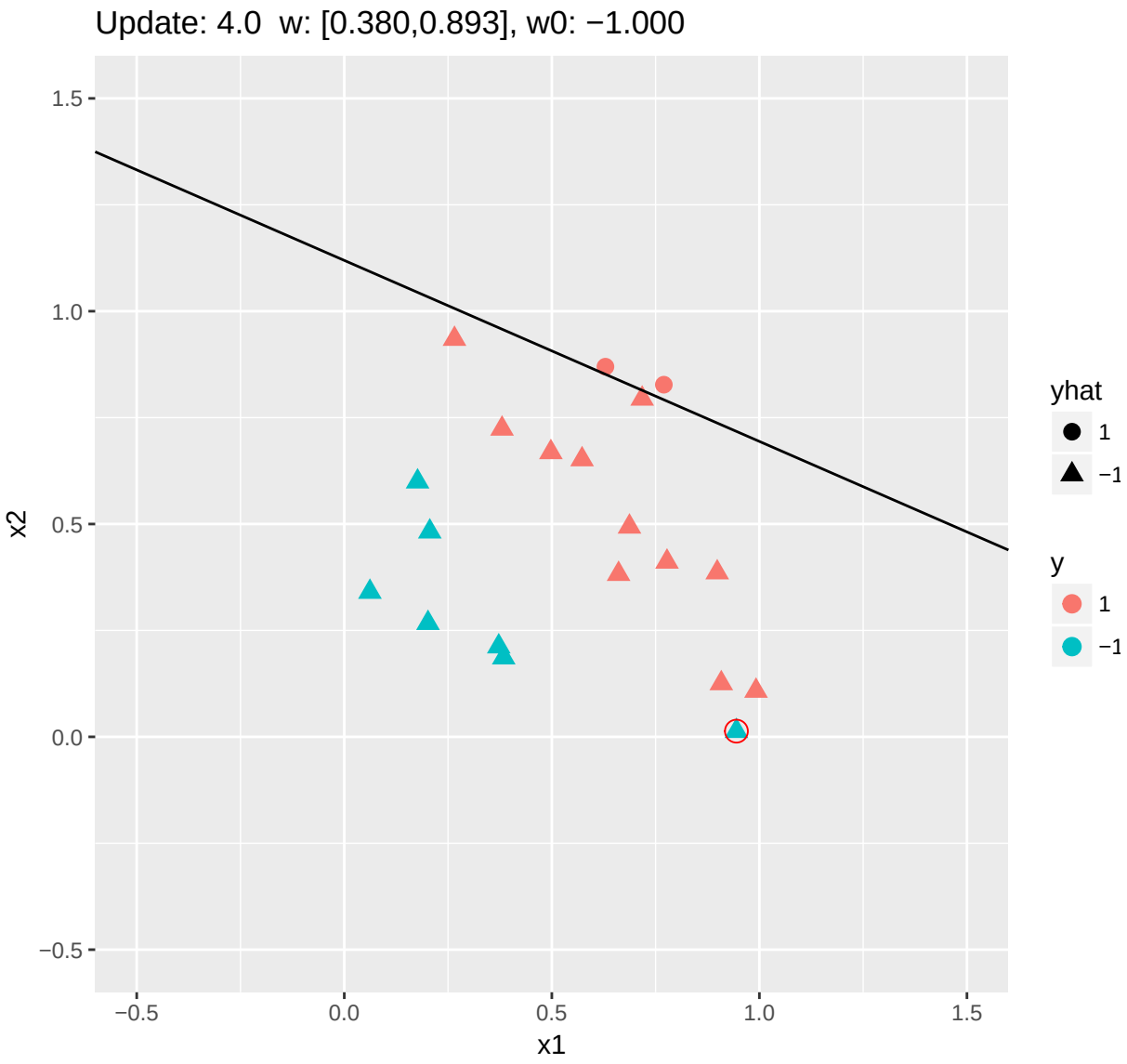

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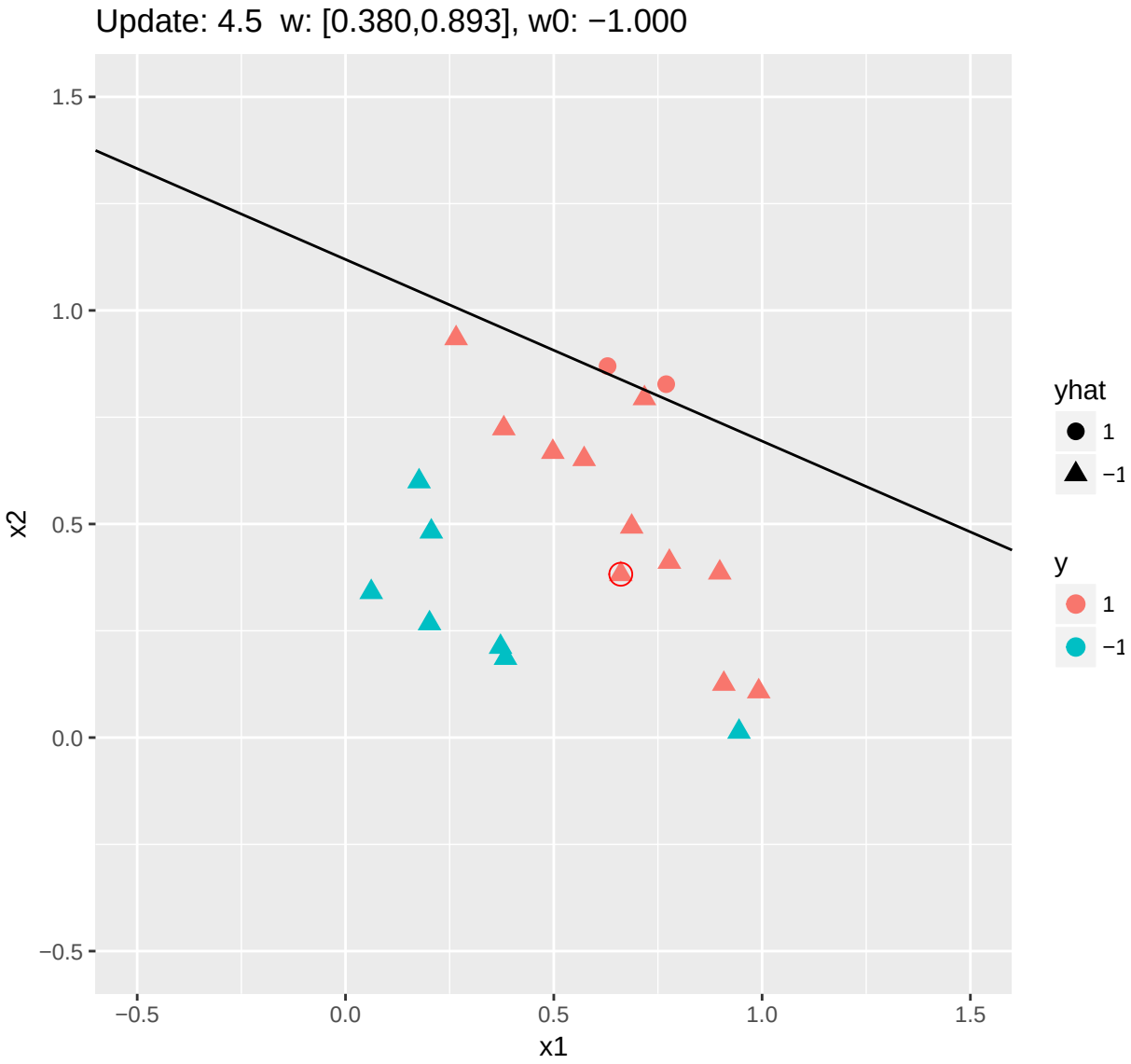

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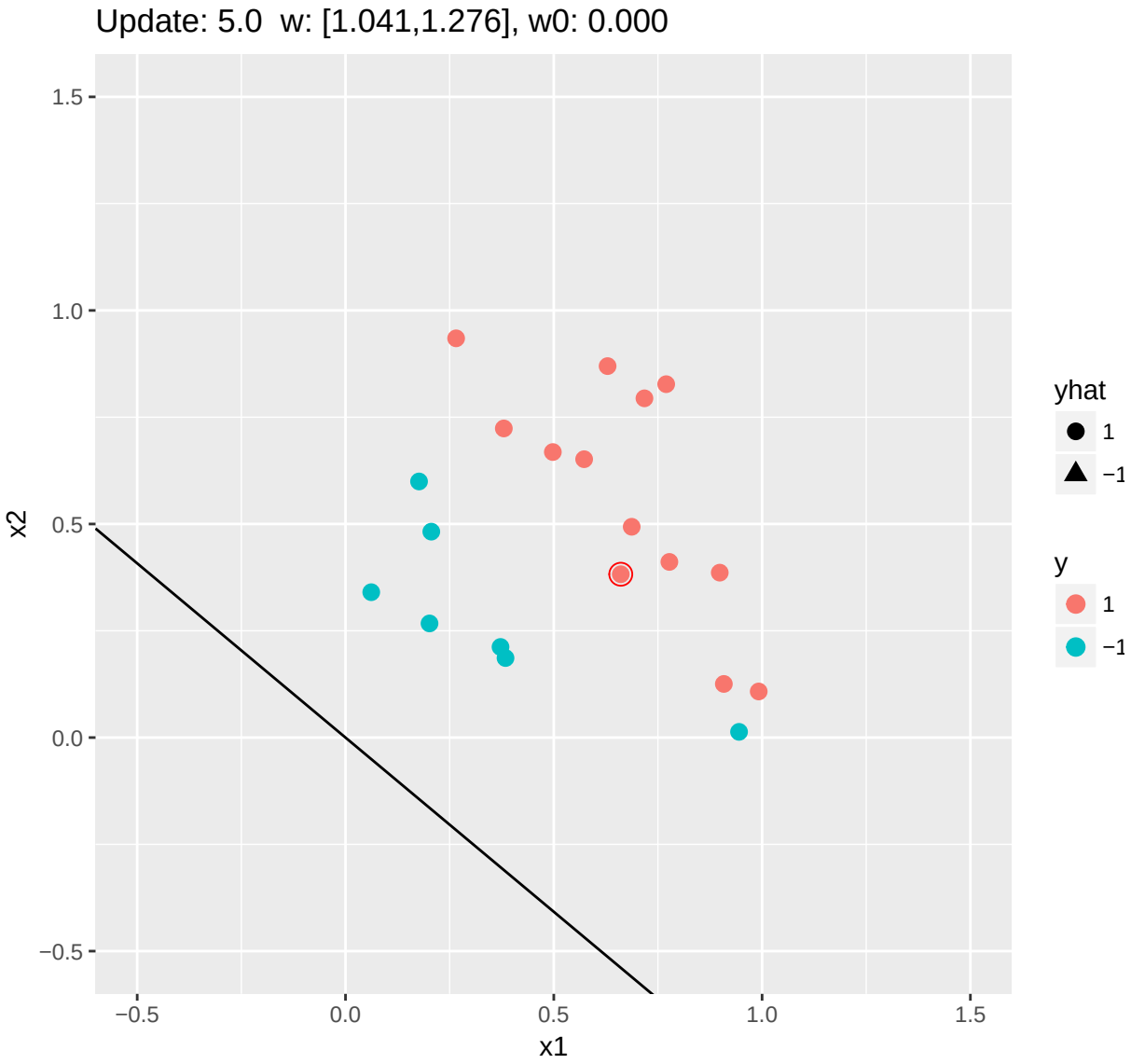

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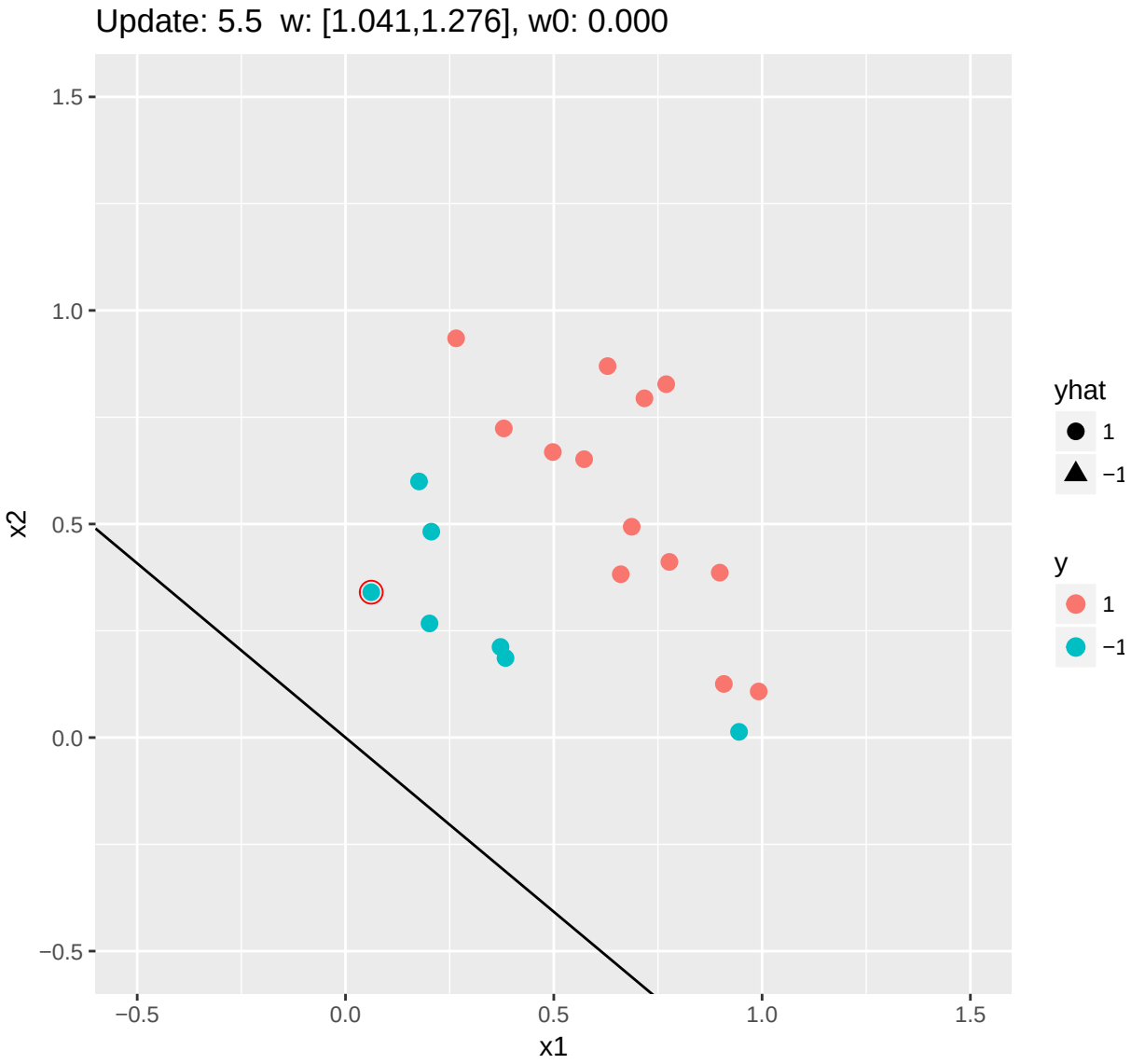

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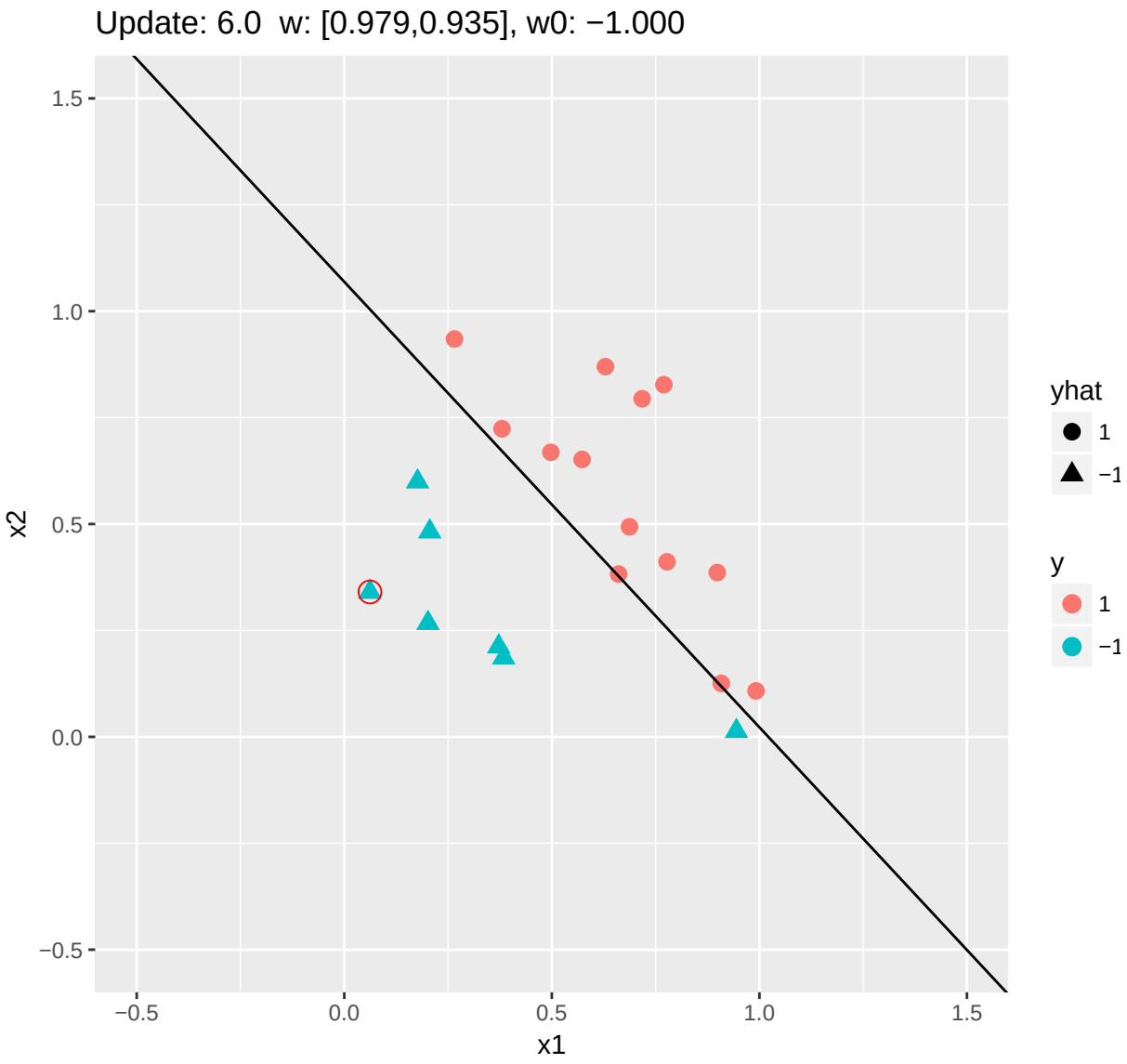


Weight as a combination of input vectors

- Recall percepton learning rule:

$$\mathbf{w} \leftarrow \mathbf{w} + \delta y_i \mathbf{x}_i, \qquad w_0 \leftarrow w_0 + \delta y_i$$

- If initial weights are zero, then at any step, the *weights are a linear combination of feature vectors of the examples*:

$$\mathbf{w} = \sum_{i=1}^{n} \alpha_i y_i \mathbf{x}_i, \qquad w_0 = \sum_{i=1}^{n} \alpha_i y_i$$

  where α_i is the sum of step sizes used for all updates based on example i.

- This is called the *dual representation* of the classifier.

- Even by the end of training, some examples may have never participated in an update, just by chance. So their corresponding $\alpha_i = 0$.

Examples used (bold) and not used (faint) in updates

Comment: Solutions are nonunique

Perceptron summary

- Perceptrons can be learned to fit linearly separable data, using a gradient descent rule.
- Blindingly fast
- Solutions are non-unique

Support Vector Machines

- Support vector machines (SVMs) for binary classification can be viewed as a way of training perceptrons
- Three main new ideas:
  - A optimization criterion (the "margin") guarantees uniqueness and has theoretical advantages
  - Natural handling nonseparable data by allowing mistakes
  - An efficient way of operating in expanded feature spaces: "kernel trick"
- SVMs can also be used for multiclass classification and regression.

Returning to the non-uniqueness issue

- Consider a linearly separable binary classification data set
- There is an infinite number of hyperplanes that separate the classes:
- Which plane is best?
- For a given plane, for which points should we be most confident in the classification?

The margin, and linear SVMs

- For a given separating hyperplane, the *margin* is two times the (Euclidean) distance from the hyperplane to the nearest training example.
- Width of the "strip" around the decision boundary containing no training examples.
- A linear SVM is a perceptron for which we choose $\mathbf{w}, w_0$ so that margin is maximized

Distance to the decision boundary

- Suppose we have a decision boundary that separates the data.
- Let γ_i be the distance from instance $\mathbf{x}_i$ to the decision boundary.
- How can we write γ_i in terms of $\mathbf{x}_i, y_i, \mathbf{w}, w_0$?

Distance to the decision boundary (II)

- $\mathbf{w}$ is orthogonal to boundary, $\frac{\mathbf{w}}{||\mathbf{w}||}$ is the unit vector orthogonal to the boundary
- Vector from B to $\mathbf{x}_i$ is $\gamma_i \frac{\mathbf{w}}{||\mathbf{w}||}$.
- B, the point on the boundary nearest $\mathbf{x}_i$, is $\mathbf{x}_i - \gamma_i \frac{\mathbf{w}}{||\mathbf{w}||}$.
- Since B is on the boundary,

$$\left(\mathbf{x}_i - \gamma_i \frac{\mathbf{w}}{||\mathbf{w}||}\right)^\mathsf{T} \mathbf{w} + w_0 = 0$$

- Solving for γ_i yields

$$\gamma_i = \frac{\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0}{||\mathbf{w}||}$$

The margin HTF Ch. 4.5, Ch 12

- The *margin of the hyperplane* is $2M$, where $M = \min_i y_i\gamma_i$
- The most direct statement of the problem of finding a maximum margin separating hyperplane is thus

$$\max_{\mathbf{w},w_0} \min_i y_i\gamma_i \equiv \max_{\mathbf{w},w_0} \min_i y_i \frac{\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0}{||\mathbf{w}||}$$

- This turns out to be inconvenient for optimization, however

Treating the γ_i as constraints

- From the definition of margin, we have:

$$M \leq y_i\gamma_i = y_i \frac{\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0}{||\mathbf{w}||} \quad \forall i$$

- This suggests:

maximize M	with respect to $M, \mathbf{w}, w_0$
	subject to $M \leq y_i \frac{\mathbf{x}_i^\mathsf{T}\mathbf{w}+w_0}{||\mathbf{w}||}$ for all i

- Problems:
  - $\mathbf{w}$ appears nonlinearly in the constraints.
  - This problem is underconstrained. If $(\mathbf{w}, w_0, M)$ is an optimal solution, then so is $(\beta\mathbf{w}, \beta w_0, M)$ for any $\beta > 0$.

Adding a constraint

Let's add the constraint that $M = 1/\|\mathbf{w}\|$:

- This allows us to rewrite the objective function:
- This is really nice because the constraints are linear.

maximize $\frac{1}{\|\|\mathbf{w}\|\|}$	with respect to $\mathbf{w}, w_0$
	subject to $\frac{1}{\|\|\mathbf{w}\|\|} \leq y_i \frac{\mathbf{x}_i^\mathsf{T}\mathbf{w}+w_0}{\|\|\mathbf{w}\|\|}$ for all i

which is the same as

maximize $\frac{1}{\|\|\mathbf{w}\|\|}$	with respect to $\mathbf{w}, w_0$
	subject to $1 \leq y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0)$ for all i

Final formulation

- Let's minimize $\|\mathbf{w}\|^2$ instead of maximizing $\frac{1}{\|\|\mathbf{w}\|\|}$. (Taking the square is a monotone transformation, as $\|\mathbf{w}\|$ is postive, so this doesn't change the optimal solution.)
- This gets us to:

minimize	$\|\mathbf{w}\|^2$ w.r.t. $\mathbf{w}, w_0$
subject to	$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1$

- This we can solve! How?
  - It is a convex *quadratic programming* (QP) problem—a standard type of optimization problem for which many efficient packages are available.

Example

We have a solution, but no "support vectors" yet…

What are "Support Vectors"?

minimize	$\frac{1}{2}\|\mathbf{w}\|^2$ w.r.t. $\mathbf{w}, w_0$	subject to	$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1$

- Turns out (HTF Ch. 4.5.2) we can write:

$$\mathbf{w} = \sum_i \alpha_i y_i \mathbf{x}_i, \quad \text{where } \alpha_i \geq 0$$

- As for the perceptron with zero initial weights, the optimal solution for $\mathbf{w}$ and w_0 is a linear combination of the $\mathbf{x}_i$.
- The output is therefore:

$$h_{\mathbf{w},w_0}(\mathbf{x}) = \text{sign}\left(\sum_{i=1}^{n} \alpha_i y_i (\mathbf{x}_i \cdot \mathbf{x}) + w_0\right)$$

- Output depends on weighted dot product of input vector with training examples

Solving "the dual"

- We can actually solve directly for the α_i (again see HTF Ch. 4.5.2):

$$\max_{\boldsymbol{\alpha}} \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j \alpha_i \alpha_j (\mathbf{x}_i \cdot \mathbf{x}_j)$$

  with constraints: $\alpha_i \geq 0$ and $\sum_i \alpha_i y_i = 0$

- This is also a QP

The support vectors

- Suppose we find optimal $\boldsymbol{\alpha}$s (e.g., using a standard QP package)
- The α_i will be > 0 only for the points for which $y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) = 1$
- These are the points lying on the edge of the margin, and they are called *support vectors*, because they define the decision boundary
- The output of the classifier for query point $\mathbf{x}$ is computed as:

$$\text{sgn}\left[\left(\sum_{i=1}^{n} \alpha_i y_i (\mathbf{x}_i \cdot \mathbf{x})\right) + w_0\right]$$

  Hence, the output is determined by computing the *dot product of the point with the support vectors*

Example

Support vectors are in bold

But why all this work?

- SVMs are a state-of-the-art for classification when you don't need probability estimates
- Inuitively, the large-margin property makes sense. Theory backs this up.
- SVMs offer "off-the-shelf" *non*-linear classification without having to do explicit feature construction, as we will see.

Soft margin classifiers

- Recall that in the linearly separable case, we compute the solution to the following optimization problem:

min	$\frac{1}{2}\|\mathbf{w}\|^2$	w.r.t. $\mathbf{w}, w_0$
s.t.	$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1$	

- What if we can't satisfy the constraints?

Soft margin classifiers

- To allow misclassifications, we relax the constraints to:

$$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1 - \xi_i$$

- If $\xi_i \in (0, 1)$, the data point is within the margin
- If $\xi_i \geq 1$, then the data point is misclassified
- We define the *soft error* as $\sum_i \xi_i$; each ξ_i is a *slack variable*

Problem formulation with soft errors

- Instead of:

min	$\frac{1}{2}\|\mathbf{w}\|^2$ w.r.t. $\mathbf{w}, w_0$
s.t.	$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1$

  we want to solve:

min	$\frac{1}{2}\|\mathbf{w}\|^2 + C\sum_i \xi_i$ w.r.t. $\mathbf{w}, w_0, \xi_i$
s.t.	$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1 - \xi_i,\ \xi_i \geq 0$

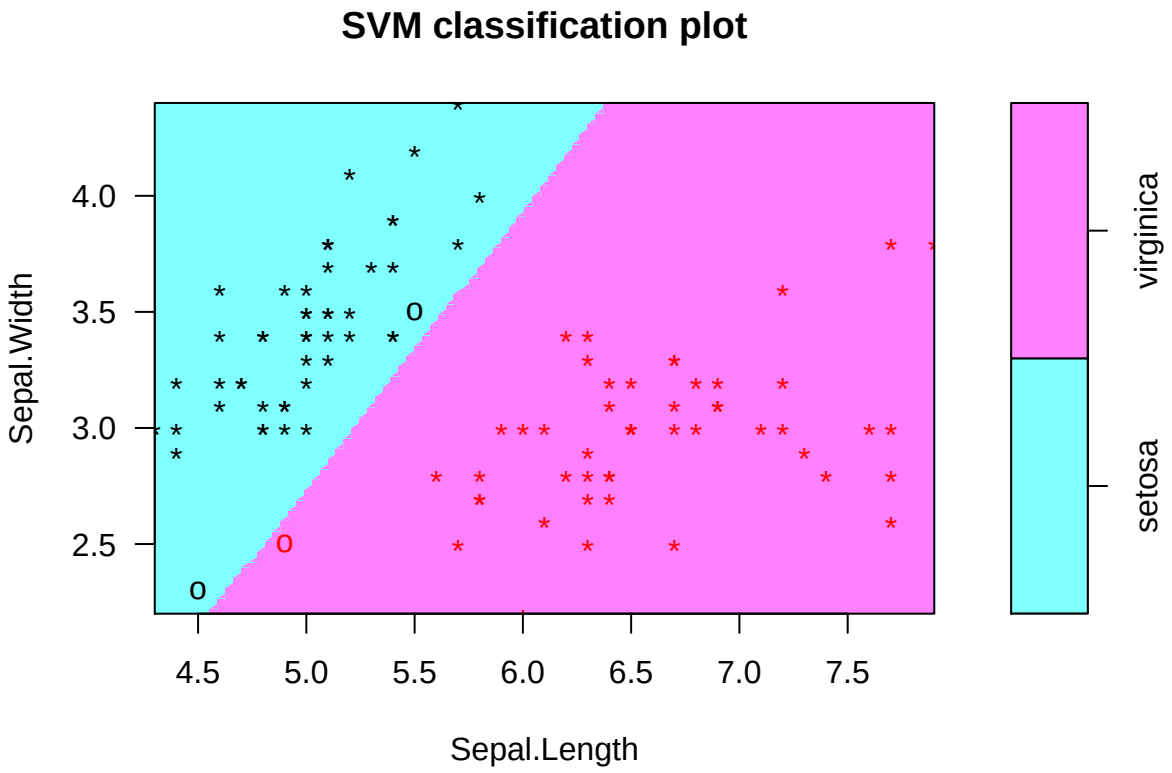
- Note that soft errors include points that are misclassified, as well as points within the margin
- There is a linear penalty for both categories
- The choice of the *constant C controls boundary-fitting*

A built-in boundary-fitting knob

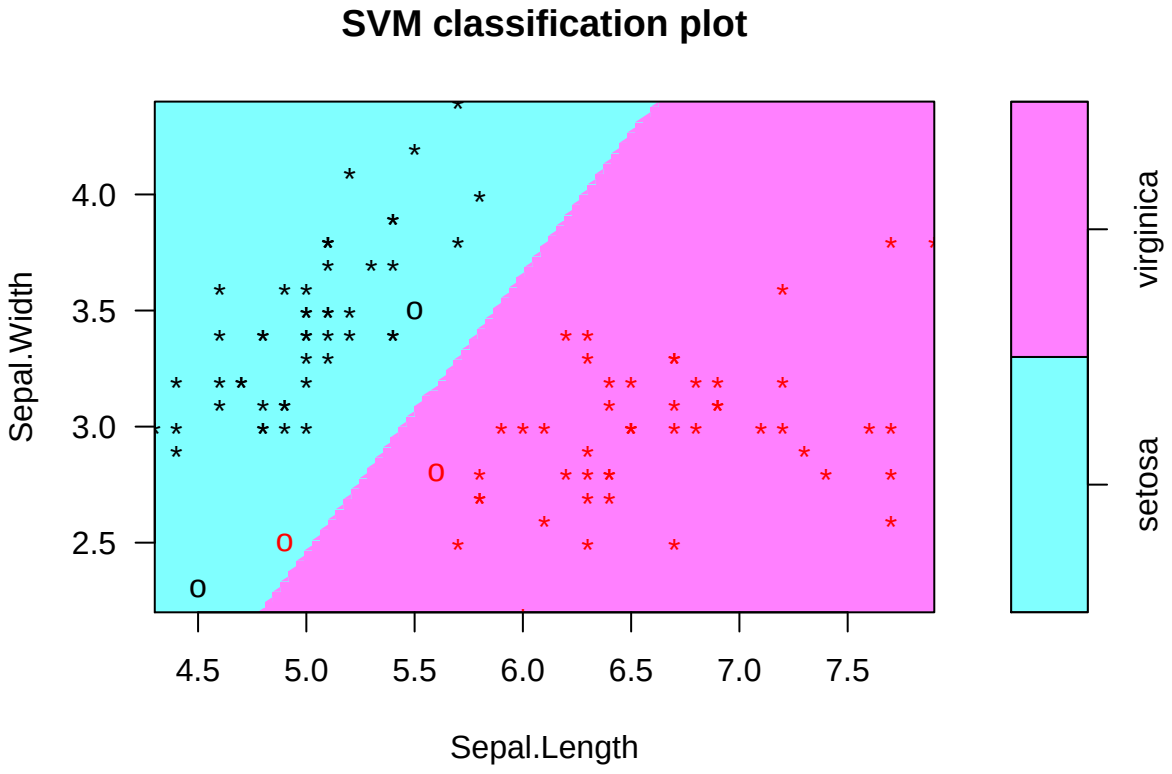
min	$\frac{1}{2}\|\mathbf{w}\|^2 + C\sum_i \xi_i$
w.r.t.	$\mathbf{w}, w_0, \xi_i$
s.t.	$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1 - \xi_i,\ \xi_i \geq 0$

- If C is very small, there is almost no penalty for soft errors, so the focus is on maximizing the margin, even if this means more mistakes
- If C is very large, the emphasis on the soft errors will decrease the margin, if this helps to classify more examples correctly.
- How could we choose C?

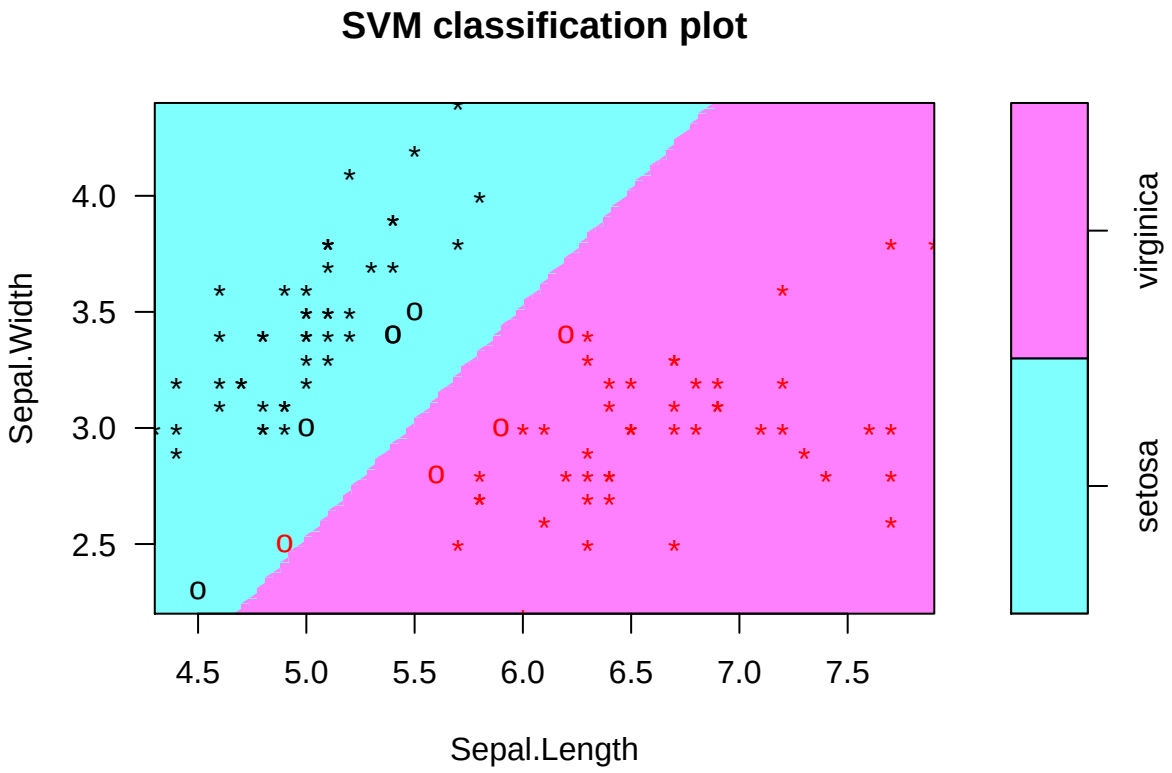
Example, C = 100



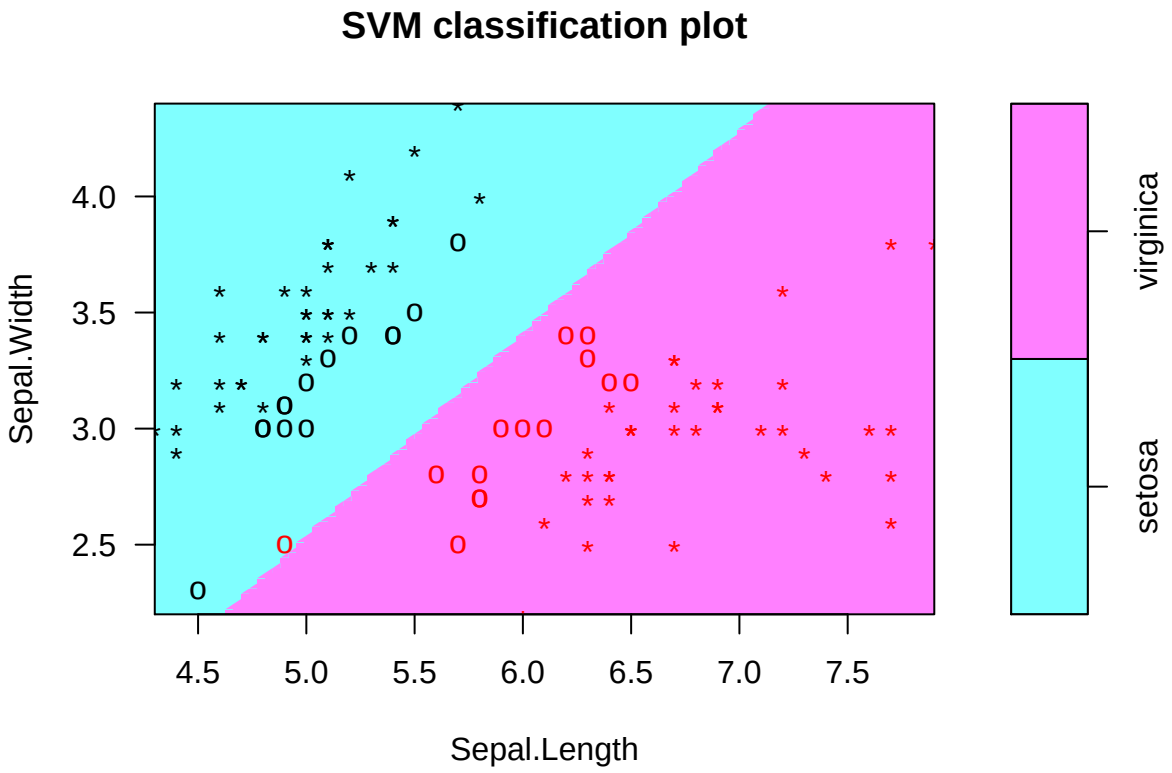
Example, C = 10



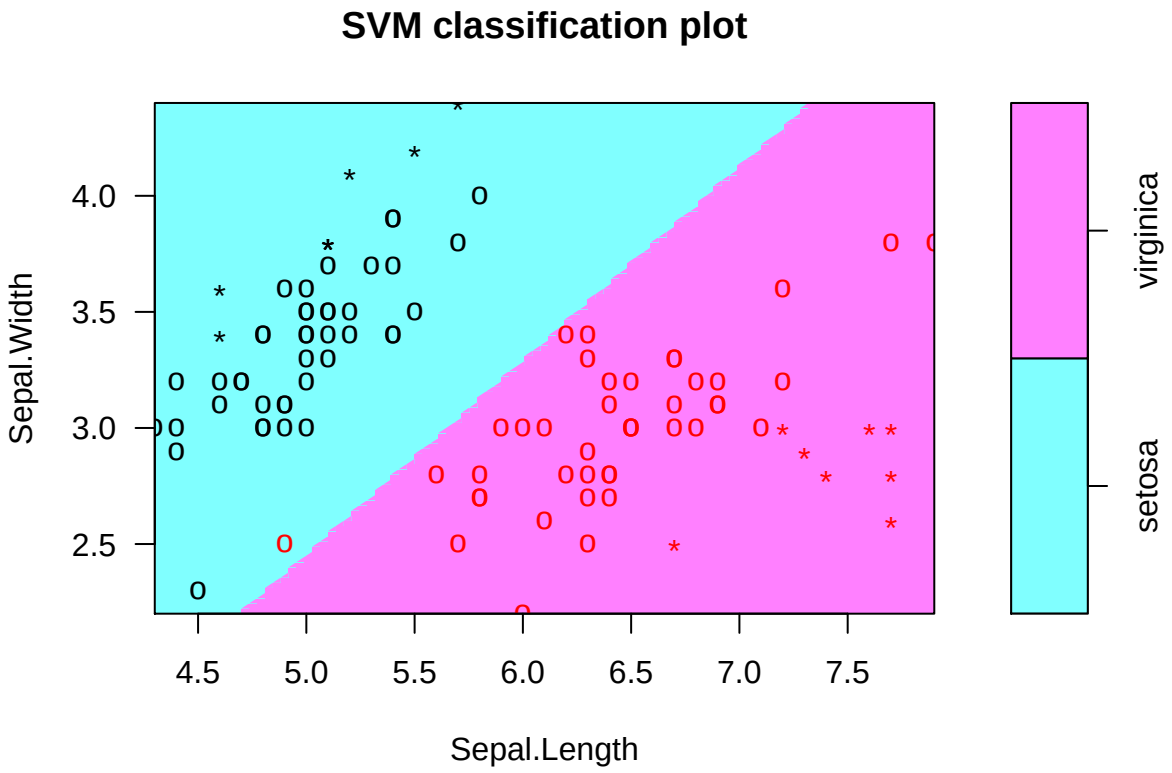
Example, C = 1



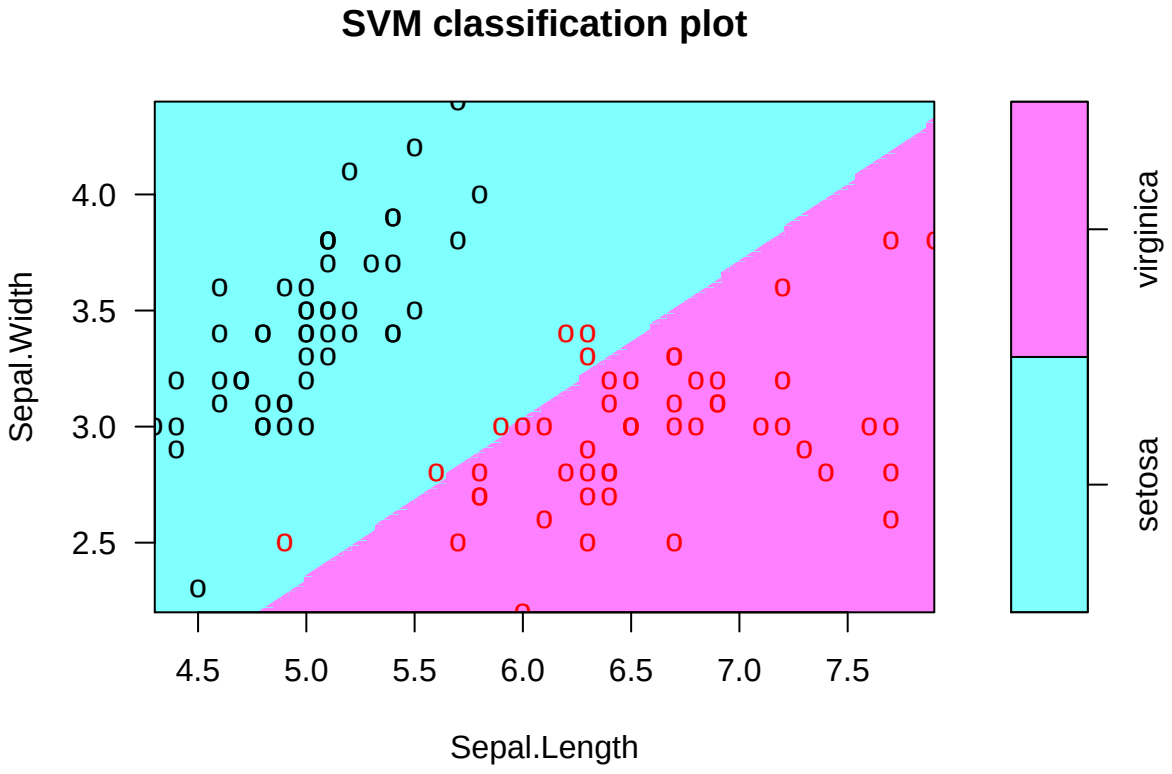
Example, C = 0.1



Example, C = 0.01



Example, C = 0.001



Dual form for the soft margin problem

- Like before, we can formulate a "dual" problem that identifies the support vectors:

	Primal form:
min	$\|\mathbf{w}\|^2 + C\sum_i \xi_i$ w.r.t. $\mathbf{w}, w_0, \xi_i$
s.t.	$y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq (1 - \xi_i)$, $\xi_i \geq 0$

	Dual form:
max	$\sum_{i=1}^n \alpha_i - \frac{1}{2}\sum_{i,j=1}^n y_i y_j \alpha_i \alpha_j (\mathbf{x}_i \cdot \mathbf{x}_j)$ w.r.t. α_i
s.t.	$0 \leq \alpha_i \leq C$, $\sum_{i=1}^n \alpha_i y_i = 0$

- All the previously described machinery can be used to solve this problem

Supervised Learning Methods: "Objective-driven"

Mthd.	Form	Objective
OLS	$h_w(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{w}$	$\sum_{i=1}^n (h_\mathbf{w}(\mathbf{x}_i) - y_i)^2$
	$\approx E[Y = y\|\mathbf{X} = \mathbf{x}]$...	...using a linear function
LR	$h_\mathbf{w}(\mathbf{x}) = \frac{1}{1+\mathrm{e}^{-\mathbf{x}^\mathsf{T}\mathbf{w}}}$	$-\sum_{i=1}^n y_i \log h_\mathbf{w}(\mathbf{x}_i) + (1 - y_i)\log(1 - h_\mathbf{w}(\mathbf{x}_i))$
	$\approx P(Y = y\|\mathbf{X} = \mathbf{x})$...	...using sigmoid of a linear function
SVM	$h_\mathbf{w}(\mathbf{x}) = \mathrm{sgn}(\mathbf{x}^\mathsf{T}\mathbf{w})$	$\frac{1}{2}\|\mathbf{w}\|^2 + C\sum_i \xi_i$ $y_i(\mathbf{x}_i^\mathsf{T}\mathbf{w} + w_0) \geq 1 - \xi_i$, $\xi_i \geq 0$
	$\approx$ decision boundary	...using a linear separator