CS 4424
Newton iteration
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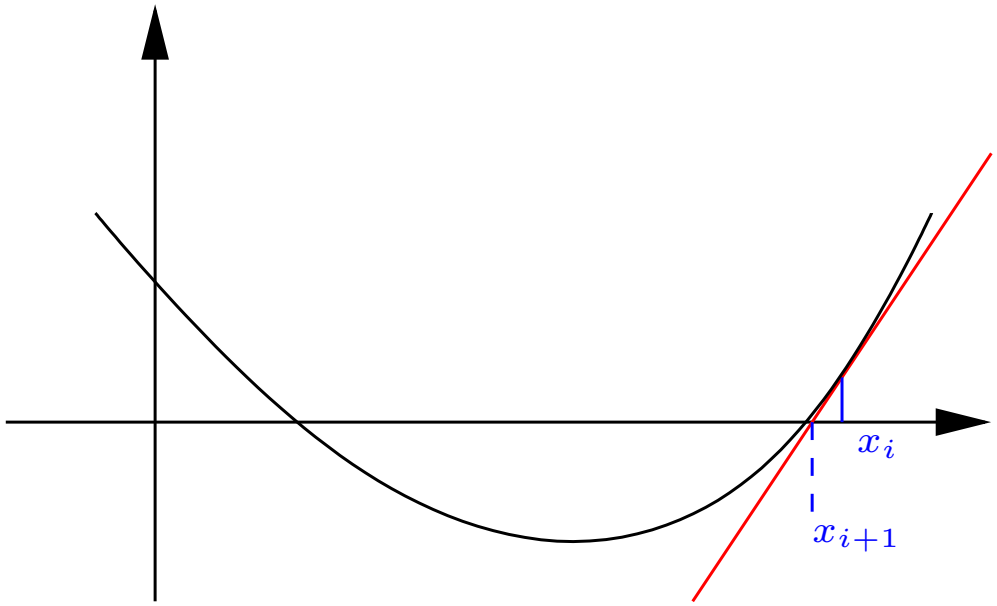
What this is about

Newton iteration is a way to compute approximate solutions to various problems.

It is classically defined in analysis: to compute a root of $P(x)$, we use the iteration

$$x_0 = \text{random}, \qquad x_{i+1} = x_i - \frac{P(x_i)}{P'(x_i)}.$$



Here, we will use it for power series computations.

Approximations

There are strong analogies between real numbers

$$a = 0.934938476304 96 \cdots = \sum_{i \geq 1} a_i \frac{1}{10^i}$$

and power series

$$S = \sum_{i \geq 0} s_i x^i.$$

- both are infinite expansions;
- computationally, we are interested in computing truncations at finite precision;
- similar techniques apply.

Remark: series are easier to handle than real numbers, because there is no carry.

Newton iteration

The example we saw can be vastly generalized and applied to series computations.

In a nutshell:

- applies to compute exponential, inverses, logarithms, square roots, ... of series,
- more generally, solutions of polynomial or differential equations.

Main feature: efficiency!

- typical behaviour: the number of correct terms doubles each step;
- combined with polynomial multiplication $\Longrightarrow$ quasi-optimal.

Multiplication

Reminder: polynomial multiplication

Let $\mathsf{M}(d)$ denote the cost of polynomial multiplication in degree d:

- $\mathsf{M}(d) \in O(d^2)$ for a naive algorithm
- $\mathsf{M}(d) \in O(d^{1.6})$ for Karatsuba algorithm
- $\mathsf{M}(d) \in O(d \log d)$ using Fast Fourier Transform (if the field has roots of 1)
- $\mathsf{M}(d) \in O(d \log d \log \log d)$ using Fast Fourier Transform in general.

Technically, we ask $\mathsf{M}(d + d') \geq \mathsf{M}(d) + \mathsf{M}(d')$.

A few rules for estimating complexity

We know that

$$\mathbf{1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1 \leq 2^n}.$$

We have similar estimates for polynomial multiplication:

$$\mathsf{M}(1) + \mathsf{M}(2) + \mathsf{M}(4) + \cdots + \mathsf{M}(2^{n-1}) \leq \mathsf{M}(2^n).$$

Proof.

$$\begin{aligned}
\mathsf{M}(a) + \mathsf{M}(b) \leq \mathsf{M}(a+b) \quad &\Longrightarrow \quad 2\mathsf{M}(a) \leq \mathsf{M}(2a) \\
&\Longrightarrow \quad 2^k \mathsf{M}(a) \leq \mathsf{M}(2^k a) \\
&\Longrightarrow \quad 2^k \mathsf{M}(2^n/2^k) \leq \mathsf{M}(2^n) \\
&\Longrightarrow \quad \mathsf{M}(2^n/2^k) \leq 2^{-k}\mathsf{M}(2^n)
\end{aligned}$$

Corollary. If $T(2n) \leq T(n) + C\mathsf{M}(n)$, then $T(n) \in O(\mathsf{M}(n))$.

Inversion

Iteration for the inverse

Given a series

$$f = f_0 + f_1 x + f_2 x^2 + \cdots, \quad f_0 \neq 0$$

we want to compute the coefficients of

$$g = g_0 + g_1 x + g_2 x^2 + \cdots$$

such that $fg = 1$.

Basic idea:

- compute one term after the other, by identification.
- slow: $O(n^2)$ operations for n terms.
- at least, this proves that g exists and is unique.

Newton's iteration

Idea: g is a root of $P(g) = 0$, with

$$P(g) = \frac{1}{g} - f.$$

With this P, the Newton iteration becomes

$$h_0 = 1/f_0, \quad h_{(i+1)} = 2h_{(i)} - h_{(i)}^2 f \bmod x^{2^{i+1}}.$$

So we consider the operation $h \mapsto N(h) = 2h - h^2 f$.

- Suppose $h = g \bmod x^k$.
- Multiplying by f, we get $hf \bmod x^k = 1$. This means $hf = 1 + x^k R$.
- Then, $N(h)f = 2hf - h^2 f^2 = 1 - (hf - 1)^2 = 1 - x^{2k} R^2$.
- So $N(h) = g \bmod x^{2k}$.

Cost analysis

The previous argument shows that

$$h_{(i)} = g \bmod x^{2^i}.$$

Cost of a single step:

- To get $h_{(i+1)}$ from $h_{(i)}$, we compute $2h_{(i)} - h_{(i)}^2 f$ modulo $x^{2^{i+1}}$.
- This costs $O(\mathsf{M}(2^{i+1}))$.

Cumulated cost:

- To get $h_{(i)}$ from $h_{(0)} = 1$, the cost is

$$O(\mathsf{M}(2) + \mathsf{M}(4) + \cdots + \mathsf{M}(2^i)) = O(\mathsf{M}(2^i)).$$

- In other words: to get $1/f \bmod x^n$, the cost is $O(\mathsf{M}(n))$.

Algebraic series

Roots of polynomial

Def.

- A series

$$f = \sum_{i \geq 0} f_i x^i$$

  is algebraic if there exists a polynomial

$$P(x, z) \quad \text{such that} \quad P(x, f) = 0.$$

Examples.

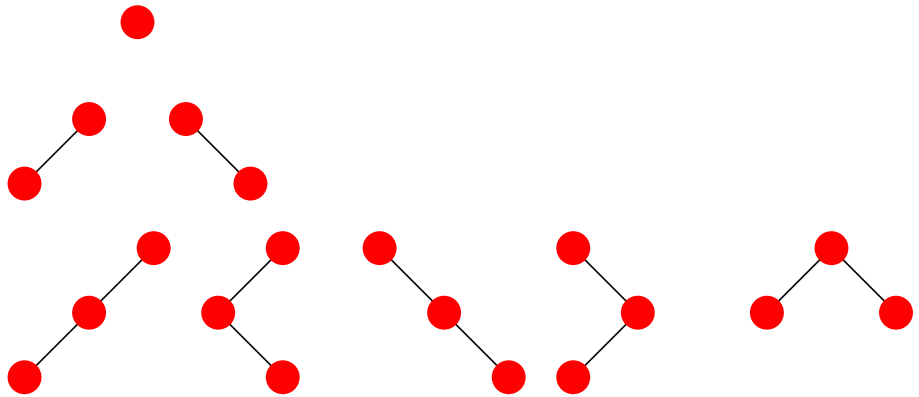
- rational series $f(x) = n(x)/d(x)$
- $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 + \cdots$
- $\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$ is not algebraic.

Examples from combinatorics

A lot of sequences arising from enumeration problems satisfy nice properties, like having an algebraic generating series.

Example: Catalan numbers.

Let C_n be the number of binary trees with n nodes.



$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5, \ldots$

Recurrence relation

To build a tree with n nodes, you

- set the root (so you have $n-1$ nodes left)
- put p nodes on the left
- and $n-1-p$ nodes on the right.

This gives

$$C_n = \sum_{p=0}^{n-1} C_p C_{n-1-p}$$

for $n \geq 1$.

The generating series

Let $f = \sum_{i \geq 0} C_i x^i$. Then,

$$\sum_{p=0}^{n-1} C_p C_{n-1-p}$$

is the coefficient of x^{n-1} in f^2.

Multiplying the recurrence relation by x^n and summing for $n \geq 1$, we get

$$f - 1 = x f^2.$$

So f is algebraic, with

$$P(x, z) = xz^2 - z + 1$$

Extracting the coefficients

In this case, we have an explicit formula

$$f = \frac{1 - \sqrt{1 - 4\,x}}{2x},$$

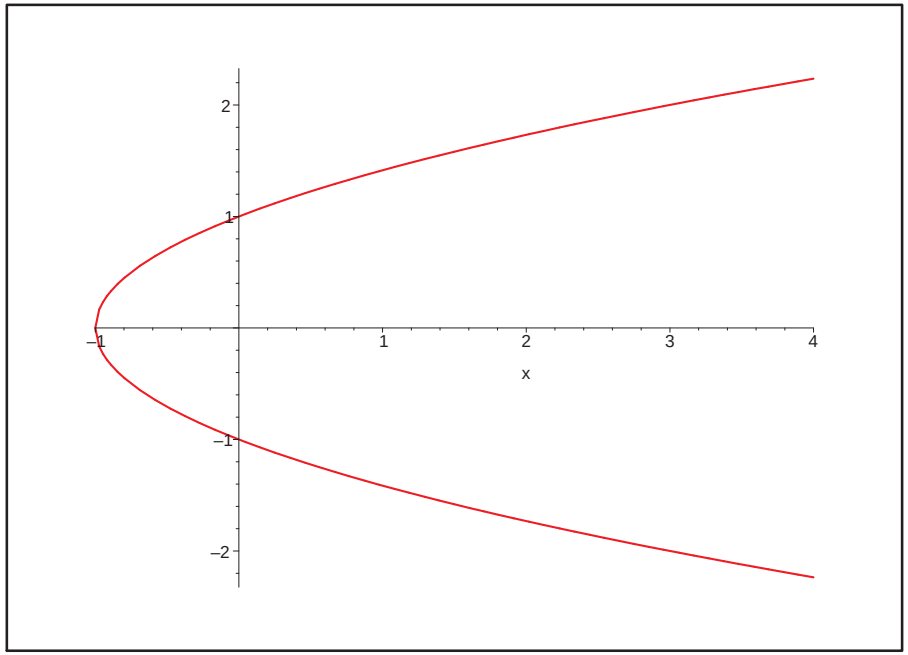
from which one can deduce

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In general, though, there are no closed formula.

Link to geometry

Let $f(x) = \sqrt{1+x}$. The power series expansion of f gives successive approximations at $x = 0$.



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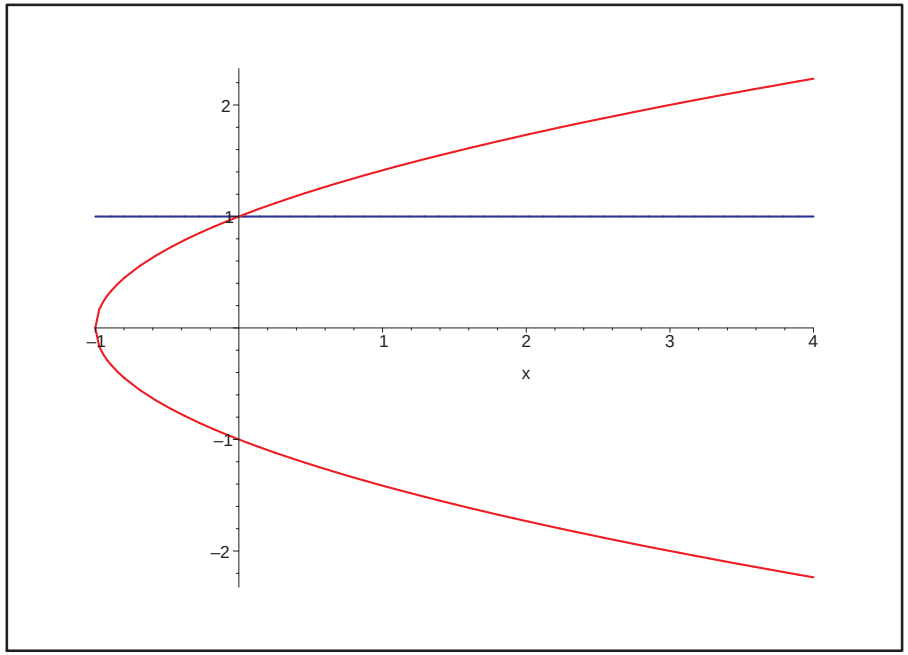

$f \bmod x = 1$

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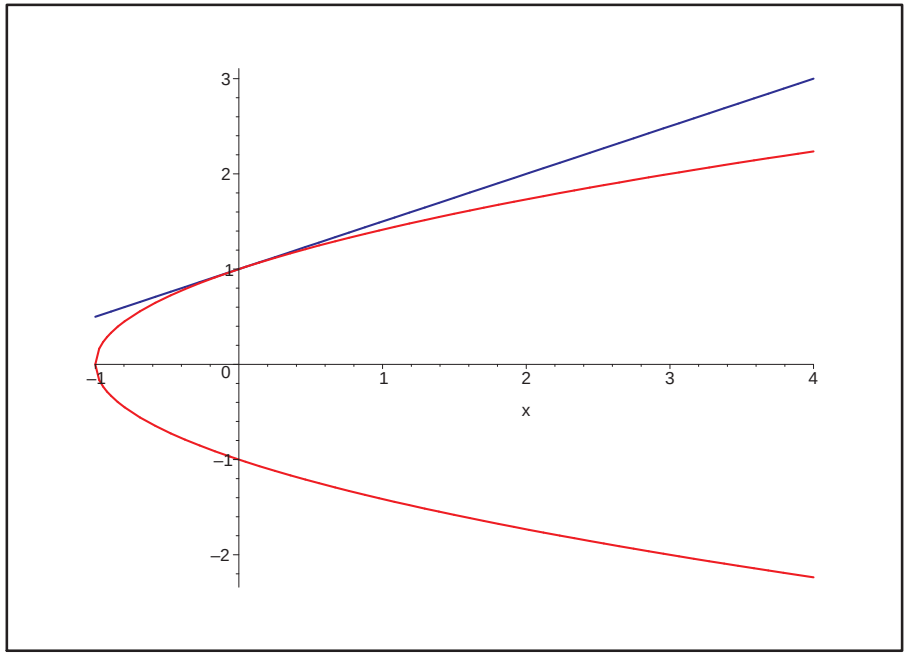

$$f \bmod x^2 = 1 + \frac{1}{2}x$$

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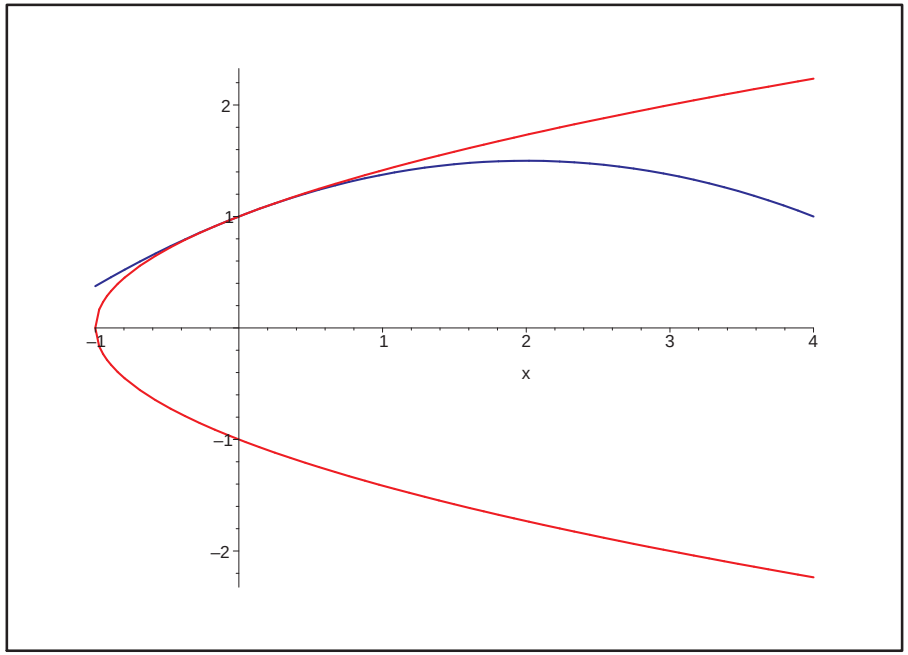

$$f \bmod x^3 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

Computing the expansion

We will compute the expansion of f such that

$$P(x, f) = 0$$

subject to the following:

- the constant term f_0 of f is known

  we need it to start the process

- the partial derivative $\frac{\partial P}{\partial z}(0, f_0)$ is not zero.

  at the starting point, the tangent to the curve exists, and is not vertical

The slow algorithm

Suppose that we know the first terms

$$f_{\text{init}} = f_0 + \cdots + f_{n-1}x^{n-1},$$

such that

$$P(x, f_{\text{init}}) \bmod x^n = 0.$$

Basic step

- we look for a single extra term $f_n x^n$, to get

  $$f_{\text{next}} = f_0 + \cdots + f_{n-1}x^{n-1} + f_n x^n,$$

  such that

  $$P(x, f_{\text{next}}) \bmod x^{n+1} = 0.$$

- we get it by identification.

Getting the next term

For any k, we have

$$\begin{aligned}(f_0 + \cdots + f_{n-1}x^{n-1} + f_n x^n)^k &= \text{known stuff in } f_0, \ldots, f_{n-1} \\ &+ k f_0^{k-1} f_n x^n \\ &+ \text{higher order terms.}\end{aligned}$$

If we write

$$P(x, z) = p_d(x)z^d + \cdots + p_1(x)z + p_0(x),$$

then the coefficient of x^n in $P(x, f_{\text{next}})$ is

$$\text{known stuff} + \left(d p_d(0) f_0^{d-1} + \cdots + p_1(0) f_0\right) f_n = \text{known stuff} + \frac{\partial P}{\partial z}(0, f_0) f_n.$$

So we can solve for f_n.

Newton iteration

Computing the nth term requires at least n operations, whence a cumulated cost of at least n^2 for $f_1, \ldots, f_n$ (disregarding the dependency in d).

Newton iteration:

$$f_{(0)} = f_0 \quad f_{(i+1)} = f_{(i)} - \frac{P(x, f_{(i)})}{\frac{\partial P}{\partial z}(x, f_{(i)})} \bmod x^{2^{i+1}}.$$

Prop.

- this correctly computes the expansion of f;
- the cost is $O(d\mathsf{M}(n))$ for order n.

Warm-up

The slow construction showed that given the initial condition f_0, f exists and is unique.

Prop.

- For a polynomial g of degree $< k$, equivalence between

$$P(x, g) \bmod x^k = 0 \quad \text{and} \quad g = f \bmod x^k$$

Proof.

- If $g = f \bmod x^k$ then $P(x, g) \bmod x^k = P(x, f) \bmod x^k = 0$.
- Converse by induction, using the explicit construction.

Why it works

Taylor formula

- For any h, g, we have

$$P(x, h+g) \;=\; P(x,h) \;+\; \frac{\partial P}{\partial z}(x,h)g \;+\; g^2 R.$$

Application

- $f_{(i)} = f \bmod x^{2^i} \implies P(x, f_{(i)}) \bmod x^{2^i} = 0.$
- take

$$h = f_{(i)} \quad \text{and} \quad g = -\frac{P(x, f_{(i)})}{\frac{\partial P}{\partial z}(x, f_{(i)})} \bmod x^{2^{i+1}}.$$

- so $h + g = f_{(i+1)}$
- remark

$$g \bmod x^{2^i} \;=\; 0 \quad \text{so} \quad g^2 \bmod x^{2^{i+1}} \;=\; 0.$$

- so $P(x, f_{(i+1)}) \bmod x^{2^{i+1}} = 0 \implies f_{(i+1)} = f \bmod x^{2^{i+1}}.$