CS 4424

Hypergeometric summation

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Hypergeometric sequences

A hypergeometric sequence u_k is a sequence that satisfies

$$\frac{u_{k+1}}{u_k} = \frac{p(k)}{q(k)},$$

with $p(k), q(k)$ polynomials.

Example. Let

$$u_k = \frac{k-1}{k(k+1)} 2^k$$

so

$$\frac{u_{k+1}}{u_k} = \frac{k 2^{k+1}}{(k+1)(k+2)} \, \frac{k(k+1)}{(k-1)2^k} = \frac{2k^2}{(k-1)(k+2)}$$

and u_k is hypergeometric.

2^k is hypergeometric, $k!$ is hypergeometric, k^2 is hypergeometric, $\ldots$

Definite and indefinite sums

Indefinite sums

- sums where the summation bounds are variables

$$\sum_{k=0}^{n} \frac{1}{k!} = F(n) \qquad \sum_{k=1}^{n} \frac{k-1}{k(k+1)} 2^k = G(n).$$

Definite sums

- these are the sums where the summation bounds are explicit (usually, $\pm\infty$).

  Nice cases: summands have two variables.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{hard}$$

$$\sum_{k=0}^{\infty} \binom{n}{k} = 2^n \qquad \text{easy}$$

Gosper's algorithm

Algorithm for indefinite summations of hypergeometric functions.

- given a hypergeometric sequence u_k,
- finds whether there is a hypergeometric sequence v_k with $u_k = v_{k+1} - v_k$ or equivalently

$$\sum_{k=1}^{n} u_k = v_{n+1} - v_1.$$

Examples.

$$\sum_{k=0}^{n} \frac{1}{k!} \qquad \text{no hypergeometric sum.}$$

$$\sum_{k=1}^{n} \frac{k-1}{k(k+1)} 2^k = \frac{2^{n+1}}{n+1} - 2.$$

What the solutions look like

Prop.

- Suppose that $v_{k+1} = R(k)\, v_k$. Then

$$v_k = r(k)\, u_k, \quad \text{with} \quad r(k) = \frac{1}{R(k) - 1}.$$

- $r(k)$ is a solution of the linear recurrence

$$r(k+1)\frac{u_{k+1}}{u_k} - r(k) \;=\; 1.$$

So we have to find a rational function solution of a non-homogeneous linear recurrence.

Example: $u'_k = 2^k$ so $u'_{k+1} = 2u'_k$ and

$$v'_{k+1} - v'_0 = 1 + 2 + \cdots + 2^k.$$

The fraction $r'(k)$ must satisfy

$$2r'(k+1) - r'(k) = 1.$$

The Gosper-Petkovšek decomposition

First step: find the denominator. Assume for a moment that we can write

$$\frac{u_{k+1}}{u_k} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}$$

such that

$$a(k) \quad \text{and} \quad b(k),\ b(k+1),\ b(k+2),\ \ldots$$

have no common factor.

Example. For the previous u_k, we have

$$\frac{2k^2}{(k-1)(k+2)} = \frac{2k}{k+2} \frac{k}{k-1},$$

so

$$a(k) = 2k, \qquad b(k) = k+2, \qquad c(k) = k-1.$$

For $u'_{k+1} = 2u'_k$, take $a = 2$ and $b = c = 1$.

The fundamental property

Prop.

- With the same a, b, c as before, suppose we have

$$\frac{a(k)}{b(k)}\ \frac{c(k+1)}{c(k)} \;=\; \frac{A(k)}{B(k)}\ \frac{C(k+1)}{C(k)},$$

with $\gcd(A(k), C(k)) = 1$ and $\gcd(B(k), C(k+1)) = 1$.

Then $C(k)$ divides $c(k)$.

Simplifying the equation

1. We look for $r(k) = f(k)/g(k)$, with

$$r(k+1)\frac{u_{k+1}}{u_k} - r(k) \ = \ 1.$$

This gives

$$\frac{u_{k+1}}{u_k} = \frac{g(k) - f(k)}{f(k+1)} \ \frac{g(k+1)}{g(k)}.$$

By the previous property, $g(k)$ divides $c(k)$, so $r(k) = h(k)/c(k)$.

2. Plugging into our equation, we get

$$h(k+1)a(k) = (c(k) + h(k))b(k).$$

Because a and b are coprime, $b(k)$ divides $h(k+1)$, so

$$r(k) = \frac{b(k-1)\ell(k)}{c(k)}.$$

Finding the numerator

The polynomial $\ell(k)$ satisfies

$$\ell(k+1)a(k) \;-\; \ell(k)b(k-1) \;=\; c(k).$$

To find it:

- find a bound on its degree;
- find its coefficients by linear algebra.

On an example

We show this on the previous example:

$$2k\,\ell(k+1) - (k+1)\,\ell(k) = k - 1.$$

Let

$$\ell(k) = \sum_{i=0}^{d} \ell_i k^i.$$

We see that

- the leading term of $2k\ell(k+1)$ is $2\ell_d k^{d+1}$;
- the leading term of $(k+1)\ell(k)$ is $\ell_d k^{d+1}$;

so $d = 0$ and $\ell(k)$ is a constant. Finally, $\ell(k) = 1$.

Finishing the example

In this case, $\ell(k) = 1$ gives

$$r(k) = \frac{b(k-1)}{c(k)} \;=\; \frac{k+1}{k-1},$$

Remember that the sum we are looking for satisfies

$$v_k = r(k)u_k.$$

This gives

$$v_k = \frac{2^k}{k}$$

and

$$\sum_{k=1}^{n} \frac{k-1}{k(k+1)} 2^k \;= v_{n+1} - v_1 \;=\; \frac{2^{n+1}}{n+1} - 2.$$

Other examples

Let $u_k = k$, so

$$\frac{u_{k+1}}{u_k} = \frac{k+1}{k}$$

1. Decomposition: easy, we get

$$a(k) = 1, \quad b(k) = 1, \quad c(k) = k.$$

2. The auxiliary equation becomes

$$\ell(k+1) - \ \ell(k) \ = \ k.$$

We are back at our starting point, there is no miracle here.

Other examples

Let

$$\ell(k) = \sum_{i=0}^{d} \ell_i k^i.$$

- the leading terms of $\ell(k+1)$ are

$$\ell_d k^d + (d\ell_d + \ell_{d-1})k^{d-1} + \cdots$$

- the leading terms of $\ell(k)$ are

$$\ell_d k^d + \ell_{d-1} k^{d-1} + \cdots$$

- so $d = 2$, and $\ell(k) = \ell_2 k^2 + \ell_1 k + \ell_0$.
- solving $\ell(k+1) - \ell(k) = k$ gives $\ell(k) = k(k-1)/2$.

Other examples

This gives

$$r(k) = \frac{b(k-1)}{c(k)}\ell(k) = \frac{k-1}{2}.$$

We get $v(k)$:

$$v(k) = r(k)u(k) = \frac{k(k-1)}{2}.$$

Finally

$$\sum_{k=0}^{n} u_k = v(n+1) - v(0) = \frac{n(n+1)}{2}.$$

More examples

Let

$$u_k = \frac{k^4 4^k}{\binom{2k}{k}} = \frac{k^4 4^k (k!)^2}{(2k)!}.$$

Then

$$\frac{u_{k+1}}{u_k} = 2\frac{(k+1)^5}{k^4(2k+1)}$$

We get

$$a(k) = 2k+2, \quad b(k) = 2k+1, \quad c(k) = k^4$$

and the equation

$$(2k+2)\ell(k+1) - (2k-1)\ell(k) = k^4.$$

More examples

Write

$$\ell(k) = \ell_d k^d + \ell_{d-1} k^{d-1} + \cdots \quad \text{and} \quad \ell(k+1) = \ell_d k^d + (d\ell_d + \ell_{d-1} k^{d-1}) + \cdots$$

Then, the leading term of

$$(2k+2)\ell(k+1) \; - \; (2k-1)\ell(k)$$

is

$$(3\ell_d + 2d\ell_d)k^{d-1} + \cdots$$

So necessarily, $d = 5$.

More examples

Writing $\ell(k) = \ell_5 k^5 + \cdots + \ell_0$, we find

$$\ell(k) = \frac{1}{11}k^4 - \frac{20}{99}k^3 + \frac{20}{231}k^2 + \frac{26}{693}k - \frac{2}{231},$$

so

$$r(k) = \frac{b(k-1)}{c(k)}\ell(k) = \frac{(2k-1)(63k^4 - 140k^3 + 60k^2 + 26k - 6)}{693k^4}.$$

Finally,

$$\sum_{k=1}^{n-1} \frac{k^4 4^k}{\binom{2k}{k}} = \frac{(2n-1)(63n^4 - 140n^3 + 60n^2 + 26n - 6)}{693n^4} \frac{n^4 4^n}{\binom{2n}{n}} - \frac{2}{231}.$$

Finding the decomposition

Reminder

Given a rational function $f(k)/g(k)$, we want to write it as

$$\frac{f(k)}{g(k)} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}$$

such that

$$a(k) \quad \text{and} \quad b(k),\ b(k+1),\ b(k+2),\ \ldots$$

have no common factor.

- **1.** Either

  $$f(k) \quad \text{and} \quad g(k),\ g(k+1),\ g(k+2),\ \ldots$$

  have no common factor; then $a = f, b = g, c = 1$ works.

- **2.** Or

  $$f(k) \quad \text{and} \quad g(k+j)$$

  have a common factor for some j.

A recursive algorithm

Suppose we have found an integer $j \geq 0$, for which

$$q(k) = \gcd(f(k), g(k+j)) \neq 1.$$

Write

$$f(k) = f'(k)q(k), \quad g(k) = g'(k)q(k-j).$$

Then

$$\frac{f(k)}{g(k)} = \frac{f'(k)}{g'(k)} \frac{q(k)}{q(k-j)}$$

which is

$$\frac{f(k)}{g(k)} = \frac{f'(k)}{g'(k)} \frac{q(k)\, q(k-1) \cdots q(k-j+1)}{q(k-1) \cdots q(k-j+1)\, q(k-j)}.$$

Then, we continue on $f'(k)$ and $g'(k)$.

Sylvester matrix

Let

$$f = f_m k^m + \cdots + f_0, \quad g = g_n k^n + \cdots + g_0,$$

with $f_m \neq 0$, $g_n \neq 0$.

Their Sylvester matrix is

$$\mathsf{Syl}(f,g) = \left[\begin{array}{cccccc} \multicolumn{3}{c}{\overbrace{\hphantom{f_m \quad f_m \quad f_m}}^{n}} & \multicolumn{3}{c}{\overbrace{\hphantom{g_n \quad g_n \quad g_n}}^{m}} \\ f_m & & & g_n & & \\ \vdots & f_m & & \vdots & g_n & \\ \vdots & & f_m & \vdots & & g_n \\ f_0 & & \ddots \quad \vdots & g_0 & & \ddots \quad \vdots \\ & f_0 & \vdots & & g_0 & \vdots \\ & & f_0 & & & g_0 \end{array} \right]$$

Resultants

The resultant $\mathsf{res}(f, g)$ is the determinant of $\mathsf{Syl}(f, g)$.

Prop.

- $\mathsf{res}(f, g) = 0$ if and only if f and g have a common factor.

Let now $g'(k, x) = g(k + x)$ and

$$R(x) = \mathsf{res}(f, g') \in \mathbb{Q}[x].$$

Prop.

- $R(j) = 0$ if and only if f and $g(k + j)$ have a common factor.

Example

Let

$$\frac{f(k)}{g(k)} = \frac{k}{k^2 - 3k + 2}.$$

Then

$$g'(k,x) = k^2 + (-3+2x)k + x^2 + 2 - 3x.$$

The Sylvester matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ & & \\ 0 & 1 & -3+2\,x \\ 0 & 0 & x^2 - 3\,x + 2 \end{bmatrix}$$

Its determinant is $R(x) = x^2 - 3x + 2$.

Example

The resultant $R(x)$ factors as $R(x) = (x-1)(x-2)$.

1. For $j = 1$, we have

$$q(k) = \gcd(f(k), g(k+1)) = \gcd(k, k^2 - k) = k.$$

So $f'(k) = 1$, $g'(k) = k-2$ and

$$\frac{f(k)}{g(k)} = \frac{1}{k-2}\,\frac{k}{k-1}.$$

2. For $j = 2$, we have

$$q(k) = \gcd(f(k), g(k+2)) = \gcd(k, k^2 + k) = k.$$

So $f'(k) = 1$, $g'(k) = k-1$ and

$$\frac{f(k)}{g(k)} = \frac{1}{k-1}\,\frac{k}{k-2} = \frac{1}{k-1}\,\frac{k(k-1)}{(k-1)(k-2)}.$$

Finding polynomial solutions

Problem statement

Given a recurrence of the form

$$\alpha(k)\ell(k+1) \;+\; \beta(k)\ell(k) \;=\; P(k),$$

find a polynomial solution $\ell(k)$.

We rewrite the equation as

$$a(k)\left(\ell(k+1) - \ell(k)\right) \;+\; b(k)\ell(k) \;=\; P(k).$$

- Write $\ell(k) = \ell_d k^d + \cdots$.
- Then $\ell(k+1) - \ell(k) = d\ell_d k^{d-1} + \cdots$.

Degree bounds

Let

$$a(k) = a_{d_a} k^{d_a} + \cdots , \quad b(k) = b_{d_b} k^{d_b} + \cdots .$$

Then, we have the expansions

$$a(k)\big(\ell(k+1) - \ell(k)\big) = a_{d_a} d \ell_d k^{d+d_a-1} + \cdots ,$$

$$b(k)\ell(k) = b_{d_b} \ell_d k^{d+d_b} + \cdots .$$

1. If $d_a - 1 > d_b$, then $d + d_a - 1 = \deg(P)$.

2. If $d_a - 1 < d_b$, then $d + d_b = \deg(P)$.

3. If $d_a - 1 = d_b$, then

- either $d + d_a - 1 = \deg(P)$,
- or $a_{d_a} d + b_{d_b} = 0$.