CS 830
Hypergeometric summation
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Hypergeometric sequences

Recall that a hypergeometric sequence $u_k$ satisfies

$$\frac{u_{k+1}}{u_k} = \frac{p(k)}{q(k)},$$

with $p(k), q(k) \text{ polynomials}$.

**Example.** Let

$$u_k = \frac{k - 1}{k(k + 1)} 2^k$$

so

$$\frac{u_{k+1}}{u_k} = \frac{k 2^{k+1}}{(k + 1)(k + 2)} \frac{k(k + 1)}{(k - 1)2^k} = \frac{2k^2}{(k - 1)(k + 2)}$$

and $u_k$ is hypergeometric.
Definite and indefinite sums

Indefinite sums

- sums where the summation bounds are variables

\[ \sum_{k=0}^{n} \frac{1}{k!} = F(n) \quad \sum_{k=1}^{n} \frac{k - 1}{k(k + 1)} 2^k = G(n). \]

Definite sums

- these are the sums where the summation bounds are explicit (usually, $\pm \infty$).

**Nice cases:** summands have two variables.

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{hard} \]

\[ \sum_{k=0}^{\infty} \binom{n}{k} = 2^n \quad \text{easy} \]
Gosper’s algorithm

Algorithm for indefinite summations of hypergeometric functions.

- given a hypergeometric sequence $u_k$,
- finds whether there is a hypergeometric sequence $v_k$ with $u_k = v_{k+1} - v_k$ or equivalently

$$\sum_{k=1}^{n} u_k = v_{n+1} - v_1.$$ 

Examples.

$$\sum_{k=0}^{n} \frac{1}{k!} \text{ no hypergeometric sum.}$$

$$\sum_{k=1}^{n} \frac{k - 1}{k(k + 1)} 2^k = \frac{2^{n+1}}{n + 1} - 2.$$
What the solutions look like

Prop.

• Suppose that \( v_{k+1} = R(k) v_k \). Then

\[
v_k = r(k) u_k, \quad \text{with} \quad r(k) = \frac{1}{R(k) - 1}.
\]

• \( r(k) \) is a solution of the linear recurrence

\[
r(k + 1) \frac{u_{k+1}}{u_k} - r(k) = 1.
\]

So we have to find a rational function solution of a non-homogeneous linear recurrence.
The Gosper-Petkovšek decomposition

Assume for a moment that we can write

\[
\frac{u_{k+1}}{u_k} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}
\]

such that

\[ a(k) \text{ and } b(k), b(k+1), b(k+2), \ldots \]

have no common factor.

Example. For the previous \(u_k\), we have

\[
\frac{2k^2}{(k-1)(k+2)} = \frac{2k}{k+2} \frac{k}{k-1},
\]

so

\[ a(k) = 2k, \quad b(k) = k + 2, \quad c(k) = k - 1. \]
The fundamental property

Prop.

- With the same $a, b, c$ as before, suppose we have

$$
\frac{a(k)}{b(k)} \frac{c(k + 1)}{c(k)} = \frac{A(k)}{B(k)} \frac{C(k + 1)}{C(k)},
$$

with $\gcd(A(k), C(k)) = 1$ and $\gcd(B(k), C(k + 1)) = 1$.

Then $C(k)$ divides $c(k)$. 
Simplifying the equation

1. We look for \( r(k) = f(k)/g(k) \), with

\[
 r(k + 1) \frac{u_{k+1}}{u_k} - r(k) = 1.
\]

This gives

\[
 \frac{u_{k+1}}{u_k} = \frac{g(k) - f(k)}{f(k + 1)} \frac{g(k + 1)}{g(k)}.
\]

By the previous property, \( g(k) \) divides \( c(k) \), so \( r(k) = h(k)/c(k) \).

2. Plugging into our equation, we get

\[
 h(k + 1)a(k) = (c(k) + h(k))b(k).
\]

Because \( a \) and \( b \) are coprime, \( b(k) \) divides \( h(k + 1) \), so

\[
 r(k) = \frac{b(k - 1)\ell(k)}{c(k)}.
\]
Solving the equation

The polynomial $\ell(k)$ satisfies

$$\ell(k+1)a(k) - \ell(k)b(k-1) = c(k).$$

To find it:

- find a **bound** on its degree;
- find its coefficients by **linear algebra**.
On an example

We show this on the previous example:

\[ 2k \ell(k + 1) - (k + 1) \ell(k) = k - 1. \]

Let

\[ \ell(k) = \sum_{i=0}^{d} \ell_i k^i. \]

We see that

- the leading term of \(2k\ell(k + 1)\) is \(2\ell_d k^{d+1}\);
- the leading term of \((k + 1)\ell(k)\) is \(\ell_d k^{d+1}\);

so \(d = 0\) and \(\ell(k)\) is a constant. Finally, \(\ell(k) = 1\).
Finishing the example

In this case, \( \ell(k) = 1 \) gives

\[
\begin{align*}
    r(k) &= \frac{b(k - 1)}{c(k)} = \frac{k + 1}{k - 1}, \\
    c(k) &= \frac{k}{k - 1},
\end{align*}
\]

Remember that the sum we are looking for satisfies

\[
v_k = r(k)u_k.
\]

This gives

\[
v_k = \frac{2^k}{k}
\]

and

\[
\sum_{k=1}^{n} \frac{k - 1}{k(k + 1)} 2^k = v_{n+1} - v_1 = \frac{2^{n+1}}{n + 1} - 2.
\]
Other examples

Let $u_k = k$, so

$$\frac{u_{k+1}}{u_k} = \frac{k + 1}{k}$$

1. Decomposition: easy, we get

$$a(k) = 1, \quad b(k) = 1, \quad c(k) = k.$$  

2. The auxiliary equation becomes

$$\ell(k + 1) - \ell(k) = k.$$  

We are back at our starting point, there is no miracle here.
Other examples

Let

$$\ell(k) = \sum_{i=0}^{d} \ell_i k^i.$$ 

- the leading terms of $\ell(k + 1)$ are

$$\ell_d k^d + (d \ell_d + \ell_{d-1}) k^{d-1} + \cdots$$

- the leading terms of $\ell(k)$ are

$$\ell_d k^d + \ell_{d-1} k^{d-1} + \cdots$$

- so $d = 2$, and $\ell(k) = \ell_2 k^2 + \ell_1 k + \ell_0$.

- solving $\ell(k + 1) - \ell(k) = k$ gives $\ell(k) = k(k - 1)/2$. 
Other examples

This gives

\[ r(k) = \frac{b(k-1)}{c(k)} \ell(k) = \frac{k-1}{2}. \]

We get \( v(k) \):

\[ v(k) = r(k)u(k) = \frac{k(k-1)}{2}. \]

Finally

\[ \sum_{k=0}^{n} u_k = v(n+1) - v(0) = \frac{n(n+1)}{2}. \]
More examples

Let

\[ u_k = \frac{k^4 4^k}{\binom{2k}{k}} = \frac{k^4 4^k (k!)^2}{(2k)!}. \]

Then

\[ \frac{u_{k+1}}{u_k} = 2 \frac{(k + 1)^5}{k^4 (2k + 1)}. \]

We get

\[ a(k) = 2k + 2, \quad b(k) = 2k + 1, \quad c(k) = k^4 \]

and the equation

\[ (2k + 2)\ell(k + 1) - (2k - 1)\ell(k) = k^4. \]
More examples

Write
\[ \ell(k) = \ell_d k^d + \ell_{d-1} k^{d-1} + \cdots \quad \text{and} \quad \ell(k + 1) = \ell_d k^d + (d\ell_d + \ell_{d-1} k^{d-1}) + \cdots \]

Then, the leading term of
\[
(2k + 2)\ell(k + 1) - (2k - 1)\ell(k)
\]
is
\[
(3\ell_d + 2d\ell_d)k^{d-1} + \cdots
\]
So necessarily, \( d = 5 \).
More examples

Writing \( \ell(k) = \ell_5 k^5 + \cdots + \ell_0 \), we find

\[
\ell(k) = \frac{1}{11} k^4 - \frac{20}{99} k^3 + \frac{20}{231} k^2 + \frac{26}{693} k - \frac{2}{231},
\]

so

\[
r(k) = \frac{b(k - 1)}{c(k)} \ell(k) = \frac{(2k - 1)(63k^4 - 140k^3 + 60k^2 + 26k - 6)}{693k^4}.
\]

Finally,

\[
\sum_{k=1}^{n-1} \frac{k^4 4^k}{(2k\choose k)} = \frac{(2n - 1)(63n^4 - 140n^3 + 60n^2 + 26n - 6)}{693n^4} \frac{n^4 4^n}{(2n\choose n)} - \frac{2}{231}.
\]
Finding the decomposition
Given a rational function \( f(k)/g(k) \), we want to write it as

\[
\frac{f(k)}{g(k)} = \frac{a(k)}{b(k)} \frac{c(k + 1)}{c(k)}
\]

such that

\[ a(k) \text{ and } b(k), b(k + 1), b(k + 2), \ldots \]

have no common factor.

- **1.** Either

  \[ f(k) \text{ and } g(k), g(k+1), g(k+2), \ldots \]

  have no common factor; then \( a = f, b = g, c = 1 \) works.

- **2.** Or

  \[ f(k) \text{ and } g(k+j) \]

  have a common factor for some \( j \).
A recursive algorithm

Suppose we have found an integer \( j \geq 0 \), for which

\[
q(k) = \gcd(f(k), g(k + j)) \neq 1.
\]

Write

\[
f(k) = f'(k)q(k), \quad g(k) = g'(k)q(k - j).
\]

Then

\[
\frac{f(k)}{g(k)} = \frac{f'(k)}{g'(k)} \frac{q(k)}{q(k - j)}
\]

which is

\[
\frac{f(k)}{g(k)} = \frac{f'(k)}{g'(k)} \frac{q(k)}{q(k - 1)} \cdots \frac{q(k - j)}{q(k - j)}.
\]

Then, we continue on \( f'(k) \) and \( g'(k) \).
Sylvester matrix

Let
\[ f = f_m k^m + \cdots + f_0, \quad g = g_n k^n + \cdots + g_0, \]
with \( f_m \neq 0, g_n \neq 0 \).

Their **Sylvester matrix** is

\[
\text{Syl}(f, g) = \begin{bmatrix}
    f_m & \cdots & f_0 \\
    \vdots & \ddots & \vdots \\
    f_0 & \cdots & f_0 \\
    g_n & \cdots & g_0 \\
    \vdots & \ddots & \vdots \\
    g_0 & \cdots & g_0 \\
\end{bmatrix}
\]
The resultant \( \text{res}(f, g) \) is the determinant of \( \text{Syl}(f, g) \).

Prop.

- \( \text{res}(f, g) = 0 \) if and only if \( f \) and \( g \) have a common factor.

Let now \( g'(k, x) = g(k + x) \) and

\[
R(x) = \text{res}(f, g') \in \mathbb{Q}[x].
\]

Prop.

- \( R(j) = 0 \) if and only if \( f \) and \( g(k + j) \) have a common factor.
Example

Let

\[
\frac{f(k)}{g(k)} = \frac{k}{k^2 - 3k + 2}.
\]

Then

\[
g'(k, x) = k^2 + (-3 + 2x)k + x^2 + 2 - 3x.
\]

The Sylvester matrix is

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -3 + 2x \\
0 & 0 & x^2 - 3x + 2
\end{bmatrix}
\]

Its determinant is \( R(x) = x^2 - 3x + 2 \).
Example

The resultant $R(x)$ factors as $R(x) = (x - 1)(x - 2)$.

1. For $j = 1$, we have

$$q(k) = \gcd(f(k), g(k + 1)) = \gcd(k, k^2 - k) = k.$$  

So $f'(k) = 1$, $g'(k) = k - 2$ and

$$\frac{f(k)}{g(k)} = \frac{1}{k - 2} \frac{k}{k - 1}.$$  

2. For $j = 2$, we have

$$q(k) = \gcd(f(k), g(k + 2)) = \gcd(k, k^2 + k) = k.$$  

So $f'(k) = 1$, $g'(k) = k - 1$ and

$$\frac{f(k)}{g(k)} = \frac{1}{k - 1} \frac{k}{k - 2} = \frac{1}{k - 1} \frac{k(k - 1)}{(k - 1)(k - 2)}.$$
Finding polynomial solutions
Problem statement

Given a recurrence of the form

\[ \alpha(k)\ell(k+1) + \beta(k)\ell(k) = P(k), \]

find a polynomial solution \( \ell(k) \).

We rewrite the equation as

\[ a(k)\left(\ell(k+1) - \ell(k)\right) + b(k)\ell(k) = P(k). \]

- Write \( \ell(k) = \ell_d k^d + \cdots \).
- Then \( \ell(k+1) - \ell(k) = d\ell_d k^{d-1} + \cdots \).
Degree bounds

Let

\[ a(k) = a_{d_a} k^{d_a} + \cdots, \quad b(k) = b_{d_b} k^{d_b} + \cdots. \]

Then, we have the expansions

\[ a(k)(\ell(k + 1) - \ell(k)) = a_{d_a} d\ell d_k^{d + d_a - 1} + \cdots, \]
\[ b(k)\ell(k) = b_{d_b} \ell d_k^{d + d_b} + \cdots. \]

1. If \( d_a - 1 > d_b \), then \( d + d_a - 1 = \deg(P) \).
2. If \( d_a - 1 < d_b \), then \( d + d_b = \deg(P) \).
3. If \( d_a - 1 = d_b \), then
   - either \( d + d_a - 1 = \deg(P) \),
   - or \( a_{d_a} d + b_{d_b} = 0 \).