CS 830
Operator algebras

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Towards summation and integration in general

Definite and indefinite sums

Indefinite sums

- sums where the summation bounds are variables

$$\sum_{k=0}^{n} \frac{1}{k!} = F(n) \qquad \sum_{k=1}^{n} \frac{k-1}{k(k+1)} 2^k = G(n).$$

Definite sums

- these are the sums where the summation bounds are explicit (usually, $\pm\infty$).

  Nice cases: summands have two variables.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{hard}$$

$$\sum_{k=0}^{\infty} \binom{n}{k} = 2^n \qquad \text{easy}$$

Definite and indefinite integrals

Indefinite integrals

- integrals where a bound is variable

$$\int_{-\infty}^{y} \frac{1+x^2}{1-x^2} dx \;=\; F(y) \qquad \int_{-\infty}^{y} \frac{x-1}{x(x+1)} 2^x dx \;=\; G(y).$$

Definite integrals

- these are the integrals where the bounds are explicit (usually, $\pm\infty$).

  Nice cases: integrand have two variables.

$$\int_{-\infty}^{\infty} \exp -x^2/2dx = \frac{\sqrt{\pi}}{2} \qquad \text{hard}$$

$$\int_{-\infty}^{\infty} \frac{1-x}{1+xy} dx = F(y)$$

What we have seen

Previous lectures

- Gosper: indefinite summation of hypergeometric sequences in k.
- Zeilberger: definite summation of hypergeometric sequences in n, k.

Both algorithms use heavily the hypergeometric assumption.

Generalizations to

- more operators,
- of higher order,
- in more variables.

Back to the basics

D-finite series

Def.

- A series $f(x)$ is D-finite if there exists a linear differential equation with polynomial coefficients such that

$$q_d(x)f^{(d)} \ + \ q_{d-1}(x)f^{(d-1)} \ + \ \cdots \ + \ q_0(x)f = 0.$$

- Equivalently, we can take rational functions as coefficients.

Examples.

- polynomials,
- rational functions,
- algebraic series (e.g., $\sqrt{1+x^2}$)
- exp, sin, cos,
- a lot more

P-recursive sequences

Def.

- A sequence u_n is P-recursive if it satisfies a recurrence with polynomial coefficients

$$p_d(n)u_{n+d} + p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n = 0$$

Examples.

- constant sequences,
- recurrences with constant coefficients,
- factorial,
- hypergeometric $u_{n+1}/u_n = a(n)/b(n)$.

Differential operators

Instead of writing an equation like

$$(x+1)h' = xh$$

we prefer

$$\Big(\ (x+1)\partial_x - x\ \Big)(h) = 0.$$

Operators such as

$$(x+1)\partial_x - x$$

can be used as a data structure.

- They somehow behave like polynomials (multiplication, division, etc).
- Algorithms will generalize many known ones (from Euclid to Gröbner bases).
- Remember: multiplication by ∂_x is differentiation.

Multiplication

Example: multiplication of differential operators

If $(x+1)h' - xh = 0$, then its derivative is zero as well:

$$\Big((x+1)h' - xh\Big)' = 0.$$

This gives

$$(x+1)h'' + h' - xh' - h = 0 \iff (x+1)h'' + (1-x)h' - h = 0.$$

In operator notation, this becomes:

- Suppose $((x+1)\partial_x - x)(h) = 0$.
- Then $\partial_x(((x+1)\partial_x - x)(h)) = 0$.
- Or $(\partial_x((x+1)\partial_x - x))(h) = 0$.
- Or $(\partial_x(x+1)\partial_x - \partial_x x)(h) = 0$.

Commutation rules

Leibniz formula

$$(Ah)' = Ah' + A'h \quad \iff \quad \partial_x(Ah) = (A\partial_x + A')(h).$$

So the previous formula becomes

$$((x+1)\partial_x + 1)\partial_x - x\partial_x - 1)(h) = 0 \iff ((x+1)\partial_x^2 + (1-x)\partial_x - 1)(h) = 0.$$

So:

- of course, we recover the same result as before;
- but we don't need to carry h in our computations.

More commutation rules

We have just seen

$$\partial_x A = A\partial_x + A'.$$

Continuing:

$$\partial_x^2 A = A\partial_x^2 + 2A'\partial_x + A'',$$

and more generally

$$\partial_x^n A = \sum_{k=0}^{n} \binom{n}{k} A^{(k)} \partial_x^{n-k}.$$

Multiplication rules

Given an operator

$$P = p_d(x)\partial_x^d + \cdots + p_1(x)\partial_x + p_0(x)$$

we can multiply on the left

- by any polynomial in x ...
- by any power of ∂_x ...
- more generally, by any other operator Q:

$$\Big(q_e(x)\partial_x^d + \cdots + q_1(x)\partial_x + q_0(x)\Big)\Big(p_d(x)\partial_x^d + \cdots + p_1(x)\partial_x + p_0(x)\Big)$$

$$= \sum_{i\le d}\sum_{j\le e} q_j(x)\partial_x^j p_i(x)\partial_x^i = \sum_{k\le d+e} r_k(x)\partial_x^k.$$

Then if $P(h) = 0$, $QP(h) = 0$.

Recurrence operators

Instead of writing an equation like

$$(n+1)u_{n+1} = nu_n$$

we prefer

$$\Big(\ (n+1)E - n\ \Big)(u_n) = 0,$$

where E is the shift $E(u_n) = u_{n+1}$. Operators such as

$$(n+1)E - n$$

can be used as a data structure.

- They somehow behave like polynomials (multiplication, division, etc).
- Algorithms will generalize many known ones (from Euclid to Gröbner bases).
- Remember: multiplication by E is shift.

Example: multiplication of recurrence operators

If $(n+1)u_{n+1} - nu_n = 0$, then its shift is zero as well:

$$\Big((n+2)u_{n+2} - (n+1)u_{n+1}\Big) = 0.$$

In operator notation, this becomes:

- Suppose $((n+1)E - n)(u_n) = 0$.
- Then $E(((n+1)E - n)(u_n)) = 0$.
- Or $(E((n+1)E - n))(u_n) = 0$.
- Or $(E(n+1)E - En)(u_n) = 0$.

Commutation rules

Discrete Leibniz formula

$$[A_n u_n]_{n+1} = A_{n+1} u_{n+1} \qquad \Longleftrightarrow \qquad E(A_n u_n) = A_{n+1} E(u_n).$$

So the previous formula becomes

$$(E(n+1)E - En)(u_n) = ((n+2)E^2 - (n+1)E)(u_n).$$

So:

- of course, we recover the same result as before;
- but we don't need to carry u_n in our computations.

More commutation rules

We have just seen

$$EA_n = A_{n+1}E.$$

Continuing:

$$E^2A_n = A_{n+2}E^2.$$

and more generally

$$E^kA_n = A_{n+k}E^k.$$

Multiplication rules

Given an operator

$$P = p_d(n)E^d + \cdots + p_1(n)E + p_0(n)$$

we can multiply on the left

- by any polynomial in n ...
- by any power of E ...
- more generally, by any other operator Q:

$$\Big(q_e(n)E^d + \cdots + q_1(n)E + q_0(n)\Big)\Big(p_d(n)E^d + \cdots + p_1(n)E + p_0(n)\Big)$$

$$= \sum_{i \le d} \sum_{j \le e} q_j(n)E^j p_i(n)E^i = \sum_{k \le d+e} r_k(n)E^k.$$

Then if $P(u_n) = 0$, $QP(u_n) = 0$.

Complexity

Prop.

- If P and Q are differential operators with

$$\deg(P,\partial_x) \leq n, \quad \deg(P,x) \leq n, \quad \deg(Q,\partial_x) \leq n, \quad \deg(Q,x) \leq n,$$

then one can compute the coefficients of

$$R = PQ = \sum_{i \leq 2n} r_i(x)\partial_x^i$$

in $O(n^\omega)$ operations.

Prop.

- If P and Q are recurrences ... I don't know.

Division

Euclidean division of differential operators

Consider two differential operators

$$P = p_d(x)\partial_x^d + \cdots \quad \text{and} \quad Q = q_e(x)\partial_x^e + \cdots ,$$

with $d \geq e$.

Euclidean division step

- multiply Q by ∂_x^{d-e} and reorganize:

$$\partial_x^{d-e} Q = \tilde{q}_d(x)\partial_x^d + \cdots$$

- compute $R = \tilde{q}_d(x)P - p_d(x)\partial_x^{d-e}Q$:

$$R = r_{d-1}(x)\partial_x^{d-1} + \cdots$$

Eventually, we get an equality $a(x)P = B(x, \delta_x)Q + S(x, \delta_x)$, with $\deg(S, \delta_x) < e$.

Application

The function $f(x) = \cos(x^2)$ satisfies the linear differential equation

$$xf'' - f' + 4x^3 f = 0,$$

so $P(f) = 0$ with

$$P = x\partial_x^2 - \partial_x + 4x^3.$$

Do we have

$$f^{(4)} - xf^{(3)} = 0?$$

We compute the remainder of the Euclidean division of

$$Q = \partial_x^4 - x\partial_x^3 \quad \text{by} \quad P.$$

Application

1. All elementary steps.

$$\begin{array}{rcl} R_1 & = & xQ - \partial_x^2 P \\ R_2 & = & xR_1 + (1+x^2)\partial_x P \\ R_3 & = & xR_2 + 4x^4 P \end{array} \quad \begin{array}{l} = -(1+x^2)\partial_x^3 - 4x^3\partial_x^2 - 24x^2\partial_x - 24x \\ = -4x^4\partial_x^2 + (-20x^3 + 4x^5)\partial_x + 12x^4 - 12x^2 \\ = (4x^6 - 24x^4)\partial_x + 16x^7 + 12x^5 - 12x^3. \end{array}$$

2. Putting them together.

$$\begin{array}{rcl} x^2 R_1 & = & x^3 Q - x^2\partial_x^2 P \\ xR_2 & = & x^2 R_1 + x(1+x^2)\partial_x P \\ R_3 & = & xR_2 + 4x^4 P \end{array}$$

3. The whole division equality

$$R_3 = x^3 Q + (-x^2\partial_x^2 + x(1+x^2)\partial_x + 4x^4)P.$$

Main ideas

Cf. computations with polynomial systems

- you don't handle the roots explicitly;
- instead, you perform manipulations on the equations themselves.

Example: suppose that x is a root of $P = T^{30} + 128T^3 - 3$. Do you have

$$\frac{x^{12}}{3x-1} = \frac{1-x^{40}}{1+23x}?$$

There's no need to compute the roots numerically. The property is equivalent to

$$(1+23x)x^{12} = (1-x^{40})(3x-1) \iff 3x^{41} - x^{40} + 23x^{13} + x^{12} - 3x + 1 = 0.$$

This polynomial has no common root with P, so the answer is no.

GCD and LCM

Using Euclidean division, we can write a Euclidean algorithm for differential operators. Given $P(x, \partial_x) = P_0, Q(x, \partial_x) = P_1$,

- $P_2 \leftarrow P_0$ rem P_1
- $P_3 \leftarrow P_1$ rem P_2
- $\dots$

1. The last non-zero remainder G is the right gcd of P and Q:

$$a(x)P = AG, \quad b(x)Q = BG.$$

2. One can deduce cofactors $P'(x, \partial_x), Q'(x, \partial_x)$ such that

$$P'(x, \partial_x)P(x, \partial_x) + Q'(x, \partial_x)Q(x, \partial_x) = 0.$$

Euclidean division of recurrences

Everything works similarly for recurrences. Consider

$$P = p_d(n)E^d + \cdots \quad \text{and} \quad Q = q_e(n)E^e + \cdots \ ,$$

with $d \geq e$.

Euclidean division step

- multiply Q by E^{d-e} and reorganize:

$$E^{d-e}Q = \tilde{q}_d(n)E^d + \cdots$$

- compute $R = \tilde{q}_d(n)P - p_d(n)E^{d-e}Q$:

$$R = r_{d-1}(n)E^{d-1} + \cdots$$

Eventually, we get an equality $a(n)P = B(n, E)Q + S(n, E)$, with $\deg(S, E) < e$.

GCD and cofactors are defined as before.

Multivariate operators

Multivariate operators

Next, we define multivariate operators and recurrences.

1. The rules defining hypergeometric sequences $u_{n,k}$

$$u_{n+1,k} = r(n,k)u_{n,k} \quad u_{n,k+1} = s(n,k)u_{n,k}$$

can be rewritten

$$E_n - r(n,k), \quad E_k - s(n,k),$$

where

- E_n is shift by 1 in n;
- E_k is shift by 1 in k.

Multivariate operators

2. Functions satisfying linear PDE's can be defined using multivariate differential operators.

The function

$$F = \exp\left(-\frac{x^2}{y^2} - y^2\right)$$

satisfies

$$\frac{\partial F}{\partial x} = -\frac{2x}{y^2}F, \qquad \frac{\partial F}{\partial y} = (\frac{2x^2}{y^3} - y)F.$$

The equations can be represented by the operators

$$\partial_x + \frac{2x}{y^2}, \qquad \partial_y - \frac{2x^2}{y^3} + y.$$

Multivariate operators

3. One can mix recurrences and differential equations.

Example: the Jacobi polynomials.

$$P_n^{(a,b)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n+a}{k} \binom{n+b}{n-k} (x-1)^{n-k} (x+1)^k.$$

They satisfy relations of the form

$$a(n,x) P_{n+2}^{(a,b)} + b(n,x) P_{n+1}^{(a,b)} + c(n,x) P_n^{(a,b)} = 0$$

$$A(n,x) {P_{n+1}^{(a,b)}}' + B(n,x) P_{n+1}^{(a,b)} + C(n,x) P_n^{(a,b)} = 0$$

$$\alpha(n,x) {P_n^{(a,b)}}'' + \beta(n,x) {P_n^{(a,b)}}' + \gamma(n,x) P_n^{(a,b)} = 0$$

Multivariate operators

3. One can mix recurrences and differential equations.

Example: the Jacobi polynomials.

The system of defining operators is

$$a(n,x)E^2 + b(n,x)E + c(n,x)$$

$$A(n,x)\partial E + B(n,x)E + C(n,x)$$

$$\alpha(n,x)\partial^2 + \beta(n,x)\partial + \gamma(n,x)$$

Computing with multivariate operators

In general, we can have

- coefficients in variables $n_1, n_2, \ldots$ and $x_1, x_2, \ldots$
- several shift operators $E_{n_1}, E_{n_2}, \ldots$
- several differentiation operators $\partial_{x_1}, \partial_{x_2}, \ldots$
- and maybe more complex operators too (but not in this lecture).

multiplication rules: all multiplications are done as before, except

- $E_{n_i} p(\ldots, n_i, \ldots) = p(\ldots, n_i + 1, \ldots) E_{n_i}$
- $\partial_{x_i} p = p \partial_{x_i} + \frac{\partial p}{\partial x_i}$.

Jacobi polynomials, again

Remember that we have three relations on the Jacobi polynomials

$$\mathbf{J}_1 = a(n,x)E^2 + b(n,x)E + c(n,x)$$

$$\mathbf{J}_2 = A(n,x)\partial E + B(n,x)E + C(n,x)$$

$$\mathbf{J}_3 = \alpha(n,x)\partial^2 + \beta(n,x)\partial + \gamma(n,x)$$

Let's see how we can deduce the last one from the other ones.

Main ingredient

- We saw that given $P(x,\partial)$ and $Q(x,\partial)$, we could find $P'(x,\partial)$ and $Q'(x,\partial)$ with
$$P'P + Q'Q = 0.$$
- The same works for $P(n,x,\partial)$ and $Q(n,x,\partial)$.

Jacobi polynomials, again

We apply Euclid's algorithm in the variable E.

1. We multiply the second relation by E:

$$\tilde{\mathbf{J}}_2 = \tilde{A}(n,x)\partial E^2 + \tilde{B}(n,x)E^2 + \tilde{C}(n,x)E.$$

The first one is still

$$\mathbf{J}_1 = a(n,x)E^2 + b(n,x)E + c(n,x).$$

2. We find $a'(n,x,\partial)$ and $A'(n,x,\partial)$ such that $a'a + A'(\tilde{A}\partial + \tilde{B}) = 0$ and compute

$$a'\mathbf{J}_1 + A'\tilde{\mathbf{J}}_2 = p(n,x,\delta)E + q(n,x,\delta).$$

3. We do one more step to get rid of E.

Elimination and summation

Creative telescoping in general

Suppose that $f(x_1, x_2, \ldots, n_1, n_2, \ldots)$ is defined by a set of relations

$$a_i(x_1, x_2, \ldots, n_1, n_2, \ldots, \partial_1, \partial_2, \ldots, E_1, E_2, \ldots)(f) = 0.$$

1. Let

$$F = \int_{x_1=-\infty}^{\infty} f dx_1.$$

We want to compute some relations

$$A_i(x_2, \ldots, n_1, n_2, \ldots, \partial_2, \ldots, E_1, E_2, \ldots)(F) = 0.$$

2. Let

$$G = \sum_{n_1=-\infty}^{\infty} f.$$

We want to compute some relations

$$B_i(x_1, x_2, \ldots, n_2, \ldots, \partial_1, \partial_2, \ldots, E_2, \ldots)(G) = 0.$$

Creative telescoping for integrals

Suppose we can find an operator P

- that is a combination of the a_i:

$$P = \sum b_i(x, n, \partial, E) a_i(x, n, \partial, E),$$

  with $x = x_1, \ldots$, $n = n_1, \ldots$, etc

- that does not depend on x_1.

1. $P(f) = 0$.

2. Because P has no x_1, ∂_{x_1} can be permuted with all other variables:

$$P = P_0(x', n, \partial', E) + \partial_{x_1} P_1(x', n, \partial, E),$$

with $x' = x_2, \ldots,$ and $\partial' = \partial_{x_2}, \ldots.$

Creative telescoping for integrals

3. Rewrite $P(f) = 0$ as

$$P_0(x', n, \partial', E)(f) \; + \; \partial_{x_1} P_1(x', n, \partial, E)(f) = 0.$$

4. Integrate

$$\int_{x_1=-\infty}^{\infty} P_0(x', n, \partial', E)(f) dx_1 \; + \; \int_{x_1=-\infty}^{\infty} \partial_{x_1} P_1(x', n, \partial, E)(f) dx_1 = 0.$$

5. Permute and clean

$$P_0(x', n, \partial', E)(F) \; + \; [P_1(x', n, \partial, E)(f)]_{-\infty}^{\infty} = 0.$$

6. Usually, the last term is zero, so

$$P_0(x', n, \partial', E)(F) = 0.$$

Creative telescoping for sums

Suppose we can find an operator Q

- that is a combination of the a_i:

$$Q = \sum c_i(x, n, \partial, E) a_i(x, n, \partial, E),$$

  with $x = x_1, \dots$, $n = n_1, \dots$, etc

- that does not depend on n_1.

1. $Q(f) = 0$.

2. Because Q has no n_1, it can be rewritten

$$Q = Q_0(x, n', \partial, E') + (E_1 - 1) Q_1(x, n', \partial, E),$$

with $n' = n_2, \dots,$ and $E' = E_2, \dots$.

We saw this form already!

Creative telescoping for sums

3. Rewrite $Q(f) = 0$ as

$$Q_0(x, n', \partial, E')(f) \; + \; (E_1 - 1)Q_1(x, n', \partial, E)(f) = 0.$$

4. Sum

$$\sum_{n_1=-\infty}^{\infty} Q_0(x, n', \partial, E')(f) \; + \; \sum_{n_1=-\infty}^{\infty} (E_1 - 1)Q_1(x, n', \partial, E)(f) = 0.$$

5. Permute and clean

$$Q_0(x, n', \partial, E')(G) = 0.$$