Assignment 2

September 30, 2008

1. Give the steps of the naive Euclidean division algorithm for the division of $3+x-x^2+x^3$ by $1-x$.

2. Give the steps of the fast Euclidean division algorithm for the division of $3+x-x^2+x^3$ by $1-x$. You should get the same result as in the previous problem.

   For all such computational questions, you are free to do the computations by hand, or to implement the algorithm and run it. If you implement in a language like C or java, you can use `float`*s or* `int`*s as coefficients.*

3. Prove that the naive algorithm for the inversion of a power series

   $$F = 1 + f_1 x + f_2 x^2 + \cdots$$

   takes $O(n^2)$ operations to compute n terms of $1/F$.

4. Consider a power series

   $$F = 1 + f_1 x + f_2 x^2 + \cdots .$$

   In this problem, you are asked to prove that there exists a unique power series

   $$G = 1 + g_1 x + g_2 x^2 + \cdots$$

   such that $F = G^2$; we will write $G = \sqrt{F}$. To prove it, compute the coefficients of G one after the other, by extracting coefficients in the equality $F = G^2$. How many operations does it take to compute n terms of G?

5. Briefly explain how to use the algorithm given in the *Algebraic Series* part of Lecture 3 to compute G.

6. You are going to write down a specific Newton iteration for computing G, instead of using the general form of the previous problem. To do it, we actually use an indirect computation, by computing $H = 1/G$ first.

   (a) Prove that H satisfies $F - 1/H^2 = 0$.

(b) Show that the Newton iteration for the previous equation is $H_0 = 1$ and

$$H_{(i+1)} = \frac{H_{(i)}(3 - FH_{(i)}^2)}{2} \text{ rem } x^{2^{i+1}}$$

(c) (bonus difficult question) Prove correctness: if $H_{(i)}$ is such that $H_{(i)} = H \text{ rem } x^{2^i}$, prove that $H_{(i+1)}$ is such that $H_{(i+1)} = H \text{ rem } x^{2^{i+1}}$.

(d) Prove that $H \text{ rem } x^n$ can be computed in $O(\mathsf{M}(n))$ operations.

(e) Prove that $G \text{ rem } x^n$ can be computed in $O(\mathsf{M}(n))$ operations.

7. Give the first 10 terms of $\sqrt{1 + 2x}$; your result can be either floating point or exact. How did you compute it?

   Don't hesitate to use any kind of trick.

8. How much time did you spend on the assignment?