Midterm: some algorithms to compute Fibonacci numbers

9:30 - 12:15

November 6, 2012

Here are some algorithms for computing Fibonacci numbers (which can be adapted to compute slightly more exciting things). The problems are probably a bit too long; if so, it may be possible to get full marks without answering all questions.

All polynomials and power series we consider will have coefficients in $\mathbb{Q}$; all complexity estimates will count operations in $\mathbb{Q}$ at unit cost.

Remember that the Fibonacci numbers are defined by

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+2} - f_{n+1} - f_n = 0.$$

We will admit that $f_n \simeq 0.447 \cdot 1.61^n$ for $n \to \infty$.

1. Compute $f_0, f_1, f_2, f_3, f_4, f_5, f_6$.

2. Consider the algorithm

   $\mathsf{fib}(n)$

   - if $n = 0$ then return 0
   - if $n = 1$ then return 1
   - return $\mathsf{fib}(n-2) + \mathsf{fib}(n-1)$

   Let F_n be the number of operations in $\mathbb{Q}$ done in this algorithm (there are only additions, actually). "If" and "return" are free.

   (a) Show that

   $$F_0 = 0, \ F_1 = 0, \ F_2 = 1, \ F_3 = 2, \ F_4 = 4, \ F_5 = 7$$

   and that the sequence (F_n) satisfies the recurrence

   $$F_n - F_{n-1} - F_{n-2} = 1 \qquad (1)$$

   for $n = 2, 3, 4, \ldots$.

(b) By comparing these values to the Fibonacci numbers, guess a relation between the sequences (F_n) and (f_n). Assuming your guess is correct (I'm not asking you for a proof, but I won't mind if you give one), deduce that the running time of fib is exponential in n.

The goal of this problem is to hint at ways to derive this kind of result without guessing anything.

(c) (Useful for the next questions) Prove that

$$x^2 + x^3 + x^4 + x^5 + \cdots = \frac{x^2}{1-x}.$$

(d) We are going to find another recurrence for F_n. Let S be the generating series

$$S = F_0 + F_1 x + F_2 x^2 + \cdots = \sum_{n \geq 0} F_n x^n.$$

Use (1) as follows

$$\begin{aligned}
F_2 - F_1 - F_0 = 1 \quad &\Longrightarrow \quad F_2 x^2 - F_1 x^2 - F_0 x^2 = x^2 \\
F_3 - F_2 - F_1 = 1 \quad &\Longrightarrow \quad F_3 x^3 - F_2 x^3 - F_1 x^3 = x^3 \\
F_4 - F_3 - F_2 = 1 \quad &\Longrightarrow \quad F_4 x^4 - F_3 x^4 - F_2 x^4 = x^4 \\
&\cdots
\end{aligned}$$

to prove that

$$S - (F_0 + F_1 x) - x(S - F_0) - x^2 S = \frac{x^2}{1-x}$$

and (because $F_0 = F_1 = 0$)

$$S - xS - x^2 S = \frac{x^2}{1-x}.$$

Hint: we did something similar in class.

(e) Use the previous question to find a recurrence of order 3 for F_n of the form $F_{n+3} + aF_{n+2} + bF_{n+1} + cF_n = 0$, for some constants a, b, c. Verify it on the values you computed!

Why do we do this? This is an illustration of the fact that generating series often make it possible to analyze more or less automatically the behaviour of sequences such as F_n. Here, we saw that S is rational, which was not obvious from the first recurrence we got for F_n; the order-3 recurrence would then make it possible to find the asymptotic equivalent of F_n.

3. Give a (very simple) algorithm $\mathsf{fib}'(n)$ that computes f_n in time $O(n)$ (and don't forget to justify why it takes time $O(n)$).

4. Let $P = x^2 - x - 1$.

(a) Compute

$$\Gamma_0 = 1 \text{ rem } P,\ \Gamma_1 = x \text{ rem } P,\ \Gamma_2 = x^2 \text{ rem } P,\ \Gamma_3 = x^3 \text{ rem } P,\ \Gamma_4 = x^4 \text{ rem } P,\ \Gamma_5 = x^5 \text{ rem } P.$$

Hint: you can use the fact (without proving it) that $\Gamma_{i+1} = (x\Gamma_i) \text{ rem } P$.

Do you recognize the coefficients in the Γ_i?

(b) More generally, let $\Gamma_n = x^n \text{ rem } P$. Prove by induction on n that $\Gamma_n = f_{n-1} + f_n x$ for $n \geq 1$, using the same fact as in the previous question.

(c) Give an algorithm $\mathsf{fib}''(n)$ that computes f_n in time $O(\log(n))$. I'm not asking for an extremely detailed proof for the running time.

Hint: case distinction on n even ($n = 2k$) or n odd ($n = 2k + 1$).