Predicate Calculus

(*Alternative names:* predicate logic, first order logic, elementary logic, restricted predicate calculus, restricted functional calculus, relational calculus, theory of quantification, theory of quantification with equality, etc.)

In propositional logic only the logical forms of compound propositions are analyzed. A simple proposition is an unanalyzed whole which is either true or false.

There are certain arguments that seem to be perfectly logical, yet they cannot be expressed by using propositional calculus.

Example

1. All cats have tails.
2. Tom is a cat.

From these two sentences, one should be able to conclude that

3. Tom has a tail.

To show that this argument is sound, we must be able to identify individuals, such as Tom, together with their properties and predicates. This is the objective of *predicate calculus*.

- Generally, predicates are used to describe certain properties or relationships between individuals or objects.

  Example: In "Mary and Jane are sisters", the phrase "are sisters" is a predicate. The entities connected this way, Mary and Jane, are called *terms*.

- Terms play a similar role in predicate calculus as nouns and pronouns do in the English language.

- In addition to terms and predicates, one uses *quantifiers*. Quantifiers indicate how frequently a certain statement is true. Specifically, the *universal* quantifier is used to indicate that a statement is always true, whereas the *existential* quantifier indicates that a statement is sometimes true.

  Example: In "All cats have tails", the word "all" indicates that the statement "cats have tails" is universally true.

- Predicate calculus is a generalization of propositional calculus. Hence, besides terms, predicates, and quantifiers, predicate calculus contains propositional variables, constants and connectives as part of the language.

- An important part is played by functions which are essential when discussing equations.

Predicate calculus in Computer Science

- Predicate calculus gives the underpinnings to the languages of logic programming, such as *Prolog.*
- Predicate calculus is increasingly used for specifying the requirements of computer applications.
- In the area of proving program correctness, predicate calculus allows one to precisely state under which conditions a program gives the correct output.

Predicate Calculus: Syntax

1. The Domain (universe of discourse)

Example.

1. Jane is Paul's mother.
2. Jane is Mary's mother.
3. Any two persons having the same mother are siblings.

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4. Paul and Mary are siblings.

The truth of the statement "Jane is Paul's mother" can only be assessed within a certain context. There are many people named Jane and Paul, and without further information the statement in question can refer to many different people, which makes it ambiguous.

To prevent such ambiguities we introduce the concept of a *domain* or *universe of discourse.*

Definition The *universe of discourse* or *domain* is the collection of all persons, ideas, symbols, data structures, and so on, that affect the logical argument under consideration. The elements of the domain are called *individuals.*

In the argument concerning Mary and Paul, the universe of discourse (domain) may, for instance, consist of the people living in a particular house or a particular block.

Many arguments involve numbers and, in this case, one must stipulate whether the domain is the set of natural numbers, the set of integers, the set of real numbers, or the set of complex numbers.

The truth of a statement may depend on the domain selected. The statement "there is a smallest number" is true in the domain of natural numbers, but false in the domain of integers.

- The elements of the domain are called *individuals*. An individual can be a person, a number, a data structure, or anything else one wants to reason about.
- To avoid trivial cases, one stipulates that every domain must contain at least one individual. Hence, the set of all natural numbers less than 0 does not constitute a domain (universe of discourse) because there is no negative number.
- Instead of the word individual one sometimes uses the word *object*, such as in "the domain must contain at least one object".
- To refer to a particular individual or object, identifiers must be used. These identifiers are called *individual constants*.

  If the universe of discourse consists of persons, the individual constants may be their names. In the case of natural numbers the individual constants are the digits representing these numbers. Each individual constant must uniquely identify a particular individual and no other one.

2. Predicates

Generally, predicates make statements about individuals:

Mary and Paul are siblings.
Jane is the mother of Mary.
Tom is a cat.
The sum of 2 and 3 is 5.

- In each of these statements, there is a list of individuals, which is given by the *argument list*, together with phrases that describe certain relations among or properties of the individuals mentioned in the argument list.
- These properties or relations are referred to as *predicates*.

- In the statement "Mary and Paul are siblings", the argument list is given by Mary and Paul, in that order, whereas the predicate is described by the phrase "are siblings".
- Similarly, the statement "Tom is a cat" has an argument list with the single element "Tom" in it, and its predicate is described by "is a cat".
- The entries of the argument list are called *arguments*.
- The arguments can be either variables or individual constants, but since we have not discussed variables yet, we restrict our attention to the case when all arguments are individual constants.

- In predicate calculus, each predicate is given a name, which is followed by the list of arguments.
- The list of arguments is enclosed in parantheses.
- To express "Jane is the mother of Mary" one could choose an identifier, say "mother" to express the predicate "is the mother of", and one would write *mother (Jane, Mary)*.
- Many logicians use only single letters for predicate names and constants. They would write, for instance $M(j, m)$ instead of *mother(Jane, Mary)*; that is, they would use M as a name for the predicate "is the mother of", j for Jane and m for Mary. To save space, we will often use this convention.
- Note that the order of arguments is important. Clearly, the statements mother(Mary, Jane) and mother(Jane, Mary) have a completely different meaning.

The number of elements in the argument list of a predicate is called the *arity* of the predicate. For instance, mother(Jane, Mary) has arity 2. The arity of a predicate is fixed. For example a predicate cannot have two arguments in one case and three in another. Alternatively, one can consider two predicates different if their arity is different. The following statement illustrates this:

The sum of 2 and 3 is 5.
The sum of 2, 3 and 4 is 9.

To express these statements in predicate calculus, one can either use two predicate names such as "sum2" and "sum3" and write sum2(2, 3, 5) and sum3(2, 3, 4, 9) respectively, or one can use the same symbol, say "sum" with the implicit understanding that the name "sum" in sum(2, 3, 5) refers to a different predicate than in sum(2, 3, 4, 9).

- A predicate with arity n is often called an n-place predicate. A one-place predicate is called a *property*.

  Example: The predicate "is a cat" is a one-place predicate, or a property. The predicate "is the mother of", as in "Jane is the mother of Mary" is a two-place predicate; that is, its arity is 2. The predicate in the statement "The sum of 2 and 3 is 6" (which is false) contains the three-place predicate "is the sum of".

- A predicate name, followed by an argument list in parantheses is called an *atomic formula* . The atomic formulas can be combined by logical connectives like propositions. For instance, if cat(Tom) and hastail(Tom) are two atomic formulas, expressing that Tom is a cat and that Tom has a tail respectively, one can form

$$\text{cat(Tom)} \;\rightarrow\; \text{hastail(Tom)}.$$

- If all arguments of a predicate are individual constants, then the resulting atomic formula must be either true or false. This is part of the definition of the predicate.
- For instance, if the domain consists of Jane, Doug, Mary and Paul, we have to know for each ordered pair of individuals whether or not the predicate "is the mother of" is true. This can be done in the form of a table.
- The method that assigns truth values to all possible combinations of individuals of a predicate is called an *assignment*. For instance, the following table is an assignment of the predicate "mother".

Example

Assignment for the Predicate "mother":

	Doug	Jane	Mary	Paul
Doug	0	0	0	0
Jane	0	0	1	1
Mary	0	0	0	0
Paul	0	0	0	0

Another example of an assignment is as follows. The domain consists of the four numbers 1, 2, 3, 4. The predicate "greater" is true if the first argument is greater than the second argument. Hence, greater(4, 3) is true and greater(3, 4) is false.
Assignment for the Predicate"greater":

	1	2	3	4
1	0	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	1	1	0

In a finite domain (universe of discourse), one can represent the assignments of predicates with arity n by n-dimensional arrays.

Note that the mathematical symbols $>, <$ are predicates. However, these predicates are normally used in *infix* notation. By this, we mean that they are placed between the arguments. For instance, to express that 2 is greater than 1, we write $2 > 1$ rather than $> (2, 1)$.

3. Variables and instantiation

- Often one does not want to associate the arguments of an atomic formula with a particular individual. To avoid this, variables are used.
- Variables are frequently chosen from the end of the alphabet; that is x, y and z, with or without subscripts, suggest variable names.
- Examples:

  $\text{cat}(x) \rightarrow \text{hastail}(x)$

  $\text{dog}(y) \wedge \text{brown}(y)$

  $\text{grade}(x) \rightarrow (x \geq 0) \wedge (x \leq 100)$
- Clearly, the first and third formulas contain the variable x, and the second the variable y.

- As in propositional calculus, formulas can be given names. For instance, one can define A as follows:

$$A = \text{cat}(x) \rightarrow \text{hastail}(x).$$

which means that when we write A we really mean "cat(x) $\rightarrow$ hastail(x)".

- Syntactically, one can use variables in any place where one is allowed to use constants.
- The word *term* is therefore used to refer to either a constant or a variable. More generally, a term is anything that can be used in place of an individual (formal definition later).

Instantiation

If A is a formula, one often has to replace all occurrences of a particular variable by a term.
For example, in the expression cat(x) $\rightarrow$ hastail(x), one may want to replace all instances of x by the term Tom which yields

$$\text{cat(Tom)} \;\rightarrow\; \text{hastail(Tom)}.$$

Generally, if A is a formula, the formula obtained by replacing all variables x in A by the term t is denoted by $S^x_t A$.
Specifically, if A is defined as previously, then

$$S^x_{Tom} A$$

stands for

$$\text{cat(Tom)} \rightarrow \text{hastail(Tom)}.$$

Definition

Let A represent a formula, x represent a variable, and t represent a term. Then $S_t^x A$ represents the formula obtained by replacing all occurrences of x in A by t. $S_t^x A$ is called an *instantiation* of A, and t is said to be an *instance* of x.

Example:

Let a, b, c be individual constants, P, Q, be predicate symbols, and x and y be variables. Find

$$S_a^x(P(a) \rightarrow Q(x))$$

$$S_b^y(P(y) \vee Q(y))$$

$$S_a^y Q(a).$$

$$S_a^y(P(x) \rightarrow Q(x)).$$

S_t^x is an operation that can be performed on predicates; therefore it is not a predicate itself, and this makes S_t^x a meta-formula.

4. Quantifiers

Consider the following three statements:
1. All cats have tails.
2. Some people like their meat raw.
3. Everyone gets a break once in a while.

All these statements indicate how frequently certain things are true. In predicate calculus one uses quantifiers in this context.

Definition.
Let A represent a formula, and let x represent a variable. If we want to indicate that A is true for all possible values of x, we write $\forall x A$. Here, $\forall x$ is called *universal quantifier*, and A is called the *scope* of the quantifier. The variable x is said to be *bound* by the quantifier. The symbol $\forall$ is pronounced "for all".

Universal quantifiers contd.

The quantifier and the bounded variable that follows have to be treated as a unit, and this unit acts somewhat like a unary connective. Statements containing words like "every", "each", and "everyone" usually indicate universal quantification. Such statements must typically be reworded such that they start with "for every x", which is then translated into $\forall x$.

Example: Express "Everyone gets a break once in a while" in predicate calculus.

Solution: We define B to mean "gets a break once in a while". Hence, $B(x)$ means that x gets a break once in a while. The word everyone indicates that this is true for all x. This leads to the following translation:

$$\forall x B(x)$$

Example: Express "All cats have tails" in predicate calculus.

Existential quantifier

Definition
Let A represent a formula, and let x represent a variable. If we want to indicate that A is true for at least one value x, we write $\exists xA$. This statement is pronounced "There exists an x such that A." Here, $\exists x$ is called the *existential quantifier,* and A is called the *scope* of the quantifier. The variable x is said to be *bound* by the quantifier.

Statements containing such phrases as "some", and "at least one" suggest existential quantifiers. They should be rephrased as "there is an x such that" which is translated by $\exists x$.
Example: Let P be the predicate "like their meat raw". Then $\exists xP(x)$ can be translated as "There exist people who like their meat raw" or "Some people like their meat raw."

Example: If the universe of discourse (domain) is a collection of things, $\exists x$ blue(x) should be understood as "There exists objects that are blue" or "Some objects are blue."

Comments 1

- $\forall x$ and $\exists x$ have to be treated like unary connectives.
- The quantifiers are given a higher precedence than all binary connectives. For instance, if $P(x)$ and $Q(x)$ means that x is living and that x is dead, respectively, then one has to write

$$\forall x(P(x) \vee Q(x))$$

to indicate that everything is either living or dead. $\forall xP(x) \vee Q(x)$ means that either everything is living, or x is dead.

- The variable x in a quantifier is just a placeholder, and it can be replaced by any other variable name not appearing elsewhere in the formula. For instance $\forall xP(x)$ and $\forall yP(y)$ mean the same thing: they are logically equivalent.
- The expression $\forall yP(y)$ is a *variant* of $\forall xP(x)$.

  Definition A formula is called a *variant* of $\forall xA$ if it is of the form $\forall yS_y^xA$ where y is any variable name and S_y^xA is the formula obtained from A by replacing all instances of x by y. Similarly, $\exists xA$ and $\exists yS_y^xA$ are variants of one another.

Comments 2

- Quantifiers may be nested, as demonstrated by the following example.
- Example: Translate the sentence "There is somebody who knows everyone" into the language of predicate calculus. To do this, use $K(x, y)$ to express the fact that x knows y.

  Solution. The best way to solve this problem is to go in steps. We write informally

  $$\exists x(x \text{ knows everybody})$$

  Here, "x knows everybody" is still in English and means that for all y is it true that x knows y. Hence

  $$x \text{ knows everybody } = \forall y K(x, y)$$

  We now add the existential quantifier and obtain

  $$\exists x \forall y K(x, y).$$

- Example: Translate "Everybody has somebody who is his or her mother."

Comments 3

- The English statement "Nobody is perfect" also includes a quantifier, "nobody" which is the absence of an individual with a certain property.
- In predicate calculus, the fact that nobody has property P cannot be expressed directly.
- To express the fact that there is no x for which an expression A is true one can either use $\neg\exists x A$ or $\forall x \neg A$.
- If P represents the property of perfection, both $\neg\exists x P(x)$ and $\forall x \neg P(x)$ indicate that nobody is perfect. They correspond to "It is not the case that there is somebody who is perfect", respectively "for everyone, it is not the case that he or she is perfect"
- The two methods to express that nobody is A must of course be logically equivalent,

$$\neg\exists x A \;\; \models\!\!\!\!\mid \; \forall x \neg A.$$

- There are many quantifiers in English, such as "a few", "most", and "about a third", that are useful in daily language, but are not precise and cannot be used in logic. We do not consider them further.

Bound and free variables

- The variable appearing in the quantifier is said to be *bound.*
  For instance, in the expression $\forall x(P(x) \rightarrow Q(x))$, the variable x appears three times and each time x is a bound variable.
- Any variable that is not bound is said to be *free.* Later we will see that the same variable can occur both bound and free in a formula. For this reason it is also important to indicate the position of the variable in question.
- Example: Find the free variables in

$$\forall z(P(z) \land Q(x)) \lor \exists y Q(y).$$

  Solution: Only one variable x is free. All occurrences of z are bound, and so are all occurrences of the variable y.
- Note that the status of a variable changes as formulas are divided into subformulas. For instance, in $\forall x P(x)$, x occurs twice and it is bound both times. This formula contains $P(x)$ as subformula. Nevertheless, in $P(x)$ the variable x is free.

- Instantiation only affects free variables. Specifically, if A is a formula, $S^x_t A$ only affects the free occurrences of the variable x in A.

  For instance, $S^x_y \forall x P(x)$ is still $\forall x P(x)$; that is, the variable x is not instantiated.

  However, $S^x_y (Q(x) \wedge \forall x P(x))$ yields $Q(y) \wedge \forall x P(x)$.
- Hence, instantiation treats the variable x differently, depending on whether it is free or bound, even if this variable appears twice in the same expression.
- Obviously, two things are only identical if they are treated identically. This implies that, if a variable appears both free and bound within the same formula, we have in fact two different variables that happen to have the same name.

Comments

- We can consider the bound variables to be local to the scope of the quantifier just as parameters and locally declared variables in PASCAL procedures are local to the procedure in which they are declared.
- The analogy to PASCAL can be extended further if we consider the variable name in the quantifier as a declaration. This analogy also suggests that, if several quantifiers use the same bound variable for quantification, then all these variables are local to their scope and they are therefore distinct.
- When forming variants, one must be careful not to interfere with local definitions. To illustrate this, consider the statement "y has a mother". If M is a predicate name for "is mother of" then this statement translates into $\exists x M(x, y)$. One obviously must not form the statement $\exists y M(y, y)$, which means that y is her own mother.

- For similar reasons there are restrictions to instantiation. For example, the instantiation $S_x^y(\exists x M(x, y))$ is illegal because it results in $\exists x M(x, x)$. In such cases, one tampers with the way in which a variable is defined, and this has undesired side effects.
- We will refer to instances in which a variable becomes bound, or otherwise changes scope, as *variable clashes*.
- All variable chashes must be avoided.

Restrictions of quantifiers

- Sometimes, quantification is over a subset of the universe of discourse. Suppose, for instance, that animals form the universe of discourse. How can one express sentences such as "All dogs are mammals" and "Some dogs are brown"?
- Consider the first statement "All dogs are mammals". Since the quantifier should be restricted to dogs, one rephrases the statement as "If x is a dog, then x is a mammal" which immediately leads to

$$\forall x(\mathrm{dog}(x) \rightarrow \mathrm{mammal}(x)).$$

- Generally, the sentence

$$\forall x(P(x) \rightarrow Q(x))$$

can be translated as "All individuals with property P also have property Q.".

- Consider now the statement “Some dogs are brown”. This means that there are some animals that are dogs and that are brown. Of course, the statement “x is a dog and x is brown” can be translated as

$$\mathrm{dog}(x) \wedge \mathrm{brown}(x).$$

“There are some brown dogs” can be now translated as

$$\exists x(\mathrm{dog}(x) \wedge \mathrm{brown}(x)).$$

- The statement

$$\exists x(P(x) \wedge Q(x))$$

can in general be interpreted as “Some individuals with property P have also property Q.”

- Note that if the universal quantifier is to apply only to individuals with a given property, we use the conditional to restrict the domain.
- If we similarly want to restrict application of the existential quantifier, we use the conjunction.
- Consider statements containing the word "only" such as "only dogs bark". To convert this into predicate calculus, this must be reworded as "It barks only if it is a dog" or, equivalently "If it barks, then is is a dog". One has therefore

$$\forall x(\text{ barks}(x) \rightarrow \text{dog}(x)).$$