Pseudo-Identities and Bordered Words

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Abstract. This paper investigates the notions of θ-bordered words and θ-unbordered words for various pseudo-identity functions θ. A θ-bordered word is a non-empty word u such that there exists a word v which is a prefix of u while $\theta(v)$ is a suffix of u. The case where θ is the identity function corresponds to the classical notions of bordered and unbordered words. Here we explore cases where θ is a pseudo-identity function, such as a morphism or antimorphism with the property $\theta^n = I$, $n \geq 2$, or a literal morphism or antimorphism. We explore properties of θ-bordered and θ-unbordered words in this context.

Keywords: Bordered words, unbordered words, antimorphic involution, pseudo-bordered words, pseudo-identity

1 Introduction

Periodicity, primitivity, and repetitions of words are fundamental properties in combinatorics on words and formal language theory. Their applications include pattern-matching algorithms (see e.g. [3], and [4]) and data-compression algorithms (see, e.g., [23]). Sometimes motivated by their applications, these classical notions have been modified in various ways that, in essence, replace the identity function with a pseudo-identity, and the notion of repetition with the notion of pseudo-repetition. A representative example is the "weak periodicity" of [5] whereby a word is called *weakly periodic* if it consists of repetitions of words with the same Parikh vector. This type of period was also called *Abelian period* in [2]. Carpi and de Luca extended the notion of periodic words to that of periodic-like words, according to the extendability of factors of a word [1].

Czeizler, Kari, and Seki have proposed and investigated the notion of *pseudo-primitivity* (and pseudo-periodicity) of words in [6, 20], motivated by the properties of information encoded as DNA strands. One of the particularities of information encoded as DNA strands is that a word u over the DNA alphabet $\{A, C, G, T\}$ contains basically the same information as its Watson-Crick complement, denoted here by $\theta(u)$. This led to natural as well as theoretically

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interesting extensions of the notion of "identity", leading to several new notions in combinatorics on words and formal language theory such as pseudo-palindrome [7], pseudo-commutativity [18], as well as hairpin-free and bond-free languages (e.g., [13–15, 19, 21]). In this context, Watson-Crick complementarity has been modeled mathematically by an antimorphic involution θ over an alphabet Σ, i.e., a function that is an antimorphism, $\theta(uv) = \theta(v)\theta(u)$, $\forall u, v \in \Sigma^*$, and an involution, $\theta(\theta(x)) = x$, $\forall x \in \Sigma^*$.

In [16], given a morphic or antimorphic involution θ, a nonempty word u was defined to be θ-bordered if there exists $v \in \Sigma^+$ that is a proper prefix of u, while $\theta(v)$ is a proper suffix of u. A nonempty word u was called θ-unbordered if it was not θ-bordered, and properties of θ-bordered and θ-unbordered words were investigated in [16], [17]. Other generalizations of the classical notions of bordered and unbordered words include pseudo-knot-bordered words, defined in [19] as nonempty words w with the property that $w = xy\alpha = \beta\theta(yx)$ for some words x, y, α, and β.

In [8–10], studies of θ-periodicity have been extended to consider the cases where the morphism or antimorphism θ is literal, non-erasing or uniform. We continue this line of study by extending the investigation of θ-bordered words from the case of morphic or antimorphic involutions θ to cases where θ^n is the identity function, for some $n \geq 2$, and the case where θ is a literal morphism or antimorphism. We study properties of θ-(un)bordered words in Section 3, some properties of the set of θ-(un)bordered words where θ is a morphic involution in Section 4, and conclude with several directions of further research in Section 5.

2 Basic definitions and notations

An alphabet Σ is a finite non-empty set of symbols. Σ^* denotes the set of all words over Σ, including the empty word λ. Σ^+ is the set of all non-empty words over Σ. The length of a word $u \in \Sigma^*$ (i.e. the number of symbols in a word) is denoted by $|u|$. By Σ^m we denote the set of all words of length $m > 0$ over Σ. The complement of a language $L \subseteq \Sigma^*$ is $L^c = \Sigma^* \backslash L$. A word is called *primitive* if it cannot be expressed as a power of another word. Let Q denote the set of all primitive words. A function $\theta : \Sigma^* \to \Sigma^*$ is said to be a *morphism* if for all words $u, v \in \Sigma^*$ we have that $\theta(uv) = \theta(u)\theta(v)$, an *antimorphism* if $\theta(uv) = \theta(v)\theta(u)$ and an *involution* if θ^2 is an identity on Σ^*. If for all $a \in \Sigma$, $|\theta(a)| = 1$, then θ is called *literal* (anti)morphism[1]. A θ-power of a word u is a word of the form $u_1 u_2 \cdots u_n$ for $n \geq 1$ where $u_1 = u$ and $u_i \in \{u, \theta(u)\}$ for $2 \leq i \leq n$. A word is called θ-*primitive* if it cannot be expressed as a θ-power of another word. Let Q_θ denote the set of all θ-primitive words.

For a language $L \subseteq \Sigma^*$, the *principal congruence* P_L determined by L is defined as follows: for any $x, y \in \Sigma^*$ such that $x \neq y$, $x \equiv y(P_L)$ if and only if $uxv \in L \Leftrightarrow uyv \in L$ for all $u, v \in \Sigma^*$. The index of P_L is the number of equivalence classes of P_L. L is said to be *disjunctive* if P_L is the identity, i.e., for

[1] By (anti)morphism we mean either a morphism or an antimorphism.

any $x \neq y \in \Sigma^*$ there exists $u, v \in \Sigma^*$ such that $uxv \in L$ and $uyv \notin L$ or vice versa.

A language $L \subseteq \Sigma^*$ is said to be *dense* if for all $u \in \Sigma^*$, $L \cap \Sigma^* u \Sigma^* \neq \emptyset$.

Definition 1. *1. For $v, w \in \Sigma^*$, $w \leq_p v$ iff $v \in w\Sigma^*$.*
2. For $v, w \in \Sigma^*$, $w \leq_s v$ iff $v \in \Sigma^* w$.
3. $\leq_d = \leq_p \cap \leq_s$.
4. For $u \in \Sigma^*$, $v \in \Sigma^*$ is said to be a border of u if $v \leq_d u$, i.e., $u = vx = yv$.
5. For $v, w \in \Sigma^*$, $w <_p v$ iff $v \in w\Sigma^+$.
6. For $v, w \in \Sigma^*$, $w <_s v$ iff $v \in \Sigma^+ w$.
7. $<_d = <_p \cap <_s$.
8. For $u \in \Sigma^*$, $v \in \Sigma^*$ is said to be a proper border of u if $v <_d u$.
9. For $u \in \Sigma^+$, $L_d(u) = \{v \in \Sigma^* | v <_d u\}$.
10. $\nu_d(u) = |L_d(u)|$.
11. $D(i) = \{u \in \Sigma^+ | \nu_d(u) = i\}$.
12. A word $u \in \Sigma^+$ is said to be a bordered word if there exists $v \in \Sigma^+$ such that $v <_d u$, i.e., $u = vx = yv$ for some $x, y \in \Sigma^+$.
13. A non-empty word which is not bordered is called unbordered.

For a word w, $\mathrm{Pref}(w) = \{u \in \Sigma^+ | \exists v \in \Sigma^*, w = uv\}$ and $\mathrm{Suff}(w) = \{u \in \Sigma^+ | \exists v \in \Sigma^*, w = vu\}$ denotes the set of all prefixes and suffixes respectively. Similarly, the set of proper prefixes and proper suffixes of a word w can be defined as $\mathrm{PPref}(w) = \{u \in \Sigma^+ | \exists v \in \Sigma^+, w = uv\}$ and $\mathrm{PSuff}(w) = \{u \in \Sigma^+ | \exists v \in \Sigma^+, w = vu\}$ respectively.

Definition 2. *[16] Let θ be either a morphism or an antimorphism on Σ^*.*

1. For $v, w \in \Sigma^*$, $w \leq_p^\theta v$ iff $v \in \theta(w)\Sigma^*$.
2. For $v, w \in \Sigma^*$, $w \leq_s^\theta v$ iff $v \in \Sigma^* \theta(w)$.
3. $\leq_d^\theta = \leq_p \cap \leq_s^\theta$.
4. For $u \in \Sigma^*$, $v \in \Sigma^*$ is said to be a θ-border of u if $v \leq_d^\theta u$, i.e., $u = vx = y\theta(v)$.
5. For $w, v \in \Sigma^*$, $w <_p^\theta v$ iff $v \in \theta(w)\Sigma^+$.
6. For $w, v \in \Sigma^*$, $w <_s^\theta v$ iff $v \in \Sigma^+ \theta(w)$.
7. $<_d^\theta = <_p \cap <_s^\theta$.
8. For $u \in \Sigma^*$, $v \in \Sigma^*$ is said to be a proper θ-border of u if $v <_d^\theta u$.
9. For $u \in \Sigma^+$, define $L_d^\theta(u) = \{v \in \Sigma^* | v <_d^\theta u\}$.
10. $\nu_d^\theta(u) = |L_d^\theta(u)|$.
11. $D_\theta(i) = \{u \in \Sigma^+ | \nu_d^\theta(u) = i\}$.
12. A word $u \in \Sigma^+$ is said to be θ-bordered if there exists $v \in \Sigma^+$ such that $v <_d^\theta u$, i.e., $u = vx = y\theta(v)$ for some $x, y \in \Sigma^+$.
13. A nonempty word which is not θ-bordered is called θ-unbordered. Thus, $D_\theta(1)$ is the set of all θ-unbordered words over Σ.

For $u, v \in \Sigma^*$, [11] calls $u <_d x_1 <_d x_2 <_d \cdots <_d v$ a $u - v$ chain. A $u - v$ chain, $u = x_1 <_d x_2 <_d \cdots <_d x_n = v$ is said to be *maximal* if for $u' \in \Sigma^*$, $u <_d u' <_d v$ implies $u' = x_i$ for some $1 < i < n$. Similarly, we can define $u -_\theta v$ chain as a sequence $u = x_1 <_d^\theta x_2 <_d^\theta \cdots <_d^\theta x_n = v$. The notion of maximal chain can be extended to that of θ-maximal chain in a similar fashion.

3 Properties of Pseudo-(Un)Bordered Words

In this section, we study some basic properties of θ-bordered and θ-unbordered words where θ is a (anti)morphism with the property that $\theta^n = I$ on Σ^* for $n \geq 2$ or any literal (anti)morphism. In the case where $\theta^n = I$ and θ is an antimorphism, it is clear that n has to be an even number.

The following result was proved in [11], and can be easily generalized to the case of morphic involutions.

Lemma 1. *[11] Let $u \in \Sigma^+ \backslash D(1)$. Then there exists $v \in \Sigma^*$ with $|v| \leq \frac{|u|}{2}$ such that $v <_d u$.*

Lemma 2. *Let θ be a morphic or an antimorphic involution and let $u \in \Sigma^+ \backslash D_\theta(1)$. Then there exists $v \in \Sigma^*$ with $|v| \leq \frac{|u|}{2}$ such that $v <_d^\theta u$.*

The next two results, Propositions 1 and 2, establish some relations between the set of θ-borders of a word u, namely $L_d^\theta(u)$, and the set of θ-borders of $\theta(u)$, namely $L_d^\theta(\theta(u))$.

Proposition 1. *Let $u \in \Sigma^+$. Then for a morphism θ on Σ^* such that $\theta^n = I$ for $n > 2$, $L_d^\theta(\theta(u)) = \theta(L_d^\theta(u))$.*

Proof. Let $v \in L_d^\theta(\theta(u))$ which implies $\theta(u) = vx = y\theta(v)$ for some $x, y \in \Sigma^+$ which further implies $\theta^2(u) = \theta(v)\theta(x) = \theta(y)\theta^2(v)$. Continuing in this way, we will get $\theta^n(u) = \theta^{n-1}(v)\theta^{n-1}(x) = \theta^{n-1}(y)\theta^n(v)$ and thus $u = \theta^{n-1}(v)\theta^{n-1}(x) = \theta^{n-1}(y)\theta^n(v)$ which implies $\theta^{n-1}(v) \in L_d^\theta(u)$ and hence $v \in \theta(L_d^\theta(u))$. Thus, $L_d^\theta(\theta(u)) \subseteq \theta(L_d^\theta(u))$.
Conversely, let $v \in L_d^\theta(u)$ which implies $u = vx = y\theta(v)$ for $x, y \in \Sigma^+$ and hence $\theta(u) = \theta(v)\theta(x) = \theta(y)\theta^2(v)$ which further implies $\theta(v) \in L_d^\theta(\theta(u))$. Also, since $v \in L_d^\theta(u)$, $\theta(v) \in \theta(L_d^\theta(u))$. Thus, $L_d^\theta(\theta(u)) = \theta(L_d^\theta(u))$.

However, if θ is literal (anti)morphism that is not bijective, Proposition 1 does not necessarily hold, as demonstrated by Example 1.

Example 1. Let $\Sigma = \{a, b\}$ and θ be (anti)morphism such that, $\theta(a) = a, \theta(b) = a$, $u = ababaa$. Then $\theta(u) = aaaaaa$, $L_d^\theta(u) = \{\lambda, a, ab\}$, $\theta(L_d^\theta(u)) = \{\lambda, a, aa\}$, $L_d^\theta(\theta(u)) = \{\lambda, a, aa, \cdots, aaaaa\}$. Clearly, $L_d^\theta(\theta(u)) \neq \theta(L_d^\theta(u))$.

Note that the inclusion $\theta(L_d^\theta(u)) \subseteq L_d^\theta(\theta(u))$ holds in case of Example 1. Moreover, the inclusion holds in general for any literal morphism θ.

Proposition 2. *Let $u \in \Sigma^+$. Then for any literal morphism θ on Σ^*, $\theta(L_d^\theta(u)) \subseteq L_d^\theta(\theta(u))$.*

Proof. Let $v \in L_d^\theta(u)$ which implies $u = vx = y\theta(v)$ for $x, y \in \Sigma^+$ and hence $\theta(u) = \theta(v)\theta(x) = \theta(y)\theta^2(v)$ which further implies $\theta(v) \in L_d^\theta(\theta(u))$. Also, since $v \in L_d^\theta(u)$, $\theta(v) \in \theta(L_d^\theta(u))$. Thus, $\theta(L_d^\theta(u)) \subseteq L_d^\theta(\theta(u))$.

It is known, [16], that, for an antimorphic involution θ, the relation $<_d^\theta$ is transitive.

Lemma 3. *[16] Let $u \in \Sigma^*$ and $v, w \in \Sigma^+$ such that $u <_d^\theta w$ and $w <_d^\theta v$. Then for a morphic involution θ, we have $u <_d v$ and for an antimorphic involution θ, we have $u <_d^\theta v$.*

The statement of Lemma 3 does not necessarily hold in the case when θ is a morphism which is literal and not bijective, as demonstrated by Example 2.

Example 2. Let $\Sigma = \{a, b\}$ and θ be a morphism such that $\theta(a) = a$, $\theta(b) = a$, $u = ab$, $w = abaa$, $v = abaabbaaaa$. Then $u <_d^\theta w$ and $w <_d^\theta v$ but $u \not<_d v$.

The following proposition demonstrates the transitivity of relation $<_d^\theta$ for literal antimorphisms θ.

Proposition 3. *If θ is any literal antimorphism on Σ^*, then the relation $<_d^\theta$ is transitive, i.e. for $u \in \Sigma^*$ and $v, w \in \Sigma^+$ such that $u <_d^\theta w$ and $w <_d^\theta v$, we have $u <_d^\theta v$.*

Proof. Let θ be any literal antimorphism such that $u <_d^\theta w$ and $w <_d^\theta v$ which implies $w = ux = y\theta(u)$ and $v = w\alpha = \beta\theta(w)$ for some $x, y, \alpha, \beta \in \Sigma^+$, hence $v = ux\alpha = \beta\theta(ux)$ which further implies $v = ux\alpha = \beta\theta(x)\theta(u)$. Hence $u <_d^\theta v$.

Corollary 1. *Let $v \in L_d^\theta(u)$ and $w \in \Sigma^+$. Then for any literal antimorphism θ on Σ^*, if $w <_d^\theta v$ then $w \in L_d^\theta(u)$.*

The converse of the Corollary 1 does not hold in general. In fact, in the case of an antimorphism, Proposition 5 holds.

The next results describe relations between the θ-borders of a word u when θ is a morphism with $\theta^n = I, n > 2$, (Proposition 4) or literal (anti)morphisms (Proposition 5).

Proposition 4. *Let $u, v, w \in \Sigma^+$, $u \neq v$ and $u <_d^\theta w, v <_d^\theta w$. If θ is a morphism on Σ^* such that $\theta^n = I$ for $n > 2$, then either $v <_d u$ or $u <_d v$.*

Proof. Let θ be a morphism such that $\theta^n = I$ and $u <_d^\theta w, v <_d^\theta w$ which implies $w = ux = y\theta(u)$ and $w = v\alpha = \beta\theta(v)$ for some $x, y, \alpha, \beta \in \Sigma^+$. If $|u| > |v|$, then $u = vp$ and $\theta(u) = q\theta(v)$ for some $p, q \in \Sigma^+$ which imply $\theta^n(u) = \theta^{n-1}(q)\theta^n(v) = \theta^{n-1}(q)v$. Thus, we get $u = vp = \theta^{n-1}(q)v$ which implies $v <_d u$. Similarly, if $|u| < |v|$ then $v = up'$ and $\theta(v) = q'\theta(u)$ for some $p', q' \in \Sigma^+$ which imply $\theta^n(v) = \theta^{n-1}(q')\theta^n(u) = \theta^{n-1}(q')u$. Thus, we get $v = up' = \theta^{n-1}(q')u$ which implies $u <_d v$.

Proposition 4 does not necessarily hold if θ is a literal (anti)morphism that is not bijective, as demonstrated by Example 3.

Example 3. Let $\Sigma = \{a, b\}$, and θ be a morphism or antimorphism such that $\theta(a) = a, \theta(b) = a$, $u = ab$, $v = abaa$, and $w = abaabbaaaa$. Then $u <_d^\theta w, v <_d^\theta w$ but neither $v <_d u$ nor $u <_d v$.

Proposition 5. *Let $u, v, w \in \Sigma^+$, $u \neq v$ and $u <_d^\theta w, v <_d^\theta w$. Then for any literal morphism θ on Σ^*, either $\theta(v) <_d \theta(u)$ or $\theta(u) <_d \theta(v)$. If θ is any literal antimorphism, then either $v <_p u$ or $u <_p v$.*

Proof. Let θ be any literal morphism and $u <_d^\theta w, v <_d^\theta w$ which imply $w = ux = y\theta(u)$ and $w = v\alpha = \beta\theta(v)$ for some $x, y, \alpha, \beta \in \Sigma^+$. If $|u| > |v|$, then $u = vp$ and $\theta(u) = q\theta(v)$ for some $p, q \in \Sigma^+$ which imply $\theta(u) = \theta(v)\theta(p) = q\theta(v)$. Thus, we get $\theta(v) <_d \theta(u)$. Similarly, if $|u| < |v|$ then $v = up'$ and $\theta(v) = q'\theta(u)$ for some $p', q' \in \Sigma^+$ which imply $\theta(v) = \theta(u)\theta(p') = q'\theta(u)$. Thus, we get $\theta(u) <_d \theta(v)$.

Let θ be any literal antimorphism and $u <_d^\theta w, v <_d^\theta w$ which imply that $w = ux = y\theta(u)$ and $w = v\alpha = \beta\theta(v)$ for some $x, y, \alpha, \beta \in \Sigma^+$. Hence, we have, $ux = v\alpha$. If $|u| > |v|$, $v <_p u$ and if $|v| > |u|$ then $u <_p v$.

Corollary 2. *Let $u, v, w \in \Sigma^+$, $u \neq v$ and $u <_d^\theta w$, $v <_d^\theta w$. Then for any literal antimorphism θ on Σ^*, either $\theta(v) <_s \theta(u)$ or $\theta(u) <_s \theta(v)$.*

Corollary 3. *Let $u \in \Sigma^+$. Then*

1. *For any morphism θ on Σ^* such that $\theta^n = I$ for $n > 2$, $L_d^\theta(u)$ is a totally ordered set with $<_d$, i.e. $L_d^\theta(u) = \{\lambda <_d u_1 <_d u_2 <_d \cdots <_d u_{i-1}\}$.*
2. *For any literal morphism θ on Σ^*, $\theta(L_d^\theta(u))$ is a totally ordered set with $<_d$.*
3. *For any literal antimorphism θ on Σ^*, $L_d^\theta(u)$ is a totally ordered set with $<_p$, i.e. $L_d^\theta(u) = \{\lambda <_p u_1 <_p u_2 <_p \cdots <_p u_{i-1}\}$ and $\theta(L_d^\theta(u))$ is a totally ordered set with $<_s$.*

Proof. Statement 1 follows from Proposition 4, statement 2 from Proposition 5 and statement 3 from Proposition 5 and Corollary 2, respectively.

The next two propositions (Proposition 6, 7) list some properties of θ-unbordered words for (anti)morphisms θ such that $\theta^n = I, n > 2$.

Proposition 6. *Let θ be a morphism on Σ^* such that $\theta^n = I$ for $n > 2$. Then for all $x, y \in D_\theta(1)$ such that $x \neq y$, we have that $xy \neq \theta^{n-1}(y)x$.*

Proof. Let $x, y \in D_\theta(1)$. As $D_\theta(i) \subseteq \Sigma^+$ for $i \geq 1$, both x and y are non-empty. Suppose $xy = \theta^{n-1}(y)x$, then we have following three cases to consider.

Case 1: $|x| = |y|$. Then $x = \theta^{n-1}(y)$ and $y = x$, which is a contradiction since $x \neq y$.

Case 2: $|x| > |y|$. Then there exists $p \in \Sigma^+$ such that $x = \theta^{n-1}(y)p$ and $x = py$ which imply that $x = \theta^{n-1}(y)p = p\theta^n(y)$, which is a contradiction since $x \in D_\theta(1)$.

Case 3: $|y| > |x|$. Then there exists $q \in \Sigma^+$ such that $\theta^{n-1}(y) = xq$ and $y = qx$ which imply that $y = qx = \theta(x)\theta(q)$, which is a contradiction since $y \in D_\theta(1)$.

Since all the three cases leads to a contradiction $xy \neq \theta^{n-1}(y)x$.

Proposition 7. *Let θ be an antimorphism on Σ^* such that $\theta^n = I$ for $n > 2$. Then for $x \in D_\theta(1)$ and $y \in \Sigma^+$ such that $x \neq y$ and $\theta(x) \neq x$, we have that $xy \neq \theta^{n-1}(y)x$.*

Proof. Let $x \in D_\theta(1)$. As $D_\theta(i) \subseteq \Sigma^+$ for $i \geq 1$, x is non-empty. Suppose $xy = \theta^{n-1}(y)x$, then we have following three cases to consider.

Case 1: $|x| = |y|$. Then $x = \theta^{n-1}(y)$ and $y = x$, which is a contradiction since $x \neq y$.

Case 2: $|x| > |y|$. Then there exists $p \in \Sigma^+$ such that $x = \theta^{n-1}(y)p$ and $x = py$ which imply that $x = \theta^{n-1}(y)p = p\theta^n(y)$, which is a contradiction since $x \in D_\theta(1)$.

Case 3: $|y| > |x|$. Then there exists $q \in \Sigma^+$ such that $\theta^{n-1}(y) = xq$ and $y = qx$ which imply that $y = qx = \theta(q)\theta(x)$, which further implies $\theta(q) = q$ and $\theta(x) = x$ which is a contradiction since $\theta(x) \neq x$.

Since all the three cases leads to a contradiction $xy \neq \theta^{n-1}(y)x$.

The following lemma provides a necessary and sufficient condition for a word to be θ-bordered, in the case when θ is a literal antimorphism.

Lemma 4. *Let θ be any literal antimorphism on Σ^*. Then $x \in \Sigma^+$ is θ-bordered iff $x = ay\theta(a)$ for some $a \in \Sigma$ and $y \in \Sigma^*$.*

The result below gives several properties of θ-unbordered words, for literal antimorphisms θ.

Proposition 8. *Let θ be any literal antimorphism on Σ^*, then*

1. *For all $u, v \in \Sigma^+$ and $w \in \Sigma^*$, we have $uwv \in D_\theta(1)$ iff $uv \in D_\theta(1)$.*
2. *If Σ is an alphabet such that there exist $a, b \in \Sigma$ with $\theta(a) \neq b$, then $D_\theta(1)$ is a dense set.*
3. *Let $a, b \in \Sigma$ such that $a \neq b$. Then for all $u \in \Sigma^+$, either ua or ub is θ-unbordered.*

Proof.
1. Suppose $uwv \in D_\theta(1)$ and $uv \notin D_\theta(1)$ which imply that $uv = ay\theta(a)$ for some $a \in \Sigma$ and $y \in \Sigma^*$. If $w = \lambda$, then clearly $uwv \notin D_\theta(1)$, a contradiction. Now, if $w \neq \lambda$, then we have three possibilities.
   Case a: $u = a, v = y\theta(a)$, hence $uwv = awy\theta(a) \notin D_\theta(1)$.
   Case b: $u = ay, v = \theta(a)$, hence $uwv = ayw\theta(a) \notin D_\theta(1)$.
   Case c: $u = ap, v = q\theta(a)$ where $y = pq$ for some $p, q \in \Sigma^*$, hence $uwv = apwq\theta(a) \notin D_\theta(1)$.
   Since all the three cases leads to a contradiction, $uv \in D_\theta(1)$.
   Conversely, suppose $uwv \notin D_\theta(1)$ which imply that $uwv = ay\theta(a)$ for some $a \in \Sigma$ and $y \in \Sigma^*$. Hence, $u = au_1$ and $v = v_1\theta(a)$ for some $u_1, v_1 \in \Sigma^*$ which further implies, $uv = au_1v_1\theta(a) \notin D_\theta(1)$, a contradiction. Hence $uwv \in D_\theta(1)$.
2. Choose $a, b \in \Sigma$ such that $\theta(a) \neq b$. Then for all $w \in \Sigma^*$, there exists $a, b \in \Sigma^*$ such that $awb \in D_\theta(1)$. Hence $D_\theta(1)$ is a dense set.
3. Let us assume that both ua and ub are θ-bordered. Then we have, $ua = a_1y_1\theta(a_1)$ and $ub = a_2y_2\theta(a_2)$ for some $a_1, a_2 \in \Sigma$ and $y_1, y_2 \in \Sigma^*$ which implies $u = a_1y_1 = a_2y_2$ and $a = \theta(a_1), b = \theta(a_2)$. This further implies that $a_1y_1 = a_2y_2$ which implies $a_1 = a_2$ and $y_1 = y_2$ which further implies $a = \theta(a_2) = b$, a contradiction. Hence, either ua or ub is θ-unbordered.

If θ is an antimorphism such that $\theta^n = I, n > 2$, the following result holds.

Proposition 9. *Let θ be an antimorphism on Σ^* such that $\theta^n = I$ for $n > 2$. Then $u \in D_\theta(1)$ iff $\theta^{n-2}(u) \in D_\theta(1)$.*

Proof. Let $u \in D_\theta(1)$ and suppose $\theta^{n-2}(u) \notin D_\theta(1)$ then we have $\theta^{n-2}(u) = ay\theta(a)$ for some $a \in \Sigma$ and $y \in \Sigma^*$ which imply that $u = \theta^n(u) = \theta^2(a)\theta^2(y)\theta^3(a)$ and thus $u \notin D_\theta(1)$, a contradiction. Hence $\theta^{n-2}(u) \in D_\theta(1)$.

Conversely, suppose $\theta^{n-2}(u) \in D_\theta(1)$ and $u \notin D_\theta(1)$. Then $u = ay\theta(a)$ for some $a \in \Sigma$ and $y \in \Sigma^*$. Since n is even and $\theta^n = I$, $n-2$ is also even and thus $\theta^{n-2}(u) = \theta^{n-2}(a)\theta^{n-2}(y)\theta^{n-1}(a) \notin D_\theta(1)$, a contradiction. Hence $u \in D_\theta(1)$.

Lemma 5. *Let θ be a morphic involution on Σ^* and $u \in \Sigma^+$ such that $u \in D(1)$, then $\theta(u) \in D(1)$.*

Proof. Let $u \in D(1)$. Suppose $\theta(u) \notin D(1)$. Then $\theta(u) = \alpha\beta_1 = \beta_2\alpha$ for $\alpha, \beta_1, \beta_2 \in \Sigma^+$. Thus, $u = \theta(\alpha)\theta(\beta_1) = \theta(\beta_2)\theta(\alpha) \notin D(1)$, a contradiction. Thus, $\theta(u) \in D(1)$.

Along similar lines, we can prove the following result concerning $D_\theta(1)$ for a morphism of the form $\theta^n = I, n \geq 2$.

Lemma 6. *Let θ be a morphism on Σ^* such that $\theta^n = I$, $n \geq 2$ and $u \in \Sigma^+$. Then the following are equivalent:*

1. $u \in D_\theta(1)$.
2. $\theta^{n-1}(u) \in D_\theta(1)$.
3. $\theta(u) \in D_\theta(1)$.

Proof. (1) $\Rightarrow$ (2): Let $u \in D_\theta(1)$ and suppose $\theta^{n-1}(u) \notin D_\theta(1)$. Then $\theta^{n-1}(u) = vx = y\theta(v)$ for some $v, x, y \in \Sigma^+$. This implies $u = \theta(v)\theta(x) = \theta(y)\theta^2(v)$, a contradiction since $u \in D_\theta(1)$. Hence $\theta^{n-1}(u) \in D_\theta(1)$.

(2) $\Rightarrow$ (3): Let $\theta^{n-1}(u) \in D_\theta(1)$ and suppose $\theta(u) \notin D_\theta(1)$. Then $\theta(u) = vx = y\theta(v)$ for some $v, x, y \in \Sigma^+$. This implies $\theta^{n-1}(u) = \theta^{n-2}(v)\theta^{n-2}(x) = \theta^{n-2}(y)\theta^{n-1}(v)$, a contradiction since $\theta^{n-1}(u) \in D_\theta(1)$. Hence $\theta(u) \in D_\theta(1)$.

(3) $\Rightarrow$ (1): Let $\theta(u) \in D_\theta(1)$ and suppose $u \notin D_\theta(1)$. Then $u = vx = y\theta(v)$ for some $v, x, y \in \Sigma^+$. This implies $\theta(u) = \theta(v)\theta(x) = \theta(y)\theta^2(v)$, a contradiction since $\theta(u) \in D_\theta(1)$. Hence $u \in D_\theta(1)$.

In fact, the implication $\theta^{n-2}(u) \in D_\theta(1) \Rightarrow u \in D_\theta(1)$ of Proposition 9 and implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) in Lemma 6 hold if θ is a literal morphism, not necessarily bijective.

Proposition 10. *Let θ be a morphism on Σ^* such that $\theta^n = I$ and $u \in \Sigma^+$. If $u \in D_\theta(i)$ for some $i \geq 2$, then for all $1 \leq k < i$, $L_d^\theta(u) \cap D(k) \neq \emptyset$.*

Proof. By Corollary 3 we have

$$L_d^\theta(u) = \{\lambda <_d u_1 <_d u_2 <_d \cdots <_d u_{i-1}\}.$$

Note that $u_k <_d^\theta u$ for all $1 \leq k \leq i-1$. Now, since $u_j \in L_d^\theta(u)$ and $|u_j| < |u_k|$ for all $1 \leq j < k$, by Proposition 4 we have that $u_j <_d u_k$. Hence,

$$L_d(u_k) = \{\lambda, u_1, \cdots u_{k-1}\}.$$

Thus $u_k \in D(k)$ and $L_d^\theta(u) \cap D(k) \neq \emptyset$.

Recall that, a $u -_\theta v$ chain, $u = x_1 <_d^\theta x_2 <_d^\theta \cdots <_d^\theta x_n = v$ is said to be θ-maximal if for $u' \in \Sigma^*$, $u <_d^\theta u' <_d^\theta v$ implies $u' = x_i$ for some $1 < i < n$.

Lemma 7. *[6] Let $u \in \Sigma^+$ be a primitive word. Then u cannot be a factor of u^2 in a nontrivial way, i.e., if $u^2 = xuy$, then necessarily either $x = \lambda$ or $y = \lambda$.*

Proposition 11. *Let θ be an antimorphic involution on Σ^* and $f \in Q$. If $f \leq_d^\theta u \leq_d^\theta f^2$, then $u = f$ or $u = f^2$, i.e., $f \leq_d^\theta f^2$ is a θ-maximal chain.*

Proof. Suppose $f \leq_d^\theta f^2$ is not a θ-maximal chain, i.e., $u \neq f$ and $u \neq f^2$. Since $f \leq_d^\theta u \leq_d^\theta f^2$, we have $u = fx = y\theta(f)$ and $f^2 = u\alpha = \beta\theta(u)$ for $x, y, \alpha, \beta \in \Sigma^*$ with $|x| = |y|$ and $|\alpha| = |\beta|$. Then,

$$f^2 = fx\alpha = y\theta(f)\alpha = \beta\theta(x)\theta(f) = \beta f\theta(y).$$

Now, since $f^2 = \beta f\theta(y)$, by Lemma 7 either $\beta = \lambda$ or $\theta(y) = \lambda$.

Case 1: Suppose, $\beta = \lambda$. This implies $f = \theta(y)$. Since, $fx\alpha = f^2$, we get $x\alpha = f = \theta(y)$. But since, $|x| = |y|$, $x = \theta(y) = f$ and thus $u = fx = f^2$, a contradiction.

Case 2: Suppose, $\theta(y) = \lambda$. This implies $\beta = f$. Since, $fx\alpha = f^2$, we get $x\alpha = f = \beta$. But since, $|\alpha| = |\beta|$, $\alpha = \beta = f$ which implies $f^2 = u\alpha = uf$ and thus $u = f$, a contradiction.

Since both the cases leads to a contradiction, $f \leq_d^\theta f^2$ is a θ-maximal chain.

The θ-unbounded annihilator $\alpha_{ub}(u)$ of a word u is defined, [12], as

$$\alpha_{ub}(u) = \{v \in \Sigma^+ | uv \in D_\theta(1)\}.$$

The following results find a relationship between the θ-unbounded annihilator of a word u and the set of catenations of suffixes of u, for θ-unbordered words u, and morphisms θ with $\theta^n = I$, $n \geq 2$ (Proposition 12) or literal antimorphisms (Proposition 13).

Proposition 12. *Let θ be a morphism on Σ^* such that $\theta^n = I, n \geq 2$. If $u \in D_\theta(1)$, then $(PSuff(u))^+ \subseteq \alpha_{ub}(u)$.*

Proof. Let $u \in D_\theta(1)$. Let $v = u_1 u_2 \cdots u_m$ for some $u_i \in \text{PSuff}(u)$ and $1 \leq i \leq m$. Suppose that $uv \notin D_\theta(1)$. Then there exists $\alpha, \alpha_1, \beta_1 \in \Sigma^+$ such that $uv = \alpha\alpha_1 = \beta_1\theta(\alpha)$. Then, we have following two cases:

Case 1: $|\alpha| > |v|$. Then, we have $\theta(\alpha) = u''v$ and $u = u'u''$ for some $u', u'' \in \Sigma^+$. This implies $u'' <_s u$. From $uv = \alpha\alpha_1$, we get $uv = \theta^{n-1}(u'')\theta^{n-1}(v)\alpha_1$. This implies $\theta^{n-1}(u'') <_p u$. This will further imply that $u \notin D_\theta(1)$, a contradiction.

Case 2: $|\alpha| \leq |v|$. Also, we have $v = u_1u_2 \cdots u_m$ for some $u_i \in \text{PSuff}(u)$ for $1 \leq i \leq m$. Thus we have following two sub-cases:

Case 2(a): $|\alpha| < |u_m|$. Then, we have $\theta(\alpha) = u_{m''}$ and $u_m = u_{m'}u_{m''}$ for some $u_{m'}, u_{m''} \in \Sigma^+$. Since, $u_m \in \text{PSuff}(u)$, we have $u = u'_m u_m = u'_m u_{m'} u_{m''}$ for some $u'_m \in \Sigma^+$. Thus, we have $u_{m''} <_s u$. From $uv = \alpha\alpha_1$, we get $uv = \theta^{n-1}(u_{m''})\alpha_1$. This implies $\theta^{n-1}(u_{m''}) <_p u$. This will further imply that $u \notin D_\theta(1)$, a contradiction.

Case 2(b): $|\alpha| \geq |u_m|$. Then, we have $\theta(\alpha) = u''_i u_{i+1} \cdots u_m$ for $u_i = u'_i u''_i$, $u'_i \in \Sigma^*$, $u''_i \in \Sigma^+$ and $i = 1, 2, \cdots, m-1$. Since, $u_i \in \text{PSuff}(u)$, we have $u = u_{i'}u_i = u_{i'}u'_i u''_i$ for some $u_{i'} \in \Sigma^+$. Thus, we have $u''_i <_s u$. From $uv = \alpha\alpha_1$, we get $uv = \theta^{n-1}(u''_i)\theta^{n-1}(u_{i+1} \cdots u_m)\alpha_1$. This implies $\theta^{n-1}(u''_i) <_p u$. This will further imply that $u \notin D_\theta(1)$, a contradiction.

Since all the cases leads to a contradiction, $(\text{PSuff}(u))^+ \subseteq \alpha_{ub}(u)$.

Proposition 13. *Let θ be any literal antimorphism on Σ^*. If $u \in D_\theta(1)$, then $(PSuff(u))^+ \subseteq \alpha_{ub}(u)$.*

Proof. Let $v = u_1u_2 \cdots u_m$ for some $u_i \in \text{PSuff}(u)$ and $1 \leq i \leq m$. Suppose, $uv \notin D_\theta(1)$. Then $uv = ay\theta(a)$ for some $a \in \Sigma$ and $y \in \Sigma^*$. This further implies, $u = ay_1$, $v = y_2\theta(a)$ and $y = y_1y_2$ for some $y_1, y_2 \in \Sigma^*$. Clearly, $a <_p u$. But, since, $v = u_1u_2 \cdots u_m = y_2\theta(a)$ where $u_m \in \text{PSuff}(u)$, we will have $u_m = u_{m'}\theta(a)$ for $u_{m'} \in \Sigma^*$. Also, $u = u'u_m = u'u_{m'}\theta(a)$ and thus $\theta(a) <_s u$. This imply $u \notin D_\theta(1)$, a contradiction.

4 Disjunctivity of the Set of θ-(Un)Bordered Words

In this section we study some properties of the set of θ-bordered and θ-unbordered words. In [11] it was shown that, for every $i \geq 1$, the set of all (un)bordered words $D(i)$ is disjunctive. Similarly, we will show that, under some conditions, if θ is a morphic involution then the set of all θ-unbordered words $D_\theta(1)$ is disjunctive, and the set of all words with exactly two θ-borders $D_\theta(2)$, are also disjunctive (Theorem 1). We also study the disjunctivity of some related languages (Theorem 2).

The following proposition provides a necessary and sufficient condition for a language to be disjunctive.

Proposition 14. *[22] Let $L \subseteq \Sigma^*$. Then the following two statements are equivalent:*

1. *L is a disjunctive language.*
2. *If $u, v \in \Sigma^+$, $u \neq v$, $|u| = |v|$, then $u \not\equiv v(P_L)$.*

The following auxiliary lemmas are needed for the main results of this section, Theorem 1 and Theorem 2.

Lemma 8. *Let θ be a morphic involution and $a, b \in \Sigma$, $a \neq b$. Let $x, y \in \Sigma^m$, $m > 0$. Then,*

1. $a^m x\theta(b) \in D_\theta(1)$.
2. *If* $a \neq \theta(a)$, $x = \theta(b)x'$, $x' \in \Sigma^*$ *and* $k \geq m$, *then* $(a^k y\theta(b))(a^k x\theta(b)) \in D_\theta(1)$.

Proof. 1. Since there does not exist any word $u \in \Sigma^+$ with $|u| \leq m$ such that $u <_d^\theta a^m x\theta(b)$, by Lemma 2, $a^m x\theta(b) \in D_\theta(1)$.
2. Let $(a^k y\theta(b))(a^k x\theta(b)) \notin D_\theta(1)$. Then there exists $u \in \Sigma^+$ such that

$$u <_d^\theta (a^k y\theta(b))(a^k x\theta(b)).$$

By Lemma 2, it is enough to consider only the case $|u| \leq m+k+1$.
Case (i): $|u| \leq k$. Then $u = a^n$ for some $n \leq k$ and $\theta(u) = \alpha''\theta(b)$ for $x = \alpha'\alpha'', \alpha' \in \Sigma^+, \alpha'' \in \Sigma^*$. Hence $a^n = \theta(\alpha'')b$ which implies $a = b$, a contradiction.
Case (ii): $k < |u| < m+k+1$. Then $u = a^k y'$ for $y = y'y''$, $y' \in \Sigma^+$, $y'' \in \Sigma^*$ and $\theta(u) = a^n x\theta(b) = a^n\theta(b)x'\theta(b)$ for $0 \leq n < k$. Hence $a^k y' = \theta(a^n)b\theta(x')b$ which implies $a = b$, a contradiction.
Case (iii): $|u| = m+k+1$. Then $u = a^k y\theta(b) = \theta(a^k)\theta(x)b$ which implies $a = \theta(a)$, a contradiction.
Since, all the three cases leads to a contradiction $(a^k y\theta(b))(a^k x\theta(b)) \in D_\theta(1)$.

Lemma 9. *Let θ be a morphic involution and let $a, b \in \Sigma$, $a \neq \theta(b)$. Let $x \neq y$, $x, y \in \Sigma^m$, $m > 0$. If $x = \theta(b)x'$, $x' \in \Sigma^*$ and $k \geq m$, then $(a^k y\theta(b))(\theta(a^k x\theta(b))) \in D_\theta(1)$.*

Proof. Let $(a^k y\theta(b))(\theta(a^k x\theta(b))) \notin D_\theta(1)$. Then there exists $u \in \Sigma^+$ such that

$$u <_d^\theta (a^k y\theta(b))(\theta(a^k x\theta(b))).$$

By Lemma 2, it is enough to consider only the case $|u| \leq m+k+1$.

Case (i): $|u| \leq k$. Then $u = a^n$ for some $n \leq k$ and $\theta(u) = \theta(\alpha'')b$ for $x = \alpha'\alpha'', \alpha' \in \Sigma^+, \alpha'' \in \Sigma^*$. Hence $a^n = \alpha''\theta(b)$ which implies $a = \theta(b)$, a contradiction.

Case (ii): $k < |u| < m+k+1$. Then $u = a^k y'$ for $y = y'y''$, $y' \in \Sigma^+$, $y'' \in \Sigma^*$ and $\theta(u) = \theta(a^n)\theta(x)b = \theta(a^n)b\theta(x')b$ for $0 \leq n < k$. Hence $a^k y' = a^n\theta(b)x'\theta(b)$ which implies $a = \theta(b)$, a contradiction.

Case (iii): $|u| = m+k+1$. Then $u = a^k y\theta(b) = a^k x\theta(b)$ which implies $y = x$, a contradiction.

Since, all the three cases lead to a contradiction $(a^k y\theta(b))(\theta(a^k x\theta(b))) \in D_\theta(1)$.

Lemma 10. *Let θ be a literal (anti)morphism on Σ^* and $a, b \in \Sigma$ such that $a \neq \theta(b)$. Let $x \neq y$, $x, y \in \Sigma^m$, $m > 0$. Then:*

1. $a^m x\theta(b) \in D(1)$.
2. *If* $x = \theta(b)x'$, $x' \in \Sigma^*$ *and* $k \geq m$, *then* $(a^k y\theta(b))(a^k x\theta(b)) \in D(1)$.

Proof. Let θ be a literal (anti)morphism.

1. Since there does not exist any word $u \in \Sigma^+$ with $|u| \leq m$ such that $u <_d a^m x\theta(b)$, by Lemma 1, $a^m x\theta(b) \in D(1)$.
2. Let $(a^k y\theta(b))(a^k x\theta(b)) \notin D(1)$. Then there exists $u \in \Sigma^+$ such that

$$u <_d (a^k y\theta(b))(a^k x\theta(b)).$$

By Lemma 1, it is enough to consider only the case $|u| \leq m + k + 1$.
Case (i): $|u| \leq k$. Then $u = a^n = \alpha''\theta(b)$ for some $n \leq k$ and $x = \alpha'\alpha'', \alpha' \in \Sigma^+, \alpha'' \in \Sigma^*$, which implies $a = \theta(b)$, a contradiction.
Case (ii): $k < |u| < m + k + 1$. Then $u = a^k y' = a^n x\theta(b) = a^n\theta(b)x'\theta(b)$ for $y = y'y''$, $y' \in \Sigma^+$, $y'' \in \Sigma^*$ and $0 \leq n < k$, which implies $a = \theta(b)$, a contradiction.
Case (iii): $|u| = m+k+1$. Then $u = a^k y\theta(b) = a^k x\theta(b)$ which implies $x = y$, a contradiction.
Since, all the three cases leads to a contradiction $(a^k y\theta(b))(a^k x\theta(b)) \in D(1)$.

Corollary 4 follows immediately from Lemma 8 and 10.

Corollary 4. *Let θ be a morphic involution on Σ^*, where Σ is an alphabet with $|\Sigma| \geq 3$ that contains letters $a \neq b$ such that $a \notin \{\theta(b), \theta(a)\}$. Let $x \neq y$, $x, y \in \Sigma^m$, $m > 0$. Then:*

1. *$a^m x\theta(b) \in D_\theta(1) \cap D(1)$.*
2. *If $x = \theta(b)x'$, $x' \in \Sigma^*$ and $k \geq m$, then $(a^k y\theta(b))(a^k x\theta(b)) \in D_\theta(1) \cap D(1)$.*

Lemma 11. *Let θ be a morphic involution and let $a, b \in \Sigma$ such that $a \notin \{b, \theta(b)\}$. Let $x \in \Sigma^m$, $m > 0$. If $x = \theta(b)x'$, $x' \in \Sigma^*$, then $(a^m x\theta(b))(\theta(a^m x\theta(b))) \in D_\theta(2)$.*

Proof. Clearly $\lambda, a^m x\theta(b) \in L_d^\theta((a^m x\theta(b))(\theta(a^m x\theta(b))))$.

Let $(a^m x\theta(b))(\theta(a^m x\theta(b))) \notin D_\theta(2)$. Then there exists $u \in \Sigma^+$ such that

$$u <_d^\theta (a^m x\theta(b))(\theta(a^m x\theta(b)))$$

and $u \notin \{\lambda, a^m x\theta(b)\}$. Then, we have following cases to consider.

Case (i): $|u| \leq m$. Then, $u = a^n$ for some $n \leq m$ and $\theta(u) = \theta(\alpha'')b$ for $x = \alpha'\alpha''$, $\alpha' \in \Sigma^+$ and $\alpha'' \in \Sigma^*$. Hence $a^n = \alpha''\theta(b)$ which implies $a = \theta(b)$, a contradiction.

Case (ii): $m < |u| < 2m + 1$. Then, $u = a^m\alpha'$ for $x = \alpha'\alpha''$, $\alpha' \in \Sigma^+$, $\alpha'' \in \Sigma^*$ and $\theta(u) = \theta(a^n)\theta(x)b = \theta(a^n)b\theta(x')b$ for $0 \leq n < m$. Hence $a^m\alpha' = a^n\theta(b)x'\theta(b)$ which implies $a = \theta(b)$, a contradiction.

Case (iii): $2m + 1 < |u| \leq 3m + 1$. Then, $u = a^m x\theta(b)\theta(a^k)$ for some $0 < k \leq m$ and $\theta(u) = \alpha''\theta(b)\theta(a^m)\theta(x)b$ for $x = \alpha'\alpha''$, $\alpha' \in \Sigma^+, \alpha'' \in \Sigma^*$. Hence, $u = a^m x\theta(b)\theta(a^k) = \theta(\alpha'')ba^m x\theta(b)$ which implies $a = b$, a contradiction.

Case (iv): $3m+1 < |u| \leq 4m+1$. Then, $u = a^m x\theta(b)\theta(a^m)\theta(\alpha')$ for $x = \alpha'\alpha''$, $\alpha' \in \Sigma^+$, $\alpha'' \in \Sigma^*$ and $\theta(u) = a^k x\theta(b)\theta(a^m)\theta(x)b$ for $0 \leq k < m$. Hence, $u = a^m x\theta(b)\theta(a^m)\theta(\alpha') = \theta(a^k)b\theta(x')ba^m x\theta(b)$ which implies $a = b$, a contradiction.

Since all the cases leads to a contradiction $(a^m x\theta(b))(\theta(a^m x\theta(b))) \in D_\theta(2)$.

Theorem 1. *Let θ be a morphic involution on Σ^*, where Σ is an alphabet with $|\Sigma| \geq 2$ that contains letters $a \neq b$ such that $a \neq \theta(b)$. Then the set of θ-unbordered words, $D_\theta(1)$ and set of words with exactly two θ-borders $D_\theta(2)$ are disjunctive.*

Proof. Let $x, y \in \Sigma^m$, $x \neq y$, $m > 0$. Without loss of generality let us assume that $x = \theta(b)x'$, $x' \in \Sigma^*$. Let $u = a^m$, $v = \theta(b)\theta(a^m x\theta(b))$. Since $a \neq b$, by Lemma 8(1), we have $a^m x\theta(b) \in D_\theta(1)$ and by Lemma 11,

$$uxv = a^m x\theta(b)\theta(a^m x\theta(b)) \in D_\theta(2).$$

Since $D_\theta(2) \cap D_\theta(1) = \emptyset$, it follows that $uxv \notin D_\theta(1)$. Further, by Lemma 6 $\theta(a^m x\theta(b)) \in D_\theta(1)$. Since $a \neq \theta(b)$, by Lemma 9,

$$uyv = a^m y\theta(b)(\theta(a^m x\theta(b))) \in D_\theta(1).$$

Since, for $x, y \in \Sigma^+$ $x \neq y$, $|x| = |y|$, we got $x \not\equiv y(P_L)$ where $L = D_\theta(1)$. Hence, by Proposition 14, we have that $D_\theta(1)$ is disjunctive. From the proof it follows that also $D_\theta(2)$ is disjunctive.

The following Lemmas are needed for the proof of Theorem 2.

Lemma 12. *Let $m \geq 1$, $x \in \Sigma^+$, $u', u'', y \in \Sigma^*$ and θ be a morphic involution on Σ^*. For any $u \in D_\theta(1) \cap D(1)$, if $(x_1y_1 \cdots x_my_m)x_{m+1} = u'uu''$, where $x_i = x$ and $y_j = y$ if i and j are odd, $x_i = \theta(x)$ and $y_j = \theta(y)$ if i and j are even for $1 \leq i \leq m+1$ and $1 \leq j \leq m$, then $|u| \leq |xy|$.*

Proof. Suppose, $|u| > |xy|$. We will prove just 3 cases here, the other cases follow similarly.

Case (i): u occurs as a subword of $y\theta(x)\theta(y)$. Then there exists $\alpha_1, \alpha_2 \in \Sigma^+$ and $\beta_1, \beta_2, \beta_1', \beta_2' \in \Sigma^*$ such that $x = \alpha_1\alpha_2$, $y = \beta_1\beta_1' = \beta_2'\beta_2$, $|\beta_2| > |\beta_1'|$, then there exists $\alpha \in \Sigma^+$ such that $\beta_1 = \beta_2'\alpha$, $\beta_2 = \alpha\beta_1'$ and we have

$$u = \beta_2\theta(\alpha_1)\theta(\alpha_2)\theta(\beta_1) = \alpha\beta_1'\theta(\alpha_1\alpha_2)\theta(\beta_2')\theta(\alpha) \notin D_\theta(1)$$

Case (ii): u occurs as a subword of $y\theta(x)\theta(y)x$. Then there exists $\alpha_1, \alpha_2 \in \Sigma^+$ and $\beta_1, \beta_2 \in \Sigma^*$ such that $x = \alpha_1\alpha_2$, $y = \beta_1\beta_2$, then

$$u = \beta_2\theta(\alpha_1)\theta(\alpha_2)\theta(\beta_1)\theta(\beta_2)\alpha_1 \notin D_\theta(1)$$

a contradiction.

Case (iii): u occurs as a subword of $y\theta(x)\theta(y)xy\theta(x)$. Then $\alpha_1, \alpha_2 \in \Sigma^+$ and $\beta_1, \beta_2 \in \Sigma^*$ such that $x = \alpha_1\alpha_2$, $y = \beta_1\beta_2$, then

$$u = \beta_2\theta(\alpha_1)\theta(\alpha_2)\theta(y)x\beta_1\beta_2\theta(\alpha_1) \notin D(1)$$

a contradiction.

All the other cases will lead to a similar contradiction, hence $|u| \leq |xy|$.

Lemma 13. *Let θ be a morphic involution on Σ^*. If $f_1 \cdots f_m = u_1 u_2 \cdots u_k$ with $u_i \in D_\theta(1) \cap D(1)$, $i = 1, 2, \cdots, k$ such that $f_j = f$ if j is odd and $f_j = \theta(f)$ if j is even, $1 \leq j \leq m$, then $|u_i| \leq |f|$ for all $1 \leq i \leq k$.*

Proof. Follows from the proof of Lemma 12 replacing y by an empty word λ.

Lemma 14. *Let $m \geq 2$, $m \geq n \geq 1$, θ be a morphic involution on Σ^*. Then for any $x \in \Sigma^+$, $y \in \Sigma^*$, $(x_1 y_1 \cdots x_m y_m) x_{m+1} \notin [D_\theta(1) \cap D(1)]^n$, where the conditions placed on x_i and y_j for $1 \leq i \leq m+1$ and $1 \leq j \leq m$ are the same as those in Lemma 12.*

Proof. Suppose $(x_1 y_1 \cdots x_m y_m) x_{m+1} \in [D_\theta(1) \cap D(1)]^n$. Then there exists $u_1, u_2, \cdots, u_n \in D_\theta(1) \cap D(1)$ such that $(x_1 y_1 \cdots x_m y_m) x_{m+1} = u_1 u_2 \cdots u_n$. By Lemma 12, we will get $|u_i| \leq |xy|$ for $1 \leq i \leq n$. However, this would further imply,

$$|u_1 u_2 \cdots u_n| \leq n|xy| \leq m|xy| < m|xy| + |x|$$

which is a contradiction. Hence $(x_1 y_1 \cdots x_m y_m) x_{m+1} \notin [D_\theta(1) \cap D(1)]^n$.

Lemma 15. *Let $m > n \geq 1$ and θ be a morphic involution on Σ^*. Then for any $f, \theta(f) \in \Sigma^+$, we have $f_1 \cdots f_m \notin [D_\theta(1) \cap D(1)]^n$, where the conditions placed on f_i for $1 \leq i \leq m$ are the same as those of Lemma 13.*

Proof. Follows from the proof of Lemma 14 replacing y by an empty word λ.

Lemma 16. *Let θ be a morphic involution on Σ^*. For any $f, \theta(f) \in D_\theta(1) \cap D(1)$ and $n \geq 2$, $f_1 \cdots f_n \notin [D_\theta(1) \cap D(1)]^{n-1}$, where the conditions placed on f_i for $1 \leq i \leq n$ are the same as those of Lemma 13.*

Proof. We will prove this result by induction on n. For $n = 2$ result holds trivially as $f\theta(f) \notin D_\theta(1) \cap D(1)$. Assume that the result holds for $n = k$, i.e., $f_1 \cdots f_k \notin [D_\theta(1) \cap D(1)]^{k-1}$. Suppose, $f_1 \cdots f_{k+1} \in [D_\theta(1) \cap D(1)]^k$, then there exists $u, v \in \Sigma^+$ such that $uv = f_1 \cdots f_{k+1}$, $u \in D_\theta(1) \cap D(1)$ and $v \in [D_\theta(1) \cap D(1)]^{k-1}$. By Lemma 13, $|u| \leq |f|$. If $|u| < |f|$, then $f = uu'$ for some $u' \in \Sigma^+$. Hence, we get

$$f_1 \cdots f_{k+1} = u_1 u_1' \cdots u_{k+1} u_{k+1}' = u_1 (u_1' u_2 \cdots u_k' u_{k+1}) u_{k+1}'$$

where $u_i u_i' = uu'$ if i is odd and $u_i u_i' = \theta(u)\theta(u')$ if i is even. But then $(u_1' u_2 \cdots u_k' u_{k+1}) u_{k+1}' \in [D_\theta(1) \cap D(1)]^{k-1}$ which is a contradiction to Lemma 14. If $|u| = |f|$, then $u = f$. Thus, $v = f_2 \cdots f_{k+1} \in [D_\theta(1) \cap D(1)]^{k-1}$, which is a contradiction to Lemma 15. Hence $f_1 \cdots f_n \notin [D_\theta(1) \cap D(1)]^{n-1}$.

Theorem 2. *Let θ be a morphic involution on Σ^*, where Σ is an alphabet with $|\Sigma| \geq 3$ that contains letters $a \neq b$ such that $a \notin \{\theta(b), \theta(a)\}$. Then the set $[D_\theta(1) \cap D(1)]^n$ is disjunctive for any even number $n \geq 2$.*

Proof. Choose $x \neq y \in \Sigma^m$, $m > 0$ with $y = \theta(b)y'$ for some $y' \in \Sigma^*$. Let $L = [D_\theta(1) \cap D(1)]^n$. By Corollary 4(1), $a^m x\theta(b) \in D_\theta(1) \cap D(1)$ and thus by Lemma 5 and 6 $\theta(a^m x\theta(b)) \in D_\theta(1) \cap D(1)$. Since $x \neq y$ and $a \neq \theta(b)$, by Lemma 9 we have $a^m x\theta(b)\theta(a^m y\theta(b)) \in D_\theta(1) \cap D(1)$, which further by Lemma 5 and 6 implies $\theta(a^m x\theta(b))a^m y\theta(b) \in D_\theta(1) \cap D(1)$. Let

$$u = (u_1 \cdots u_n)a^m, v = \theta(b).$$

where $u_i = a^m x\theta(b)$ if i is odd and $u_i = \theta(a^m x\theta(b))$ if i is even.

Since n is even, we obtain

$$uyv = (u_1 \cdots u_n)a^m y\theta(b) = (u_1 \cdots u_{n-1})(\theta(a^m x\theta(b))a^m y\theta(b)) \in L.$$

On the other hand, by Lemma 16,

$$uxv = (u_1 \cdots u_n)a^m x\theta(b) = u_1 \cdots u_{n+1} \notin L.$$

Since, for $x, y \in \Sigma^+$, $x \neq y$, $|x| = |y|$, we got $x \not\equiv y(P_L)$, by Proposition 14, L is disjunctive.

In [11], it was shown that the language $D(i) \cap Q$ is disjunctive for $i \geq 1$. However, the following example shows that there exist morphic involutions θ for which the language $D_\theta(1) \cap Q_\theta$ is not disjunctive.

Example 4. Let $\Sigma = \{A, C, G, T\}$ with θ being the morphic involution defined as $\theta(A) = T$, $\theta(T) = A$, $\theta(G) = C$ and $\theta(C) = G$. Let $u = ACT$, $v = CA$, $x = AGG$ and $y = TCA$. Then $uxv = ACTAGGCA \in D_\theta(1) \cap Q_\theta$ and $uyv = ACTTCACA \in D_\theta(1) \cap Q_\theta$, which shows that $D_\theta(1) \cap Q_\theta$ is not disjunctive.

Proposition 15. *If θ is any literal antimorphism on Σ^*, $D_\theta(1)$ is a regular language.*

Proof. We know that, for all $a \in \Sigma$, a is θ-unbordered and from Lemma 4, we have $D_\theta(1) = \Sigma \cup Y$ where $Y = \cup_{a,b \in \Sigma} a\Sigma^* b$ such that $\theta(a) \neq b$. Since Σ is finite, Y is regular and hence $D_\theta(1)$ is regular.

5 Conclusions

In this paper we investigate properties of θ-bordered words, where θ is not just the identity function or a morphic or antimorphic involution, but, more generally, a morphism or an antimorphism with the property that $\theta^n = I$, for $n \geq 2$, or a literal (anti)morphism θ. Results we obtained include the transitivity of the relation $<_d^\theta$ for literal antimorphisms θ, and the disjunctivity of the set of all θ-unbordered words for morphic involutions θ.

Future directions of research includes exploring other properties of θ-bordered and θ-unbordered words, as well as the disjunctivity of other languages related to $D_\theta(i)$.

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