

Triangular Decomposition of Semi-Algebraic Systems

Changbo Chen
University of Western Ontario
cchen252@csd.uwo.ca

Marc Moreno Maza
University of Western Ontario
moreno@csd.uwo.ca

James H. Davenport
University of Bath
J.H.Davenport@bath.ac.uk

Bican Xia
Peking University
xbc@math.pku.edu.cn

John P. May
Maplesoft
jmay@maplesoft.com

Rong Xiao
University of Western Ontario
rong@csd.uwo.ca

ABSTRACT

Regular chains and triangular decompositions are fundamental and well-developed tools for describing the complex solutions of polynomial systems. This paper proposes adaptations of these tools focusing on solutions of the real analogue: semi-algebraic systems.

We show that any such system can be decomposed into finitely many *regular semi-algebraic systems*. We propose two specifications of such a decomposition and present corresponding algorithms. Under some assumptions, one type of decomposition can be computed in singly exponential time w.r.t. the number of variables. We implement our algorithms and the experimental results illustrate their effectiveness.

Categories and Subject Descriptors

I.1.2 [Symbolic and Algebraic Manipulation]: Algorithms—*Algebraic algorithms, Analysis of algorithms*

General Terms

Algorithms, Experimentation, Theory

Keywords

regular semi-algebraic system, regular chain, triangular decomposition, border polynomial, fingerprint polynomial set

1. INTRODUCTION

Regular chains, the output of triangular decompositions of systems of polynomial equations, enjoy remarkable properties. Size estimates play in their favor [12] and permit the design of modular [13] and fast [17] methods for computing triangular decompositions. These features stimulate the development of algorithms and software for solving polynomial systems via triangular decompositions.

For the fundamental case of semi-algebraic systems with rational number coefficients, to which this paper is devoted, we observe that several algorithms for studying the real solutions of such systems take advantage of the structure of a

regular chain. Some are specialized to isolating the real solutions of systems with finitely many complex solutions [23, 10, 3]. Other algorithms deal with parametric polynomial systems via real root classification (RRC) [25] or with arbitrary systems via cylindrical algebraic decompositions (CAD) [9].

In this paper, we introduce the notion of a *regular semi-algebraic system*, which in broad terms is the “real” counterpart of the notion of a regular chain. Then we define two notions of a *decomposition of a semi-algebraic system*: one that we call *lazy triangular decomposition*, where the analysis of components of strictly smaller dimension is deferred, and one that we call *full triangular decomposition* where all cases are worked out. These decompositions are obtained by combining triangular decompositions of algebraic sets over the complex field with a special Quantifier Elimination (QE) method based on RRC techniques.

Regular semi-algebraic system. Let T be a regular chain of $\mathbb{Q}[x_1, \dots, x_n]$ for some ordering of the variables $\mathbf{x} = x_1, \dots, x_n$. Let $\mathbf{u} = u_1, \dots, u_d$ and $\mathbf{y} = y_1, \dots, y_{n-d}$ designate respectively the variables of \mathbf{x} that are free and algebraic w.r.t. T . Let $P \subset \mathbb{Q}[\mathbf{x}]$ be finite such that each polynomial in P is regular w.r.t. the saturated ideal of T . Define $P_{>} := \{p > 0 \mid p \in P\}$. Let \mathcal{Q} be a quantifier-free formula of $\mathbb{Q}[\mathbf{x}]$ involving only the variables of \mathbf{u} . We say that $R := [\mathcal{Q}, T, P_{>}]$ is a *regular semi-algebraic system* if:

- (i) \mathcal{Q} defines a non-empty open semi-algebraic set S in \mathbb{R}^d ,
- (ii) the regular system $[T, P]$ specializes well at every point u of S (see Section 2 for this notion),
- (iii) at each point u of S , the specialized system $[T(u), P(u)_{>}]$ has at least one real zero.

The zero set of R , denoted by $Z_{\mathbb{R}}(R)$, is defined as the set of points $(u, y) \in \mathbb{R}^d \times \mathbb{R}^{n-d}$ such that $\mathcal{Q}(u)$ is true and $t(u, y) = 0, p(u, y) > 0$, for all $t \in T$ and all $p \in P$.

Triangular decomposition of a semi-algebraic system. In Section 3 we show that the zero set of any semi-algebraic system \mathfrak{S} can be decomposed as a finite union (possibly empty) of zero sets of regular semi-algebraic systems. We call such a decomposition a *full triangular decomposition* (or simply *triangular decomposition* when clear from context) of \mathfrak{S} , and denote by `RealTriangularize` an algorithm to compute it. The proof of our statement relies on triangular decompositions in the sense of Lazard (see Section 2 for this notion) for which it is not known whether or not they can be computed in singly exponential time w.r.t. the number of variables. Meanwhile, we are hoping to obtain an algorithm for decomposing semi-algebraic systems (certainly under some genericity assumptions) that would fit in that complexity class. Moreover, we observe that, in practice, full triangular

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decompositions are not always necessary and that providing information about the components of maximum dimension is often sufficient. These theoretical and practical motivations lead us to a weaker notion of a decomposition of a semi-algebraic system.

Lazy triangular decomposition of a semi-algebraic system. Let $\mathfrak{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$ (see Section 3 for this notation) be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$ and $Z_{\mathbb{R}}(\mathfrak{S}) \subseteq \mathbb{R}^n$ be its zero set. Denote by d the dimension of the constructible set $\{x \in \mathbb{C}^n \mid f(x) = 0, g(x) \neq 0, \text{ for all } f \in F, g \in P \cup H\}$. A finite set of regular semi-algebraic systems $R_i, i = 1 \cdots t$ is called a *lazy triangular decomposition* of \mathfrak{S} if

- $\cup_{i=1}^t Z_{\mathbb{R}}(R_i) \subseteq Z_{\mathbb{R}}(\mathfrak{S})$ holds, and
- there exists $G \subset \mathbb{Q}[\mathbf{x}]$ such that the real-zero set $Z_{\mathbb{R}}(G) \subset \mathbb{R}^n$ contains $Z_{\mathbb{R}}(\mathfrak{S}) \setminus (\cup_{i=1}^t Z_{\mathbb{R}}(R_i))$ and the complex-zero set $V(G) \subset \mathbb{C}^n$ either is empty or has dimension less than d .

We denote by `LazyRealTriangularize` an algorithm computing such a decomposition. In the implementation presented hereafter, `LazyRealTriangularize` outputs additional information in order to continue the computations and obtain a full triangular decomposition, if needed. This additional information appears in the form of *unevaluated function calls*, explaining the usage of the adjective *lazy* in this type of decompositions.

Complexity results for lazy triangular decomposition. In Section 4, we provide a running time estimate for computing a lazy triangular decomposition of the semi-algebraic system \mathfrak{S} when \mathfrak{S} has no inequations nor inequalities, (that is, when $N_{\geq} = P_{>} = H_{\neq} = \emptyset$ holds) and when F generates a strongly equidimensional ideal of dimension d . We show that one can compute such a decomposition in time singly exponential w.r.t. n . Our estimates are not sharp and are just meant to reach a singly exponential bound. We rely on the work of J. Renegar [20] for QE. In Sections 5 and 6 we turn our attention to algorithms that are more suitable for implementation even though they rely on sub-algorithms with a doubly exponential running time w.r.t. d .

A special case of quantifier elimination. By means of triangular decomposition of algebraic sets over \mathbb{C} , triangular decomposition of semi-algebraic systems (both full and lazy) reduces to a special case of QE. In Section 5, we implement this latter step via the concept of a *fingerprint polynomial set*, which is inspired by that of a *discrimination polynomial set* used for RRC in [25, 24].

Implementation and experimental results. In Section 6 we describe the algorithms that we have implemented for computing triangular decompositions (both full and lazy) of semi-algebraic systems. Our MAPLE code is written on top of the `RegularChains` library. We provide experimental data for two groups of well-known problems. In the first group, each input semi-algebraic system consists of equations only while the second group is a collection of QE problems. To illustrate the difficulty of our test problems, and only for this purpose, we provide timings obtained with other well-known polynomial system solvers which are based on algorithms whose running time estimates are comparable to ours. For this first group we use the MAPLE command `Groebner:-Basis` for computing lexicographical Gröbner bases. For the second group we use a general purpose QE software: QPCAD B (in its non-interactive mode) [5]. Our experimental results show that our `LazyRealTriangularize` code can solve most of

our test problems and that it can solve more problems than the package it is compared to.

We conclude this introduction by computing a triangular decomposition of a particular semi-algebraic system taken from [6]. Consider the following question: when does $p(z) = z^3 + az + b$ have a non-real root $x + iy$ satisfying $xy < 1$? This problem can be expressed as $(\exists x)(\exists y)[f = g = 0 \wedge y \neq 0 \wedge xy - 1 < 0]$, where $f = \text{Re}(p(x + iy)) = x^3 - 3xy^2 + ax + b$ and $g = \text{Im}(p(x + iy))/y = 3x^2 - y^2 + a$.

We call our `LazyRealTriangularize` command on the semi-algebraic system $f = 0, g = 0, y \neq 0, xy - 1 < 0$ with the variable order $y > x > b > a$. Its first step is to call the `Triangularize` command of the `RegularChains` library on the algebraic system $f = g = 0$. We obtain one squarefree regular chain $T = [t_1, t_2]$, where $t_1 = g$ and $t_2 = 8x^3 + 2ax - b$, satisfying $V(f, g) = V(T)$. The second step of `LazyRealTriangularize` is to check whether the polynomials defining inequalities and inequations are regular w.r.t. the saturated ideal of T , which is the case here. The third step is to compute the so called *border polynomial set* (see Section 2) which is $B = [h_1, h_2]$ with $h_1 = 4a^3 + 27b^2$ and $h_2 = -4a^3b^2 - 27b^4 + 16a^4 + 512a^2 + 4096$. One can check that the regular system $[T, \{y, xy - 1\}]$ specializes well outside of the hypersurface $h_1h_2 = 0$. The fourth step is to compute the fingerprint polynomial set which yields the quantifier-free formula $\mathcal{Q} = h_1 > 0$ telling us that $[\mathcal{Q}, T, 1 - xy > 0]$ is a regular semi-algebraic system. After performing these four steps, (based on Algorithm 5, Section 6) the function call

`LazyRealTriangularize([f, g, y ≠ 0, xy - 1 < 0], [y, x, b, a])`

in our implementation returns the following:

$$\left\{ \begin{array}{ll} \{[t_1 = 0, t_2 = 0, 1 - xy > 0, h_1 > 0]\} & h_1h_2 \neq 0 \\ \% \text{LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, \\ h_1 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & h_1 = 0 \\ \% \text{LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, \\ h_2 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & h_2 = 0 \end{array} \right.$$

The above output shows that $\{\{\mathcal{Q}, T, 1 - xy > 0\}\}$ forms a lazy triangular decomposition of the input semi-algebraic system. Moreover, together with the output of the recursive calls, one obtains a full triangular decomposition. Note that the cases of the two recursive calls correspond to $h_1 = 0$ and $h_2 = 0$. Since our `LazyRealTriangularize` uses the MAPLE piecewise structure for formatting its output, one simply needs to evaluate the recursive calls with the `value` command, yielding the same result as directly calling `RealTriangularize`

$$\left\{ \begin{array}{ll} \{[t_1 = 0, t_2 = 0, 1 - xy > 0, h_1 > 0]\} & h_1h_2 \neq 0 \\ [] & h_1 = 0 \\ \left\{ \begin{array}{ll} (\{[t_3 = 0, t_4 = 0, \\ h_2 = 0, 1 - xy > 0]\}) & h_3 \neq 0 \\ [] & h_3 = 0 \end{array} \right. & h_2 = 0 \end{array} \right.$$

where $t_3 = xy + 1$, $t_4 = 2a^3x - a^2b + 32ax - 48b + 18xb^2$, $h_3 = (a^2 + 48)(a^2 + 16)(a^2 + 12)$.

From this output, after some simplification, one could obtain the equivalent quantifier-free formula, $4a^3 + 27b^2 > 0$, of the original QE problem.

2. TRIANGULAR DECOMPOSITION OF ALGEBRAIC SETS

We review in the section the basic notions related to regular chains and triangular decompositions of algebraic sets. Throughout this paper, let \mathbf{k} be a field of characteristic 0 and \mathbf{K} be its algebraic closure. Let $\mathbf{k}[\mathbf{x}]$ be the polynomial ring over \mathbf{k} and with ordered variables $\mathbf{x} = x_1 < \dots < x_n$. Let $p, q \in \mathbf{k}[\mathbf{x}]$ be polynomials. Assume that $p \notin \mathbf{k}$. Then denote by $\text{mvar}(p)$, $\text{init}(p)$, and $\text{mdeg}(p)$ respectively the greatest variable appearing in p (called the *main variable* of p), the leading coefficient of p w.r.t. $\text{mvar}(p)$ (called the *initial* of p), and the degree of p w.r.t. $\text{mvar}(p)$ (called the *main degree* of p); denote by $\text{der}(p)$ the derivative of p w.r.t. $\text{mvar}(p)$; denote by $\text{discrim}(p)$ the discriminant of p w.r.t. $\text{mvar}(p)$.

Triangular sets. Let $T \subset \mathbf{k}[\mathbf{x}]$ be a *triangular set*, that is, a set of non-constant polynomials with pairwise distinct main variables. Denote by $\text{mvar}(T)$ the set of main variables of the polynomials in T . A variable v in \mathbf{x} is called *algebraic* w.r.t. T if $v \in \text{mvar}(T)$, otherwise it is said *free* w.r.t. T . If no confusion is possible, we shall always denote by $\mathbf{u} = u_1, \dots, u_d$ and $\mathbf{y} = y_1, \dots, y_m$ respectively the free and the main variables of T . Let h_T be the product of the initials of the polynomials in T . We denote by $\text{sat}(T)$ the *saturated ideal* of T : if T is the empty triangular set, then $\text{sat}(T)$ is defined as the trivial ideal $\langle 0 \rangle$, otherwise it is the ideal $\langle T \rangle : h_T^\infty$. The *quasi-component* $W(T)$ of T is defined as $V(T) \setminus V(h_T)$. Denote $\overline{W}(T) = V(\text{sat}(T))$ as the Zariski closure of $W(T)$.

Iterated resultant. Let p and q be two polynomials of $\mathbf{k}[\mathbf{x}]$. Assume q is non-constant and let $v = \text{mvar}(q)$. We define $\text{res}(p, q, v)$ as follows: if v does not appear in p , then $\text{res}(p, q, v) := p$; otherwise $\text{res}(p, q, v)$ is the resultant of p and q w.r.t. v . Let T be a triangular set of $\mathbf{k}[\mathbf{x}]$. We define $\text{res}(p, T)$ by induction: if T is empty, then $\text{res}(p, T) = p$; otherwise let v be the greatest variable appearing in T , then $\text{res}(p, T) = \text{res}(\text{res}(p, T_v, v), T_{<v})$, where T_v and $T_{<v}$ denote respectively the polynomials of T with main variables equal to and less than v .

Regular chain. A triangular set $T \subset \mathbf{k}[\mathbf{x}]$ is a *regular chain* if: either T is empty; or (letting t be the polynomial in T with maximum main variable), $T \setminus \{t\}$ is a regular chain, and the initial of t is regular w.r.t. $\text{sat}(T \setminus \{t\})$. The empty regular chain is denoted by \emptyset . Let $H \subset \mathbf{k}[\mathbf{x}]$. The pair $[T, H]$ is a *regular system* if each polynomial in H is regular modulo $\text{sat}(T)$. A regular chain T or a regular system $[T, H]$, is *squarefree* if for all $t \in T$, the $\text{der}(t)$ is regular w.r.t. $\text{sat}(T)$.

Triangular decomposition. Let $F \subset \mathbf{k}[\mathbf{x}]$. Regular chains T_1, \dots, T_e of $\mathbf{k}[\mathbf{x}]$ form a *triangular decomposition* of $V(F)$ if either: $V(F) = \cup_{i=1}^e \overline{W}(T_i)$ (Kalkbrener's sense) or $V(F) = \cup_{i=1}^e W(T_i)$ (Lazard's sense). In this paper, we denote by **Triangularize** an algorithm, such as the one of [18], computing a triangular decomposition of the former kind.

Regularization. Let $p \in \mathbf{k}[\mathbf{x}]$. Let T be a regular chain of $\mathbf{k}[\mathbf{x}]$. Denote by **Regularize**(p, T) an operation which computes a set of regular chains $\{T_1, \dots, T_e\}$ such that (1) for each $i, 1 \leq i \leq e$, either $p \in \text{sat}(T_i)$ or p is regular w.r.t. $\text{sat}(T_i)$; (2) we have $\overline{W}(T) = \overline{W}(T_1) \cup \dots \cup \overline{W}(T_e)$, $\text{mvar}(T) = \text{mvar}(T_i)$ and $\text{sat}(T) \subseteq \text{sat}(T_i)$ for $1 \leq i \leq e$.

Good specialization [8]. Consider a squarefree regular system $[T, H]$ of $\mathbf{k}[\mathbf{u}, \mathbf{y}]$. Recall that \mathbf{y} and $\mathbf{u} = u_1, \dots, u_d$ stand respectively for $\text{mvar}(T)$ and $\mathbf{x} \setminus \mathbf{y}$. Let $z = (z_1, \dots, z_d)$ be

a point of \mathbf{K}^d . We say that $[T, H]$ *specializes well* at z if: (i) none of the initials of the polynomials in T vanishes modulo the ideal $\langle z_1 - u_1, \dots, z_d - u_d \rangle$; (ii) the image of $[T, H]$ modulo $\langle z_1 - u_1, \dots, z_d - u_d \rangle$ is a squarefree regular system.

Border polynomial [25]. Let $[T, H]$ be a squarefree regular system of $\mathbf{k}[\mathbf{u}, \mathbf{y}]$. Let bp be the primitive and square free part of the product of all $\text{res}(\text{der}(t), T)$ and all $\text{res}(h, T)$ for $h \in H$ and $t \in T$. We call bp the *border polynomial* of $[T, H]$ and denote by **BorderPolynomial**(T, H) an algorithm to compute it. We call the set of irreducible factors of bp the *border polynomial set* of $[T, H]$. Denote by **BorderPolynomialSet**(T, H) an algorithm to compute it. Proposition 1 follows from the specialization property of subresultants and states a fundamental property of border polynomials.

PROPOSITION 1. *The system $[T, H]$ specializes well at $u \in \mathbf{K}^d$ if and only if the border polynomial $bp(u) \neq 0$.*

3. TRIANGULAR DECOMPOSITION OF SEMI-ALGEBRAIC SYSTEMS

In this section, we prove that any semi-algebraic system can be decomposed into finitely many regular semi-algebraic systems. This latter notion was defined in the introduction.

Semi-algebraic system. Consider four finite polynomial subsets $F = \{f_1, f_2, \dots, f_s\}$, $N = \{n_1, n_2, \dots, n_t\}$, $P = \{p_1, p_2, \dots, p_r\}$, and $H = \{h_1, h_2, \dots, h_\ell\}$ of $\mathbb{Q}[x_1, \dots, x_n]$. Let N_{\geq} denote the set of non-negative inequalities $\{n_1 \geq 0, \dots, n_t \geq 0\}$. Let $P_{>}$ denote the set of positive inequalities $\{p_1 > 0, \dots, p_r > 0\}$. Let H_{\neq} denote the set of inequations $\{h_1 \neq 0, \dots, h_\ell \neq 0\}$. We will denote by $[F, P_{>}]$ the *basic semi-algebraic system* $\{f_1 = 0, \dots, f_s = 0, p_1 > 0, \dots, p_r > 0\}$. We denote by $\mathfrak{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$ the semi-algebraic system (SAS) which is the conjunction of the following conditions: $f_1 = 0, \dots, f_s = 0$, $n_1 \geq 0, \dots, n_t \geq 0$, $p_1 > 0, \dots, p_r > 0$ and $h_1 \neq 0, \dots, h_\ell \neq 0$.

Notations on zero sets. In this paper, we use “ Z ” to denote the zero set of a polynomial system, involving equations and inequations, in \mathbb{C}^n and “ $Z_{\mathbb{R}}$ ” to denote the zero set of a semi-algebraic system in \mathbb{R}^n .

Pre-regular semi-algebraic system. Let $[T, P]$ be a squarefree regular system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Let bp be the border polynomial of $[T, P]$. Let $B \subset \mathbb{Q}[\mathbf{u}]$ be a polynomial set such that bp divides the product of polynomials in B . We call the triple $[B_{\neq}, T, P_{>}]$ a *pre-regular semi-algebraic system* of $\mathbb{Q}[\mathbf{x}]$. Its zero set, written as $Z_{\mathbb{R}}(B_{\neq}, T, P_{>})$, is the set $(u, y) \in \mathbb{R}^n$ such that $b(u) \neq 0$, $t(u, y) = 0$, $p(u, y) > 0$, for all $b \in B$, $t \in T$, $p \in P$. Lemma 1 and Lemma 2 are fundamental properties of pre-regular semi-algebraic systems.

LEMMA 1. *Let \mathfrak{S} be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Then there exists finitely many pre-regular semi-algebraic systems $[B_{i \neq}, T_i, P_{i >}]$, $i = 1 \dots e$, s.t. $Z_{\mathbb{R}}(\mathfrak{S}) = \cup_{i=1}^e Z_{\mathbb{R}}(B_{i \neq}, T_i, P_{i >})$.*

PROOF. The semi-algebraic system \mathfrak{S} decomposes into basic semi-algebraic systems, by rewriting inequality of type $n \geq 0$ as: $n > 0 \vee n = 0$. Let $[F, P_{>}]$ be one of those basic semi-algebraic systems. If F is empty, then the triple $[P, \emptyset, P_{>}]$ is a pre-regular semi-algebraic system. If F is not empty, by Proposition 1 and the specifications of **Triangularize** and **Regularize**, one can compute finitely many squarefree regular systems $[T_i, H]$ such that $V(F) \cap Z(P_{\neq}) = \cup_{i=1}^e (V(T_i) \cap Z(B_{i \neq}))$ holds and where B_i is the border polynomial set of the regular system $[T_i, H]$. Hence, we have

$Z_{\mathbb{R}}(F, P_{>}) = \cup_{i=1}^e Z_{\mathbb{R}}(B_{i \neq}, T_i, P_{>})$, where each $[B_{i \neq}, T_i, P_{>}]$ is a pre-regular semi-algebraic system. \square

LEMMA 2. Let $[B_{\neq}, T, P_{>}]$ be a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Let h be the product of polynomials in B . The complement of the hypersurface $h = 0$ in \mathbb{R}^d consists of finitely many open cells of dimension d . Let C be one of them. Then for all $\alpha \in C$, the number of real zeros of $[T(\alpha), P_{>}(\alpha)]$ is the same.

PROOF. From Proposition 1 and recursive use of Theorem 1 in [11] on the delineability of a polynomial. \square

LEMMA 3. Let $[B_{\neq}, T, P_{>}]$ be a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. One can decide whether its zero set is empty or not. If it is not empty, then one can compute a regular semi-algebraic system $[\mathcal{Q}, T, P_{>}]$ whose zero set in \mathbb{R}^n is the same as that of $[B_{\neq}, T, P_{>}]$.

PROOF. If $T = \emptyset$, we can always test whether the zero set of $[B_{\neq}, P_{>}]$ is empty or not, for instance using CAD. If it is empty, we are done. Otherwise, defining $\mathcal{Q} = B_{\neq} \wedge P_{>}$, the triple $[\mathcal{Q}, T, P_{>}]$ is a regular semi-algebraic system. If T is not empty, we solve the quantifier elimination problem $\exists \mathbf{y}(B(\mathbf{u}) \neq 0, T(\mathbf{u}, \mathbf{y}) = 0, P(\mathbf{u}, \mathbf{y}) > 0)$ and let \mathcal{Q} be the resulting formula. If \mathcal{Q} is false, we are done. Otherwise, by Lemma 2, above each connected component of $B(\mathbf{u}) \neq 0$, the number of real zeros of the system $[B_{\neq}, T, P_{>}]$ is constant. Then, the zero set defined by \mathcal{Q} is the union of the connected components of $B(\mathbf{u}) \neq 0$ above which $[B_{\neq}, T, P_{>}]$ possesses at least one solution. Thus, \mathcal{Q} defines a nonempty open set of \mathbb{R}^d and $[\mathcal{Q}, T, P_{>}]$ is a regular semi-algebraic system. \square

THEOREM 1. Let \mathfrak{S} be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Then one can compute a (full) triangular decomposition of \mathfrak{S} , that is, as defined in the introduction, finitely many regular semi-algebraic systems such that the union of their zero sets is the zero set of \mathfrak{S} .

PROOF. It follows from Lemma 1 and 3. \square

4. COMPLEXITY RESULTS

We start this section by stating complexity estimates for basic operations on multivariate polynomials.

Complexity of basic polynomial operations. Let $p, q \in \mathbb{Q}[\mathbf{x}]$ be polynomials with respective total degrees δ_p, δ_q , and let $x \in \mathbf{x}$. Let h_p, h_q, h_{pq} and h_r be the height (that is, the bit size of the maximum absolute value of the numerator or denominator of a coefficient) of p, q , the product pq and the resultant $\text{res}(p, q, x)$, respectively. In [14], it is proved that $\text{gcd}(p, q)$ can be computed within $O(n^{2\delta+1}h^3)$ bit operations where $\delta = \max(\delta_p, \delta_q)$ and $h = \max(h_p, h_q)$. It is easy to establish that h_{pq} and h_r are respectively upper bounded by $h_p + h_q + n \log(\min(\delta_p, \delta_q) + 1)$ and $\delta_q h_p + \delta_p h_q + n \delta_q \log(\delta_p + 1) + n \delta_p \log(\delta_q + 1) + \log((\delta_p + \delta_q)!).$ Finally, let M be a $k \times k$ matrix over $\mathbb{Q}[\mathbf{x}]$. Let δ (resp. h) be the maximum total degree (resp. height) of a polynomial coefficient of M . Then $\det(M)$ can be computed within $O(k^{2n+5}(\delta + 1)^{2n}h^2)$ bit operations, see [15].

We turn now to the main subject of this section, that is, complexity estimates for a lazy triangular decomposition of a polynomial system under some genericity assumptions. Let $F \subset \mathbb{Q}[\mathbf{x}]$. A lazy triangular decomposition (as defined in the Introduction) of the semi-algebraic system $\mathfrak{S} = [F, \emptyset, \emptyset, \emptyset]$,

Algorithm 1: LazyRealTriangularize(\mathfrak{S})

Input: a semi-algebraic system $\mathfrak{S} = [F, \emptyset, \emptyset, \emptyset]$
Output: a lazy triangular decomposition of \mathfrak{S}

- 1 $\mathfrak{T} := \text{Triangularize}(F)$
- 2 **for** $T_i \in \mathfrak{T}$ **do**
- 3 $bp_i := \text{BorderPolynomial}(T_i, \emptyset)$
- 4 solve $\exists \mathbf{y}(bp_i(\mathbf{u}) \neq 0, T_i(\mathbf{u}, \mathbf{y}) = 0)$, and let \mathcal{Q}_i be the resulting quantifier-free formula
- 5 **if** $\mathcal{Q}_i \neq \text{false}$ **then** output $[\mathcal{Q}_i, T_i, \emptyset]$

which only involves equations, is obtained by the above algorithm.

Proof of Algorithm 1. The termination of the algorithm is obvious. Let us prove its correctness. Let $R_i = [\mathcal{Q}_i, T_i, \emptyset]$, for $i = 1 \dots t$ be the output of Algorithm 1 and let T_j for $j = t + 1 \dots s$ be the regular chains such that $\mathcal{Q}_j = \text{false}$. By Lemma 3, each R_i is a regular semi-algebraic system. For $i = 1 \dots s$, define $F_i = \text{sat}(T_i)$. Then we have $V(F) = \cup_{i=1}^s V(F_i)$, where each F_i is equidimensional. For each $i = 1 \dots s$, by Proposition 1, we have

$$V(F_i) \setminus V(bp_i) = V(T_i) \setminus V(bp_i).$$

Moreover, we have

$$V(F_i) = (V(F_i) \setminus V(bp_i)) \cup V(F_i \cup \{bp_i\}).$$

Hence,

$$Z_{\mathbb{R}}(R_i) = Z_{\mathbb{R}}(T_i) \setminus Z_{\mathbb{R}}(bp_i) \subseteq Z_{\mathbb{R}}(F_i) \subseteq Z_{\mathbb{R}}(F)$$

holds. In addition, since bp_i is regular modulo F_i , we have

$$\begin{aligned} Z_{\mathbb{R}}(F) \setminus \cup_{i=1}^t Z_{\mathbb{R}}(R_i) &= \cup_{i=1}^s Z_{\mathbb{R}}(F_i) \setminus \cup_{i=1}^t Z_{\mathbb{R}}(R_i) \\ &\subseteq \cup_{i=1}^s Z_{\mathbb{R}}(F_i) \setminus (Z_{\mathbb{R}}(T_i) \setminus Z_{\mathbb{R}}(bp_i)) \\ &\subseteq \cup_{i=1}^s Z_{\mathbb{R}}(F_i \cup \{bp_i\}), \end{aligned}$$

and $\dim(\cup_{i=1}^s V(F_i \cup \{bp_i\})) < \dim(V(F))$. So the R_i , for $i = 1 \dots t$, form a lazy triangular decomposition of \mathfrak{S} . \square

In this section, under some genericity assumptions for F , we establish running time estimates for Algorithm 1, see Proposition 3. This is achieved through:

- (1) Proposition 2 giving running time and output size estimates for a Kalkbrener triangular decomposition of an algebraic set, and
- (2) Theorem 2 giving running time and output size estimates for a border polynomial computation.

Our assumptions for these results are the following:

- (H₀) $V(F)$ is equidimensional of dimension d ,
- (H₁) x_1, \dots, x_d are algebraically independent modulo each associated prime ideal of the ideal generated by F in $\mathbb{Q}[\mathbf{x}]$,

- (H₂) F consists of $m := n - d$ polynomials, f_1, \dots, f_m .

Hypotheses (H₀) and (H₁) are equivalent to the existence of regular chains T_1, \dots, T_e of $\mathbb{Q}[x_1, \dots, x_n]$ such that x_1, \dots, x_d are free w.r.t. each of $\overline{T_1}, \dots, \overline{T_e}$ and such that we have $V(F) = \overline{W(T_1)} \cup \dots \cup \overline{W(T_e)}$.

Denote by δ, h respectively the maximum total degree and height of f_1, \dots, f_m . In her PhD Thesis [22], Á. Szántó describes an algorithm which computes a Kalkbrener triangular decomposition, T_1, \dots, T_e , of $V(F)$. Under Hypotheses (H₀) to (H₂), this algorithm runs in time $m^{O(1)}(\delta^{O(n^2)})^{d+1}$ counting operations in \mathbb{Q} , while the total degrees of the polynomials in the output are bounded by $n\delta^{O(m^2)}$. In addition,

T_1, \dots, T_e are square free, *strongly normalized* [18] and *reduced* [1].

From T_1, \dots, T_e , we obtain regular chains E_1, \dots, E_e forming another Kalkbrener triangular decomposition of $V(F)$, as follows. Let $i = 1 \cdots e$ and $j = (d+1) \cdots n$. Let $t_{i,j}$ be the polynomial of T_i with x_j as main variable. Let $e_{i,j}$ be the primitive part of $t_{i,j}$ regarded as a polynomial in $\mathbb{Q}[x_1, \dots, x_d][x_{d+1}, \dots, x_n]$. Define $E_i = \{e_{i,d+1}, \dots, e_{i,n}\}$. According to the complexity results for polynomial operations stated at the beginning of this section, this transformation can be done within $\delta^{O(m^4)O(n)}$ operations in \mathbb{Q} .

Dividing $e_{i,j}$ by its initial we obtain a monic polynomial $d_{i,j}$ of $\mathbb{Q}(x_1, \dots, x_d)[x_{d+1}, \dots, x_n]$. Denote by D_i the regular chain $\{d_{i,d+1}, \dots, d_{i,n}\}$. Observe that D_i is the reduced lexicographic Gröbner basis of the radical ideal it generates in $\mathbb{Q}(x_1, \dots, x_d)[x_{d+1}, \dots, x_n]$. So Theorem 1 in [12] applies to each regular chain D_i . For each polynomial $d_{i,j}$, this theorem provides height and total degree estimates expressed as functions of the *degree* [7] and the *height* [19, 16] of the algebraic set $\overline{W(D_i)}$. Note that the degree and height of $\overline{W(D_i)}$ are upper bounded by those of $V(F)$. Write $d_{i,j} = \sum_{\mu} \frac{\alpha_{\mu}}{\beta_{\mu}} \mu$ where each $\mu \in \mathbb{Q}[x_{d+1}, \dots, x_n]$ is a monomial and $\alpha_{\mu}, \beta_{\mu}$ are in $\mathbb{Q}[x_1, \dots, x_d]$ such that $\gcd(\alpha_{\mu}, \beta_{\mu}) = 1$ holds. Let γ be the lcm of the β_{μ} 's. Then for γ and each α_{μ} :

- the total degree is bounded by $2\delta^{2m}$ and,
- the height by $O(\delta^{2m}(m\bar{h} + dm \log(\delta) + n \log(n)))$.

Multiplying $d_{i,j}$ by γ brings $e_{i,j}$ back. We deduce the height and total degree estimates for each $e_{i,j}$ below.

PROPOSITION 2. *The Kalkbrener triangular decomposition E_1, \dots, E_e of $V(F)$ can be computed in $\delta^{O(m^4)O(n)}$ operations in \mathbb{Q} . In addition, every polynomial $e_{i,j}$ has total degree upper bounded by $4\delta^{2m} + \delta^m$, and has height upper bounded by $O(\delta^{2m}(m\bar{h} + dm \log(\delta) + n \log(n)))$.*

Next we estimate the running time and output size for computing the border polynomial of a regular system.

THEOREM 2. *Let $R = [T, P]$ be a squarefree regular system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$, with $m = \#T$ and $\ell = \#P$. Let bp be the border polynomial of R . Denote by δ_R, \bar{h}_R respectively the maximum total degree and height of a polynomial in R . Then the total degree of bp is upper bounded by $(\ell + m)2^{m-1}\delta_R^m$, and bp can be computed within $(n\ell + nm)^{O(n)}(2\delta_R)^{O(n)O(m)}\bar{h}_R^3$ bit operations.*

PROOF. Define $G := P \cup \{\text{der}(t) \mid t \in T\}$. We need to compute the $\ell + m$ iterated resultants $\text{res}(g, T)$, for all $g \in G$. Let $g \in G$. Observe that the total degree and height of g are bounded by δ_R and $\bar{h}_R + \log(\delta_R)$ respectively. Define $r_{m+1} := g, \dots, r_i := \text{res}(t_i, r_{i+1}, y_i), \dots, r_1 := \text{res}(t_1, r_2, y_1)$. Let $i \in \{1, \dots, m\}$. Denote by δ_i and \bar{h}_i the total degree and height of r_i , respectively. Using the complexity estimates stated at the beginning of this section, we have $\delta_i \leq 2^{m-i+1}\delta_R^{m-i+2}$ and $\bar{h}_i \leq 2\delta_{i+1}(\bar{h}_{i+1} + n \log(\delta_{i+1} + 1))$. Therefore, we have $\bar{h}_i \leq (2\delta_R)^{O(m^2)}n^{O(m)}\bar{h}_R$. From these size estimates, one can deduce that each resultant r_i (thus the iterated resultants) can be computed within $(2\delta_R)^{O(mn)+O(m^2)}n^{O(m)}\bar{h}_R^2$ bit operations, by the complexity of computing a determinant stated at the beginning of this section.

Hence, the product of all iterated resultants has total degree and height bounded by $(\ell + m)2^{m-1}\delta_R^m$ and $(\ell + m)(2\delta_R)^{O(m^2)}n^{O(m)}\bar{h}_R$, respectively. Thus, the primitive

and squarefree part of this product can be computed within $(n\ell + nm)^{O(n)}(2\delta_R)^{O(n)O(m)}\bar{h}_R^3$ bit operations, based on the complexity of a polynomial gcd computation stated at the beginning of this section. \square

PROPOSITION 3. *From the Kalkbrener triangular decomposition E_1, \dots, E_e of Proposition 2, a lazy triangular decomposition of $f_1 = \dots = f_m = 0$ can be computed in $(\delta^{n^2}n^{4^n})^{O(n^2)}\bar{h}^{O(1)}$ bit operations. Thus, a lazy triangular decomposition of this system is computed from the input polynomials in singly exponential time w.r.t. n , counting operations in \mathbb{Q} .*

PROOF. For each $i \in \{1 \cdots e\}$, let bp_i be the border polynomial of $[E_i, \emptyset]$ and let \bar{h}_{R_i} (resp. δ_{R_i}) be the height (resp. the total degree) bound of the polynomials in the pre-regular semi-algebraic system $R_i = [\{bp_i\}_{\neq}, E_i, \emptyset]$. According to Algorithm 1, the remaining task is to solve the QE problem $\exists \mathbf{y}(bp_i(\mathbf{u}) \neq 0, E_i(\mathbf{u}, \mathbf{y}) = 0)$ for each $i \in \{1 \cdots e\}$, which can be solved within $((m+1)\delta_{R_i})^{O(dm)}\bar{h}_{R_i}^{O(1)}$ bit operations, based on the results of [20]. The conclusion follows from the size estimates in Proposition 2 and Theorem 2. \square

5. QUANTIFIER ELIMINATION BY RRC

In the last two sections, we saw that in order to compute a triangular decomposition of a semi-algebraic system, a key step is to solve the following quantifier elimination problem:

$$\exists \mathbf{y}(B(\mathbf{u}) \neq 0, T(\mathbf{u}, \mathbf{y}) = 0, P(\mathbf{u}, \mathbf{y}) > 0), \quad (1)$$

where $[B_{\neq}, T, P_{>}]$ is a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. This problem is an instance of the so-called *real root classification* (RRC) [27]. In this section, we show how to solve this problem when B is what we call a *fingerprint polynomial set*.

Fingerprint polynomial set. Let $R := [B_{\neq}, T, P_{>}]$ be a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Let $D \subset \mathbb{Q}[\mathbf{u}]$. Let dp be the product of all polynomials in D . We call D a *fingerprint polynomial set* (FPS) of R if:

- (i) for all $\alpha \in \mathbb{R}^d$, for all $b \in B$ we have:
 $dp(\alpha) \neq 0 \implies b(\alpha) \neq 0$,
- (ii) for all $\alpha, \beta \in \mathbb{R}^d$ with $\alpha \neq \beta$ and $dp(\alpha) \neq 0, dp(\beta) \neq 0$, if the signs of $p(\alpha)$ and $p(\beta)$ are the same for all $p \in D$, then $R(\alpha)$ has real solutions if and only if $R(\beta)$ does.

Hereafter, we present a method to construct an FPS based on projection operators of CAD.

Open projection operator [21, 4]. Hereafter in this section, let $\mathbf{u} = u_1 < \dots < u_d$ be ordered variables. Let $p \in \mathbb{Q}[\mathbf{u}]$ be non-constant. Denote by $\text{factor}(p)$ the set of the non-constant irreducible factors of p . For $A \subset \mathbb{Q}[\mathbf{u}]$, define $\text{factor}(A) = \cup_{p \in A} \text{factor}(p)$. Let C_d (resp. C_0) be the set of the polynomials in $\text{factor}(p)$ with main variable equal to (resp. less than) u_d . The *open projection operator* (oproj) w.r.t. variable u_d maps p to a set of polynomials of $\mathbb{Q}[u_1, \dots, u_{d-1}]$ defined below:

$$\begin{aligned} \text{oproj}(p, u_d) := & C_0 \cup \bigcup_{f, g \in C_d, f \neq g} \text{factor}(\text{res}(f, g, u_d)) \\ & \cup \bigcup_{f \in C_d} \text{factor}(\text{init}(f, u_d) \cdot \text{discrim}(f, u_d)). \end{aligned}$$

Then, we define $\text{oproj}(A, u_d) := \text{oproj}(\prod_{p \in A} p, u_d)$.

Augmentation. Let $A \subset \mathbb{Q}[\mathbf{u}]$ and $x \in \{u_1, \dots, u_d\}$. Denote by $\text{der}(A, x)$ the *derivative closure* of A w.r.t. x , that is, $\text{der}(A, x) := \cup_{p \in A} \{\text{der}^{(i)}(p, x) \mid 0 \leq i < \deg(p, x)\}$. The

open augmented projected factors of A is denoted by $\text{oaf}(A)$ and defined as follows. Let k be the smallest positive integer such that $A \subset \mathbb{Q}[u_1, \dots, u_k]$ holds. Denote by C the set $\text{factor}(\text{der}(A, u_k))$; we have

- if $k = 1$, then $\text{oaf}(A) := C$;
- if $k > 1$, then $\text{oaf}(A) := C \cup \text{oaf}(\text{oproj}(C, u_k))$.

THEOREM 3. *Let $A \subset \mathbb{Q}[\mathbf{u}]$ be finite and let σ be a map from $\text{oaf}(A)$ to the set of signs $\{-1, +1\}$. Then the set $S_d := \bigcap_{p \in \text{oaf}(A)} \{u \in \mathbb{R}^d \mid p(u)\sigma(p) > 0\}$ is either empty or a connected open set in \mathbb{R}^d .*

PROOF. By induction on d . When $d = 1$, the conclusion follows from Thom's Lemma [2]. Assume $d > 1$. If d is not the smallest positive integer k such that $A \subset \mathbb{Q}[u_1, \dots, u_k]$ holds, then S_d can be written $S_{d-1} \times \mathbb{R}$ and the conclusion follows by induction. Otherwise, write $\text{oaf}(A)$ as $C \cup E$, where $C = \text{factor}(\text{der}(A, u_d))$ and $E = \text{oaf}(\text{oproj}(C, u_d))$. We have: $E \subset \mathbb{Q}[u_1, \dots, u_{d-1}]$. Denote by M the set $\bigcap_{p \in E} \{u \in \mathbb{R}^{d-1} \mid p(u)\sigma(p) > 0\}$. If M is empty then so is S_d and the conclusion is clear. From now on assume M not empty. Then, by induction hypothesis, M is a connected open set in \mathbb{R}^{d-1} . By the definition of the operator oproj , the product of the polynomials in C is delineable over M w.r.t. u_d . Moreover, C is derivative closed (may be empty) w.r.t. u_d . Therefore $\bigcap_{p \in \text{oaf}(A)} \{u \in \mathbb{R}^d \mid p(u)\sigma(p) > 0\} \subset M \times \mathbb{R}$ is either empty or a connected open set by Thom's Lemma. \square

THEOREM 4. *Let $R := [B_{\neq}, T, P_{>}]$ be a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. The polynomial set $\text{oaf}(B)$ is a fingerprint polynomial set of R .*

PROOF. Recall that the border polynomial bp of $[T, P]$ divides the product of the polynomials in B . We have $\text{factor}(B) \subseteq \text{oaf}(B)$. So $\text{oaf}(B)$ satisfies (i) in the definition of FPS. Let us prove (ii). Let dp be the product of the polynomials in $\text{oaf}(B)$. Let $\alpha, \beta \in \mathbb{R}^d$ such that both $dp(\alpha) \neq 0$, $dp(\beta) \neq 0$ hold and the signs of $p(\alpha)$ and $p(\beta)$ are equal for all $p \in \text{oaf}(B)$. Then, by Theorem 3, α and β belong to the same connected component of $dp(\mathbf{u}) \neq 0$, and thus to the same connected component of $B(\mathbf{u}) \neq 0$. Therefore the number of real solutions of $R(\alpha)$ and that of $R(\beta)$ are the same by Lemma 2. \square

From now on, let us assume that the set B in the pre-regular semi-algebraic system $R = [B_{\neq}, T, P_{>}]$ is an FPS of R . We solve the quantifier elimination problem (1) in three steps: (s_1) compute at least one sample point in each connected component of the semi-algebraic set defined by $B(\mathbf{u}) \neq 0$; (s_2) for each sample point α such that the specialized system $R(\alpha)$ possesses real solutions, compute the sign of $b(\alpha)$ for each $b \in B$; (s_3) generate the corresponding quantifier-free formulas.

In practice, when the set B is not an FPS, one adds some polynomials from $\text{oaf}(B)$, using a heuristic procedure (for instance one by one) until Property (ii) of the definition of an FPS is satisfied. This strategy is implemented in Algorithm 3 of Section 6.

6. IMPLEMENTATION

In this section, we present algorithms for `LazyRealTriangularize` and `RealTriangularize` that we have implemented on top of the `RegularChains` library in MAPLE. We also provide experimental results for test problems which are available at www.orcca.on.ca/~cchen/issac10.txt.

Algorithm 2: GeneratePreRegularSas(\mathfrak{S})

Input: a semi-algebraic system $\mathfrak{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$
Output: a set of pre-regular semi-algebraic systems $[B_{i \neq}, T_i, P_{i >}]$, $i = 1 \dots e$, such that $Z_{\mathbb{R}}(\mathfrak{S}) = \bigcup_{i=1}^e Z_{\mathbb{R}}(B_{i \neq}, T_i, P_{i >}) \cup \{\prod_{b \in B_i} b\}, N_{\geq}, P_{>}, H_{\neq}$.

```

1  $\mathfrak{T} := \text{Triangularize}(F)$ ;  $\mathfrak{T}' := \emptyset$ 
2 for  $p \in P \cup H$  do
3   for  $T \in \mathfrak{T}$  do
4     for  $C \in \text{Regularize}(p, T)$  do
5       if  $p \notin \text{sat}(C)$  then  $\mathfrak{T}' := \mathfrak{T}' \cup \{C\}$ 
6    $\mathfrak{T} := \mathfrak{T}'$ ;  $\mathfrak{T}' := \emptyset$ 
7  $\mathfrak{T} := \{[T, \emptyset] \mid T \in \mathfrak{T}\}$ ;  $\mathfrak{T}' := \emptyset$ 
8 for  $p \in N$  do
9   for  $[T, N'] \in \mathfrak{T}$  do
10    for  $C \in \text{Regularize}(p, T)$  do
11      if  $p \in \text{sat}(C)$  then
12         $\mathfrak{T}' := \mathfrak{T}' \cup \{[C, N']\}$ 
13      else
14         $\mathfrak{T}' := \mathfrak{T}' \cup \{[C, N' \cup \{p\}]\}$ 
15     $\mathfrak{T} := \mathfrak{T}'$ ;  $\mathfrak{T}' := \emptyset$ 
16  $\mathfrak{T} := \{[T, N', P, H] \mid [T, N'] \in \mathfrak{T}\}$ 
17 for  $[T, N', P, H] \in \mathfrak{T}$  do
18    $B := \text{BorderPolynomialSet}(T, N' \cup P \cup H)$ 
19   output  $[B, T, N' \cup P]$ 

```

Algorithm 3: GenerateRegularSas(B, T, P)

Input: $\mathfrak{S} = [B_{\neq}, T, P_{>}]$, a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$, where $\mathbf{u} = u_1, \dots, u_d$ and $\mathbf{y} = y_1, \dots, y_{n-d}$.
Output: A pair (D, \mathcal{R}) satisfying:
(1) $D \subset \mathbb{Q}[\mathbf{u}]$ such that $\text{factor}(B) \subseteq D$;
(2) \mathcal{R} is a finite set of regular semi-algebraic systems, s.t. $\bigcup_{R \in \mathcal{R}} Z_{\mathbb{R}}(R) = Z_{\mathbb{R}}(D_{\neq}, T, P_{>})$.

```

1  $D := \text{factor}(B \setminus \mathbb{Q})$ 
2 if  $d = 0$  then
3   if  $\text{RealRootCounting}(T, P) = 0$  then
4     return  $(D, \emptyset)$ 
5   else
6     return  $(D, \{\{true, T, P\}\})$ 
7 while true do
8    $S := \text{SamplePoints}(D, d)$ ;  $G_0 := \emptyset$ ;  $G_1 := \emptyset$ 
9   for  $s \in S$  do
10    if  $\text{RealRootCounting}(T(s), P(s)) = 0$  then
11       $G_0 := G_0 \cup \{\text{GenerateFormula}(D, s)\}$ 
12    else
13       $G_1 := G_1 \cup \{\text{GenerateFormula}(D, s)\}$ 
14  if  $G_0 \cap G_1 = \emptyset$  then
15     $Q := \text{Disjunction}(G_1)$ 
16    if  $Q = false$  then return  $(D, \emptyset)$ 
17    else return  $(D, \{\{Q, T, P\}\})$ 
18  else
19    select a subset  $D' \subseteq \text{oaf}(B) \setminus D$  by some heuristic method
20     $D := D \cup D'$ 

```

Basic subroutines. For a zero-dimensional squarefree regular system $[T, P]$, $\text{RealRootCounting}(T, P)$ [23] returns the number of real zeros of $[T, P_{>}]$. For $A \subset \mathbb{Q}[u_1, \dots, u_d]$ and a point s of \mathbb{Q}^d such that $p(s) \neq 0$ for all $p \in A$, $\text{GenerateFormula}(A, s)$ computes a formula $\bigwedge_{p \in A} (p \sigma_{p,s} > 0)$, where $\sigma_{p,s}$ is defined as $+1$ if $p(s) > 0$ and -1 otherwise. For a set of formulas G , $\text{Disjunction}(G)$ computes a logic formula Φ equivalent to the disjunction of the formulas in G .

Proof of Algorithm 2. Its termination is obvious. Let us prove its correctness. By the specification of Triangularize and Regularize , at line 16, we have

$$Z(F, P_{\neq} \cup H_{\neq}) = \cup_{[T, N', P, H] \in \mathfrak{T}} Z(\text{sat}(T), P_{\neq} \cup H_{\neq}).$$

Write $\cup_{[T, N', P, H] \in \mathfrak{T}}$ as \cup_T . Then we deduce that

$$Z_{\mathbb{R}}(F, N_{\geq}, P_{>}, H_{\neq}) = \cup_T Z_{\mathbb{R}}(\text{sat}(T), N_{\geq}, P_{>}, H_{\neq}).$$

For each $[T, N', P, H]$, at line 19, we generate a pre-regular semi-algebraic system $[B_{\neq}, T, N'_{>} \cup P_{>}]$. By Proposition 1, we have

$$\begin{aligned} & Z_{\mathbb{R}}(\text{sat}(T), N_{\geq}, P_{>}, H_{\neq}) = \\ & Z_{\mathbb{R}}(B_{\neq}, T, N'_{>} \cup P_{>}) \cup Z_{\mathbb{R}}(\text{sat}(T) \cup \{\prod_{b \in B} b\}, N_{\geq}, P_{>}, H_{\neq}), \end{aligned}$$

which implies that

$$\begin{aligned} Z_{\mathbb{R}}(\mathfrak{S}) &= \cup_T Z_{\mathbb{R}}(B_{\neq}, T, N'_{>} \cup P_{>}) \\ &\quad \cup_T Z_{\mathbb{R}}(\text{sat}(T) \cup \{\prod_{b \in B} b\}, N_{\geq}, P_{>}, H_{\neq}). \end{aligned}$$

So Algorithm 2 satisfies its specification.

Algorithm 4: SamplePoints(A, k)

Input: $A \subset \mathbb{Q}[x_1, \dots, x_k]$ is a finite set of non-zero polynomials

Output: A finite subset of \mathbb{Q}^k contained in $(\prod_{p \in A} p) \neq 0$ and having a non-empty intersection with each connected component of $(\prod_{p \in A} p) \neq 0$.

```

1 if  $k = 1$  then
2   return one rational point from each connected
   component of  $\prod_{p \in A} p \neq 0$ 
3 else
4    $A_k := \{p \in A \mid \text{mvar}(p) = x_k\}$ ;  $A' := \text{oproj}(A, x_k)$ 
5   for  $s$  in  $\text{SamplePoints}(A', k - 1)$  do
6     Collect in a set  $S$  one rational point from each
     connected component of  $\prod_{p \in A_k} p(s, x_k) \neq 0$ ;
7     for  $\alpha \in S$  do output  $(s, \alpha)$ 

```

Algorithm 5: LazyRealTriangularize(\mathfrak{S})

Input: a semi-algebraic system $\mathfrak{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$

Output: a lazy triangular decomposition of \mathfrak{S}

```

1  $\mathfrak{T} := \text{GeneratePreRegularSas}(F, N, P, H)$ 
2 for  $[B, T, P'] \in \mathfrak{T}$  do
3    $(D, \mathcal{R}) = \text{GenerateRegularSas}(B, T, P')$ 
4   if  $\mathcal{R} \neq \emptyset$  then output  $\mathcal{R}$ 

```

Proof of Algorithms 3 and 4. By the definition of oproj , Algorithm 4 terminates and satisfies its specification. By Theorem 4, $\text{oaf}(B)$ is an FPS. Thus, by the definition of an FPS, Algorithm 3 terminates and satisfies its specification.

Proof of Algorithm 5. Its termination is obvious. Let us prove the algorithm is correct. Let R_i , $i = 1 \dots t$ be the

Algorithm 6: RealTriangularize(\mathfrak{S})

Input: a semi-algebraic system $\mathfrak{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$

Output: a triangular decomposition of \mathfrak{S}

```

1  $\mathfrak{T} := \text{GeneratePreRegularSas}(F, N, P, H)$ 
2 for  $[B, T, P'] \in \mathfrak{T}$  do
3    $(D, \mathcal{R}) = \text{GenerateRegularSas}(B, T, P')$ 
4   if  $\mathcal{R} \neq \emptyset$  then output  $\mathcal{R}$ 
5   for  $p \in D$  do
6     output  $\text{RealTriangularize}(F \cup \{p\}, N, P, H)$ 

```

output. By the specification of each sub-algorithm, each R_i is a regular semi-algebraic system and we have:

$$\cup_{i=1}^t Z_{\mathbb{R}}(R_i) \subseteq Z_{\mathbb{R}}(\mathfrak{S}).$$

Next we show that there exists an ideal $\mathcal{I} \subseteq \mathbb{Q}[\mathbf{x}]$, whose dimension is less than $\dim(Z(F, P_{\neq} \cup H_{\neq}))$ and such that $Z_{\mathbb{R}}(\mathfrak{S}) \setminus \cup_{i=1}^t Z_{\mathbb{R}}(R_i) \subseteq Z_{\mathbb{R}}(\mathcal{I})$ holds.

At line 1, by the specification of Algorithm 2, we have

$$\begin{aligned} Z_{\mathbb{R}}(\mathfrak{S}) &= \cup_T Z_{\mathbb{R}}(B_{\neq}, T, P'_{>}) \\ &\quad \cup_T Z_{\mathbb{R}}(\text{sat}(T) \cup \{\prod_{b \in B} b\}, N_{\geq}, P_{>}, H_{\neq}). \end{aligned}$$

At line 3, by the specification of Algorithm 3, for each B , we compute a set D such that $\text{factor}(B) \subseteq D$ and

$$\cup_T Z_{\mathbb{R}}(D_{\neq}, T, P'_{>}) = \cup_{i=1}^t Z_{\mathbb{R}}(R_i)$$

both hold. Combining the two relations together, we have

$$\begin{aligned} Z_{\mathbb{R}}(\mathfrak{S}) &= \cup_T Z_{\mathbb{R}}(R_i) \\ &\quad \cup_T Z_{\mathbb{R}}(\text{sat}(T) \cup \{\prod_{p \in D} p\}, N_{\geq}, P_{>}, H_{\neq}). \end{aligned}$$

Therefore, the following relations hold

$$\begin{aligned} & Z_{\mathbb{R}}(\mathfrak{S}) \setminus \cup_{i=1}^t Z_{\mathbb{R}}(R_i) \\ & \subseteq \cup_T Z_{\mathbb{R}}(\text{sat}(T) \cup \{\prod_{p \in D} p\}, N_{\geq}, P_{>}, H_{\neq}) \\ & \subseteq Z_{\mathbb{R}}(\cap_T (\text{sat}(T) \cup \{\prod_{p \in D} p\})). \end{aligned}$$

Define

$$\mathcal{I} = \cap_T (\text{sat}(T) \cup \{\prod_{p \in D} p\}).$$

Since each $p \in D$ is regular modulo $\text{sat}(T)$, we have

$$\dim(\mathcal{I}) < \dim(\cap_T \text{sat}(T)) \leq \dim(Z(F, P_{\neq} \cup H_{\neq})).$$

So all R_i form a lazy triangular decomposition of \mathfrak{S} . \square

Proof of Algorithm 6. For its termination, it is sufficient to prove that there are only finitely many recursive calls to RealTriangularize . Indeed, if $[F, N, P, H]$ is the input of a call to RealTriangularize then each of the immediate recursive calls takes $[F \cup \{p\}, N, P, H]$ as input, where p belongs to the set D of some pre-regular semi-algebraic system $[D_{\neq}, T, P_{>}]$. Since p is regular (and non-zero) modulo $\text{sat}(T)$ we have:

$$\langle F \rangle \subsetneq \langle F \cup \{p\} \rangle.$$

Therefore, the algorithm terminates by the ascending chain condition on ideals of $\mathbb{Q}[\mathbf{x}]$. The correctness of Algorithm 6 follows from the specifications of the sub-algorithms. \square

Table 1. Table 1 summarizes the notations used in Tables 2 and 3. Tables 2 and 3 demonstrate benchmarks running in MAPLE 14 β 1, using an Intel Core 2 Quad CPU (2.40GHz) with 3.0GB memory. The timings are in seconds and the time-out is 1 hour.

Table 2. The systems in this group involve equations only. We report the running times for a triangular decomposition

Table 1 Notations for Tables 2 and 3

symbol	meaning
#e	number of equations in the input system
#v	number of variables in the input equations
d	maximum total degree of an input equation
G	Groebner-Basis (plex order) in MAPLE
T	Triangularize in RegularChains library of MAPLE
LR	LazyRealTriangularize implemented in MAPLE
R	RealTriangularize implemented in MAPLE
Q	QEPCAD B
> 1h	computation does not complete within 1 hour
FAIL	QEPCAD B failed due to prime list exhausted

Table 2 Timings for varieties

system	#v/#e/d	G	T	LR
Hairer-2-BGK	13/ 11/ 4	25	1.924	2.396
Collins-jsc02	5/ 4/ 3	876	0.296	0.820
Leykin-1	8/ 6/ 4	103	3.684	3.924
8-3-config-Li	12/ 7/ 2	109	5.440	6.360
Lichtblau	3/ 2/ 11	126	1.548	11
Cinquin-3-3	4/ 3/ 4	64	0.744	2.016
Cinquin-3-4	4/ 3/ 5	> 1h	10	22
DonatiTraverso-rev	4/ 3/ 8	154	7.100	7.548
Cheaters-homotopy-1	7/ 3/ 7	3527	174	> 1h
hereman-8.8	8/ 6/ 6	> 1h	33	62
L	12/ 4/ 3	> 1h	0.468	0.676
dgp6	17/19/ 2	27	60	63
dgp29	5/ 4/ 15	84	0.008	0.016

of the input algebraic variety and a lazy triangular decomposition of the corresponding real variety. These illustrate the good performance of our tool.

Table 3. The examples in this table are quantifier elimination problems and most of them involve both equations and inequalities. We provide the timings for computing a lazy and a full triangular decomposition of the corresponding semi-algebraic system and the timings for solving the quantifier elimination problem via QEPCAD B [5] (in non-interactive mode). Computations complete with our tool on more examples than with QEPCAD B.

Remark. The output of our tools is a set of regular semi-algebraic systems, which is different than that of QEPCAD B. We note also that our tool is more effective for systems with more equations than inequalities.

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Table 3 Timings for semi-algebraic systems

system	#v/#e/d	T	LR	R	Q
BM05-1	4/ 2/ 3	0.008	0.208	0.568	86
BM05-2	4/ 2/ 4	0.040	2.284	> 1h	FAIL
Solotareff-4b	5/ 4/ 3	0.640	2.248	924	> 1h
Solotareff-4a	5/ 4/ 3	0.424	1.228	8.216	FAIL
putnam	6/ 4/ 2	0.044	0.108	0.948	> 1h
MPV89	6/ 3/ 4	0.016	0.496	2.544	> 1h
IBVP	8/ 5/ 2	0.272	0.560	12	> 1h
Lafferriere37	3/ 3/ 4	0.056	0.184	0.180	10
Xia	6/ 3/ 4	0.164	191	739	> 1h
SEIT	11/ 4/ 3	0.400	> 1h	> 1h	> 1h
p3p-isosceles	7/ 3/ 3	1.348	> 1h	> 1h	> 1h
p3p	8/ 3/ 3	210	> 1h	> 1h	FAIL
Ellipse	6/ 1/ 3	0.012	> 1h	> 1h	> 1h

7. REFERENCES

- [1] P. Aubry, D. Lazard, and M. Moreno Maza. On the theories of triangular sets. *J. Symb. Comput.*, 28(1-2):105–124, 1999.
- [2] S. Basu, R. Pollack, and M-F. Roy. *Algorithms in real algebraic geometry*. Springer-Verlag, 2006.
- [3] F. Boulier, C. Chen, F. Lemaire, and M. Moreno Maza. Real root isolation of regular chains. In *Proc. ASCM'09*.
- [4] C. W. Brown. Improved projection for cylindrical algebraic decomposition. *J. Symb. Comput.*, 32(5):447–465, 2001.
- [5] C. W. Brown. QEPCAD B: a program for computing with semi-algebraic sets using cads. *SIGSAM Bull.*, 37(4):97–108, 2003.
- [6] C. W. Brown and S. McCallum. On using bi-equational constraints in cad construction. In *ISSAC'05*, pages 76–83, 2005.
- [7] P. Bürgisser, M. Clausen, and M. A. Shokrollahi. *Algebraic Complexity Theory*. Springer, 1997.
- [8] C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, and W. Pan. Comprehensive triangular decomposition. In *CASC'07*, pages 73–101, 2007.
- [9] C. Chen, M. Moreno Maza, B. Xia, and L. Yang. Computing cylindrical algebraic decomposition via triangular decomposition. In *ISSAC'09*, pages 95–102.
- [10] J.S. Cheng, X.S. Gao, and C.K. Yap. Complete numerical isolation of real zeros in zero-dimensional triangular systems. In *ISSAC '07*, pages 92–99, 2007.
- [11] G. E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *Springer Lecture Notes in Computer Science*, 33:515–532, 1975.
- [12] X. Dahan, A. Kadri, and É. Schost. Bit-size estimates for triangular sets in positive dimension. Technical report, University of Western Ontario, 2009.
- [13] X. Dahan, M. Moreno Maza, É. Schost, W. Wu, and Y. Xie. Lifting techniques for triangular decompositions. In *ISSAC'05*, pages 108–115, 2005.
- [14] J.H. Davenport, Y. Siret, and E. Tournier. *Computer Algebra*. Academic Press, 1988.
- [15] H. Hong and J. R. Sendra. Computation of variant results, B. Caviness and J. Johnson, eds, *Quantifier Elimination and Cylindrical Algebraic Decomposition*, 1998.
- [16] T. Krick, L. M. Pardo, and M. Sombra. Sharp estimates for the arithmetic Nullstellensatz. *Duke Math. J.*, 109(3):521–598, 2001.
- [17] X. Li, M. Moreno Maza, and W. Pan. Computations modulo regular chains. In *ISSAC'09*, pages 239–246, 2009.
- [18] M. Moreno Maza. On triangular decompositions of algebraic varieties. MEGA-2000, Bath, UK. <http://www.csd.uwo.ca/~moreno/books-papers.html>
- [19] P. Philippon. Sur des hauteurs alternatives III. *J. Math. Pures Appl.*, 74(4):345–365, 1995.
- [20] J. Renegar. On the computational complexity and geometry of the first-order theory of the reals. parts I–III. *J. Symb. Comput.*, 13(3):255–352, 1992.
- [21] A. Strzeboński. Solving systems of strict polynomial inequalities. *J. Symb. Comput.*, 29(3):471–480, 2000.
- [22] Á. Szántó. *Computation with polynomial systems*. PhD thesis, Cornell University, 1999.
- [23] B. Xia and T. Zhang. Real solution isolation using interval arithmetic. *Comput. Math. Appl.*, 52(6-7):853–860, 2006.
- [24] R. Xiao. *Parametric Polynomial System Solving*. PhD thesis, Peking University, Beijing, 2009.
- [25] L. Yang, X. Hou, and B. Xia. A complete algorithm for automated discovering of a class of inequality-type theorems. *Science in China, Series F*, 44(6):33–49, 2001.
- [26] L. Yang and B. Xia. *Automated proving and discovering inequalities*. Science Press, Beijing, 2008.
- [27] L. Yang and B. Xia. Real solution classifications of a class of parametric semi-algebraic systems. In *A3L'05*, pages 281–289, 2005.