A Standard Basis Free Algorithm for Computing the Tangent Cones of a Space Curve

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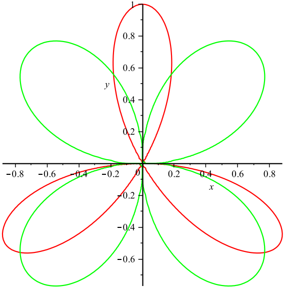
Plan

1. Intersection Multiplicities via Regular Chains
2. Reducing from dim n to dim $n-1$: using transversality
3. Tangent Cone via Limit Computation
4. Limit Points of a Quasi-Component
5. Tangent Cone via Regular Chains

> $Fs := \left[\left(x^2+y^2\right)^2+3x^2y-y^3, \left(x^2+y^2\right)^3-4x^2y^2\right]$:

> plots[implicitplot](Fs,x=-2..2,y=-2..2);



> $R := PolynomialRing\left([x, y], 101\right)$:

> $rcs := Triangularzie\left(Fs, R, normalized = \text{‘}yes\text{‘}\right)$:

> $\text{seq}\left(\text{TriangularizeWithMultiplicity}\left(Fs, T, R\right), T \textbf{ in } rcs\right)$:

$$\left[\left[1, \left\{\begin{array}{c} x-1=0 \\ y+14=0 \end{array}\right.\right]\right], \left[\left[1, \left\{\begin{array}{c} x+1=0 \\ y+14=0 \end{array}\right.\right]\right], \left[\left[1, \left\{\begin{array}{c} x-47=0 \\ y-14=0 \end{array}\right.\right]\right],$$

$$\left[\left[1, \left\{\begin{array}{c} x+47=0 \\ y-14=0 \end{array}\right.\right]\right], \left[\left[14, \left\{\begin{array}{c} x=0 \\ y=0 \end{array}\right.\right]\right]$$

$> Fs := \left[x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\right]:$

$> R := PolynomialRing\,([x, y, z], 101):$

$>$ TriangularizeWithMultiplicity (Fs, R):

$$\left[\left[1, \left\{\begin{array}{c} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{array}\right.\right]\right], \left[\left[2, \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z - 1 = 0 \end{array}\right.\right]\right],$$

$$\left[\left[2, \left\{\begin{array}{c} x = 0 \\ y - 1 = 0 \\ z = 0 \end{array}\right.\right]\right], \left[\left[2, \left\{\begin{array}{c} x - 1 = 0 \\ y = 0 \\ z = 0 \end{array}\right.\right]\right]$$

`TriangularizeWithMultiplicity`

We specify `TriangularizeWithMultiplicity`:

Input $f_1, \ldots, f_n \in \mathbf{k}[x_1, \ldots, x_n]$ such that $V(f_1, \ldots, f_n)$ is zero-dimensional.

Output Finitely many pairs $[(T_1, m_1), \ldots, (T_\ell, m_\ell)]$ where $T_1, \ldots, T_\ell$ are regular chains of $\mathbf{k}[x_1, \ldots, x_n]$ such that for all $p \in V(T_i)$

$$\mathrm{I}(p; f_1, \ldots, f_n) = m_i \text{ and } V(f_1, \ldots, f_n) = V(T_1) \uplus \cdots \uplus V(T_\ell).$$

`TriangularizeWithMultiplicity` works as follows

1. Apply `Triangularize` on $f_1, \ldots, f_n$,
2. Apply $\mathrm{IM}_n(T; f_1, \ldots, f_n)$ on each regular chain T.

$\mathrm{IM}_n(T; f_1, \ldots, f_n)$ works as follows

1. if $n = 2$ apply Fulton's algorithm extended for working at a regular chains instead of a point (S. Marcus, M., P. Vrbik; CASC 2013),
2. if $n > 2$ attempt a reduction from dimension n to $n - 1$.

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Reducing from dim n to dim $n-1$: using transversality (1/2)

Definition

The intersection multiplicity of p in $V(f_1, \ldots, f_n)$ is given by

$$I(p; f_1, \ldots, f_n) := \dim_{\overline{k}} \left(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle\right).$$

where $\mathcal{O}_{\mathbb{A}^n,p}$ and $\dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle)$ are respectively the local ring at the point p and the dimension of the vector space $\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle$.

The next theorem reduces the n-dimensional case to $n-1$, under assumptions which state that f_n does not contribute to $I(p; f_1, \ldots, f_n)$.

Reducing from dim n to dim $n-1$: using transversality (1/2)

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The next theorem reduces the n-dimensional case to $n-1$, under assumptions which state that f_n does not contribute to $I(p; f_1, \ldots, f_n)$.

Theorem 1

Assume that $h_n = V(f_n)$ is non-singular at p. Let v_n be its tangent hyperplane at p. Assume that h_n meets each component (through p) of the curve $\mathcal{C} = V(f_1, \ldots, f_{n-1})$ transversely (that is, the tangent cone $TC_p(\mathcal{C})$ intersects v_n only at the point p). Let $h \in k[x_1, \ldots, x_n]$ be the degree 1 polynomial defining v_n. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h).$$

Reducing from dim n to dim $n-1$: using transversality (2/2)

The theorem again:

Theorem

Assume that $h_n = V(f_n)$ is non-singular at p. Let v_n be its tangent hyperplane at p. Assume that h_n meets each component (through p) of the curve $\mathcal{C} = V(f_1, \ldots, f_{n-1})$ transversely (that is, the tangent cone $TC_p(\mathcal{C})$ intersects v_n only at the point p). Let $h \in k[x_1, \ldots, x_n]$ be the degree 1 polynomial defining v_n. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h).$$

How to use this theorem in practise?

Assume that the coefficient of x_n in h is non-zero, thus $h = x_n - h'$, where $h' \in k[x_1, \ldots, x_{n-1}]$. Hence, we can rewrite the ideal $\langle f_1, \ldots, f_{n-1}, h\rangle$ as $\langle g_1, \ldots, g_{n-1}, h\rangle$ where g_i is obtained from f_i by substituting x_n with h'. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p|_{x_1, \ldots, x_{n-1}}; g_1, \ldots, g_{n-1}).$$

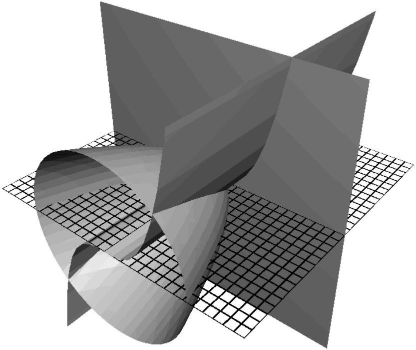
Reducing from dim n to dim $n-1$: a simple case (1/3)

Example

Consider the system

$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$

near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$



Reducing from dim n to dim $n-1$: a simple case (2/3)

Example

Recall the system

$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$

near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$.

Computing the IM using the definition

Let us compute a basis for $\mathcal{O}_{\mathbb{A}^3,o}/\langle f_1, f_2, f_3\rangle$ as a vector space over $\overline{k}$.

Setting $x = 0$ and $y = z^3$, we must have $z^2(z^4+1) = 0$ in $\mathcal{O}_{\mathbb{A}^3,o} = \overline{k}[x,y,z]_{(z,y,z)}$.

Since z^4+1 is a unit in this local ring, we see that

$$\mathcal{O}_{\mathbb{A}^3,o}/\langle f_1, f_2, f_3\rangle = \langle 1, z\rangle$$

as a vector space, so $I(o; f_1, f_2, f_3) = 2$.

Reducing from dim n to dim $n-1$: a simple case (3/3)

Example

Recall the system again
$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$
near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$.

Computing the IM using the reduction

We have
$$\mathcal{C} := V(x, x + y^2 - z^2) = V(x, (y-z)(y+z)) = TC_o(\mathcal{C})$$
and we have
$$h = y.$$
Thus $\mathcal{C}$ and $V(f_3)$ intersect transversally at the origin. Therefore, we have
$$I_3(p; f_1, f_2, f_3) = I_2((0,0); x, x - z^2) = 2.$$

Reducing from dim n to dim $n-1$: via cylindrification (1/3)

In practise, this reduction from n to $n-1$ variables does not always apply. For instance, this is the case for *Ojika 2*:

$$x^2 + y + z - 1 = x + y^2 + z - 1 = x + y + z^2 - 1 = 0.$$

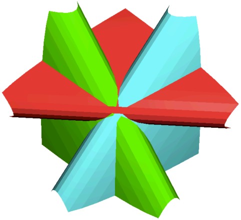


Figure: The real points of $V(x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$.

Reducing from dim n to dim $n-1$: via cylindrification (2/3)

Recall the system

$$x^2 + y + z - 1 = x + y^2 + z - 1 = x + y + z^2 - 1 = 0.$$

If one uses the first equation, that is $x^2 + y + z - 1 = 0$, to eliminate z from the other two, we obtain two bivariate polynomials $f, g \in k[x, y]$.

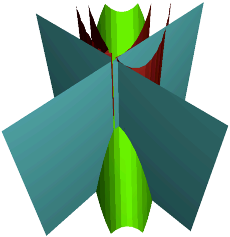


Figure: The real points of
$V(x^2 + y + z - 1, x + y^2 - x^2 - y, x - y + x^4 + 2\,x^2y - 2\,x^2 + y^2)$ near the origin.

Reducing from dim n to dim $n-1$: via cylindrification (3/3)

At any point of $p \in V(h, f, g)$ the tangent cone of the curve $V(f, g)$ is independent of z; in some sense it is "vertical". On the other hand, at any point of $p \in V(h, f, g)$ the tangent space of $V(h)$ is not vertical.

Thus, the previous theorem applies without computing any tangent cones.

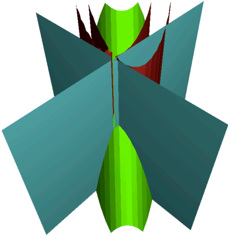


Figure: The real points of
$V(x^2+y+z-1, x+y^2-x^2-y, x-y+x^4+2\,x^2y-2\,x^2+y^2)$ near the origin.

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Tangent cones and tangent spaces

Tangent space

Let $F \subset k[x_1, \dots, x_n]$ and $p \in V(F)$. The tangent space of $V := V(F)$ at p is the algebraic set given by

$$T_p(V) := V(\{\ \mathbf{d}_p(f) : f \in \mathbf{I}(V)\})$$

where $\mathbf{d}_p(f)$ is the linear part of f at p, that is, the affine form $\frac{\partial f}{\partial x_1}(p)(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}(p)(x_n - p_n)$.

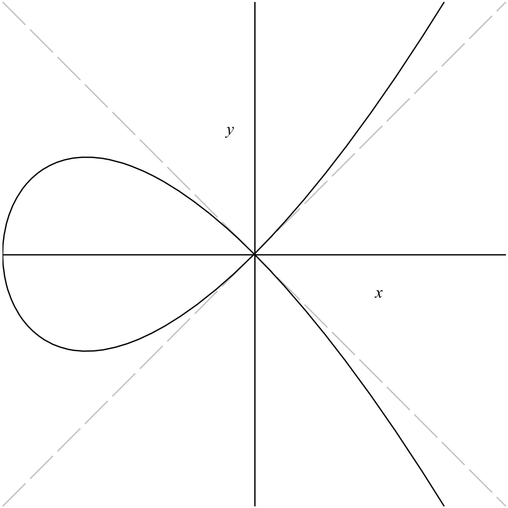
Tangent cone

The tangent cone of $V := V(F)$ at p is the algebraic set given by

$$TC_p(V) = V(\{\mathrm{HC}_p(f; \min) : f \in \mathbf{I}(V))$$

where $\mathrm{HC}_p(f; \min)$ is the homogeneous component of least degree of f in $x - p$. If V is a curve, then $TC_p(V)$ consists of finitely many lines, all intersecting at p.

Tangent cone: a basic example



The tangent cone of $V(h)$ for $h = y^2 - x^2(x+1) \in \mathbb{C}[x, y]$ is $V((y-x)(y+x))$.

Tangent cone computation as limits of secants

Previous works

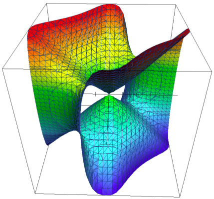
One can compute the ideal $\langle \mathrm{HC}_p(f; \min) : f \in \mathbf{I}(V) \rangle$ by means of standard bases (F. Mora 1982) or Grönber bases (T. Mora, G. Pfister & C. Traverso; 1992).

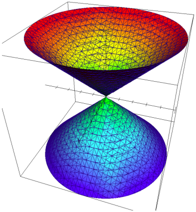
We are going to take a different route and rely on:

Theorem (Chapter 9 in (D. Cox, J. Little, & D. O'Shea; 1992))

A line L through p lies in the tangent cone $TC_p(V)$ if and only if there exists a sequence $(q_k, k \in \mathbb{N})$ of points on $V \setminus \{p\}$ converging to p and such that the secant line L_k containing p and q_k becomes L when q_k approaches p.

Tangent cone computation via tangent spaces





Assume $\overline{k} = \mathbb{C}$ and none of the $V(f_i)$ is singular at p. For each component $\mathcal{G}$ through p of $\mathcal{C} = V(f_1, \ldots, f_{n-1})$,

- There exists a neighborhood B of p such that $V(f_i)$ is not singular at all $q \in (B \cap \mathcal{G}) \setminus \{p\}$, for $i = 1, \ldots, n-1$.
- Let $v_i(q)$ be the tangent hyperplane of $V(f_i)$ at q. Regard $v_1(q) \cap \cdots \cap v_{n-1}(q)$ as a parametric variety with q as parameter.
- Then, $TC_p(\mathcal{G}) = v_1(q) \cap \cdots \cap v_{n-1}(q)$ when q approaches p.
- Finally, $TC_p(\mathcal{C})$ is the union of all $TC_p(\mathcal{G})$. This approach avoids standard basis computation and extends for working with $V(T)$ instead of p.

But how to compute the limit of $v_1(q) \cap \cdots \cap v_{n-1}(q)$ when approaches p?

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Limit points of a quasi-component

Input

- Let $R \subset \mathbb{C}[X_1, \ldots, X_s]$ be a regular chain.
- Let h_R be the product of initials of polynomials of R.
- Let $W(R)$ be the quasi-component of R, that is $V(R) \setminus V(h_R)$.

Desired output

The non-trivial limit points of $W(R)$, that is

$$\lim(W(R)) := \overline{W(R)}^{Z} \setminus W(R).$$

Puiseux expansions of a regular chain

Notation

- Let $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\} \subset \mathbb{C}[X_1 < \cdots < X_s]$ be a 1-dim regular chain.
- Assume R is strongly normalized, that is, $\mathrm{init}(R) \in \mathbb{C}[X_1]$.
- Let $\mathbf{k} = \mathbb{C}(\langle X_1^* \rangle)$.
- Then R generates a zero-dimensional ideal in $\mathbf{k}[X_2, \ldots, X_s]$.
- Let $V^*(R)$ be the zero set of R in $\mathbf{k}^{s-1}$.

Definition

We call *Puiseux expansions* of R the elements of $V^*(R)$.

Remarks

- The *strongly normalized assumption* is only for presentation ease.
- The 1-*dim* assumption is, however, harder to relax.
- One could think of generalizations of Puiseux expanions using Jung-Abhyankar theorem. More on this tomorrow and slater.

An example

A regular chain R

$$R := \begin{cases} X_1 X_3^2 + X_2 \\ X_1 X_2^2 + X_2 + X_1 \end{cases}$$

Puiseux expansions of R

$$\begin{cases} X_3 = 1 + O(X_1^2) \\ X_2 = -X_1 + O(X_1^2) \end{cases} \qquad \begin{cases} X_3 = -1 + O(X_1^2) \\ X_2 = -X_1 + O(X_1^2) \end{cases}$$

$$\begin{cases} X_3 = {X_1}^{-1} - \frac{1}{2} X_1 + O(X_1^2) \\ X_2 = -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases} \qquad \begin{cases} X_3 = -{X_1}^{-1} + \frac{1}{2} X_1 + O(X_1^2) \\ X_2 = -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases}$$

Relation between $\lim_0(W(R))$ and Puiseux expansions of R

Theorem

For $W \subseteq \mathbb{C}^s$, denote

$$\lim_0(W) := \{x = (x_1, \ldots, x_s) \in \mathbb{C}^s \mid x \in \lim(W) \text{ and } x_1 = 0\},$$

and define

$V^*_{\geq 0}(R) := \{\Phi = (\Phi^1, \ldots, \Phi^{s-1}) \in V^*(R) \mid \mathrm{ord}(\Phi^j) \geq 0, j = 1, \ldots, s-1\}.$

Then we have

$$\lim_0(W(R)) = \cup_{\Phi \in V^*_{\geq 0}(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

$$V^*_{\geq 0}(R) := \left\{ \begin{array}{lcl} X_3 & = & 1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right. \cup \left\{ \begin{array}{lcl} X_3 & = & -1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right.$$

Thus the limit ponts are $\lim_0(W(R)) = \{(0, 0, 1), (0, 0, -1)\}$.

Limit points of a quasi-component

```
> with(AlgebraicGeometryTools):
> R := PolynomialRing([x, y, t]);
> F := [t*y^2 + y + 1, (t + 2)*t*x^2 + (y +1)* (x + 1)];
> C := Chain(F, Empty(R), R);
> lm := LimitPoints(C, R, false, true);
> Display(lm, R);
```

$$R := polynomial\_ring$$

$$F := \left[t\,y^2 + y + 1,\ (t+2)\,t\,x^2 + (y+1)\,(x+1)\right]$$

$$C := regular\_chain$$

$$lm := [regular\_chain, regular\_chain, regular\_chain, regular\_chain]$$

$$\left[\begin{cases} x + 1 = 0 \\ y + \frac{1}{2} = 0 \\ t + 2 = 0 \end{cases},\ \begin{cases} x + 1 = 0 \\ y - 1 = 0 \\ t + 2 = 0 \end{cases},\ \begin{cases} x + \frac{1}{2} = 0 \\ y + 1 = 0 \\ t = 0 \end{cases},\ \begin{cases} x - 1 = 0 \\ y + 1 = 0 \\ t = 0 \end{cases}\right]$$

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Tangent cone computation with regular chains (1/2)

Algorithm principle

- Recall $\langle f_1, \ldots, f_{n-1}, f_n \rangle$ is zero-dimensional.
- We want $TC_p(\mathcal{C})$ for $p \in V(f_1, \ldots, f_{n-1}, f_n)$ and $\mathcal{C} := V(f_1, \ldots, f_{n-1})$.
- Let $m(x_1, \ldots, x_n)$ be a point on the curve $\mathcal{C}$.
- Let $\vec{u}$ be a unit vector directing the line (pm).
- The set $\{\lim_{m \to p, m \neq p} \ \vec{u}\}$ describes $TC_p(\mathcal{C})$.

Step 1

- Let T be a 0-dim regular chain defining the point p; rename its variables to $y_1, \ldots, y_n$.

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Step 1

- Let T be a 0-dim regular chain defining the point p; rename its variables to $y_1, \ldots, y_n$.
- Consider the polynomial system (S) defined by T and $f_1 = \cdots = f_{n-1} = 0$.

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- Let T be a 0-dim regular chain defining the point p; rename its variables to $y_1, \ldots, y_n$.
- Consider the polynomial system (S) defined by T and $f_1 = \cdots = f_{n-1} = 0$.
- This is a 1-dim system in the variables $y_1, \ldots, y_n, x_1, \ldots, x_n$.

Tangent cone computation with regular chains (1/2)

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- We want $TC_p(\mathcal{C})$ for $p \in V(f_1, \ldots, f_{n-1}, f_n)$ and $\mathcal{C} := V(f_1, \ldots, f_{n-1})$.
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- Let T be a 0-dim regular chain defining the point p; rename its variables to $y_1, \ldots, y_n$.
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- This is a 1-dim system in the variables $y_1, \ldots, y_n, x_1, \ldots, x_n$.
- Let $R_1, \ldots, R_e$ be regular chains decomposing the zero set V of (S).

Tangent cone computation with regular chains (2/2)

Recall

- The set $\{\lim_{m \to p, m \neq p} \ \vec{u}\}$ describes $TC_p(\mathcal{C})$
- Consider the system (S) defined by T and $f_1 = \cdots = f_{n-1} = 0$.
- Let $R_1, \ldots, R_e$ be regular chains decomposing the zero set V of (S).

Step 2

- We divide each component of $p\vec{m}$ by $x_1 - y_1$. This works only if $x_1 - y_1$ vanishes finitely many times in V.

Tangent cone computation with regular chains (2/2)

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- The set $\{\lim_{m \to p, m \neq p} \ \vec{u}\}$ describes $TC_p(\mathcal{C})$
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- Fix $i = 1 \cdots e$. If $x_1 - y_1$ is regular modulo the saturated ideal of R_i, then each compliant of $p\vec{m}$ can be divided by $x_1 - y_1$.

Tangent cone computation with regular chains (2/2)

Recall

- The set $\{\lim_{m \to p, m \neq p} \ \vec{u}\}$ describes $TC_p(\mathcal{C})$
- Consider the system (S) defined by T and $f_1 = \cdots = f_{n-1} = 0$.
- Let $R_1, \ldots, R_e$ be regular chains decomposing the zero set V of (S).

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- Let $s_2, \ldots, s_n$ be variables; extend R_j with the polynomials $s_2(x_1 - y_1) - (x_2 - y_2), \ldots, s_n(x_1 - y_1) - (x_n - y_n)$ to a chain S_j.

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- Finally $\{\lim_{m \to p, m \neq p} \ \vec{u}\}$ is given by the limit points of the S_j's, that is, the sets $\overline{W(S_j)} \setminus W(S_j)$.

Example

```
> R := PolynomialRing([x, y, z]):
  F := [x^2 + y^2 + z^2 -1, x^2 - y^2 - z*(z-1)];
  rc := Chain([z-1, y, x], Empty(R), R);
```

$$F := [x^2 + y^2 + z^2 - 1,\ x^2 - y^2 - z\,(z-1)]$$

$$rc := regular\_chain$$

```
> with (AlgebraicGeometryTools);
```

[*Cylindrify*, *IntersectionMultiplicity*, *IsTransverse*, *LimitPoints*, *RootOfToRegularChain*, *Tang*
TangentPlane, *TriangularizeWithMultiplicity*]

```
> cases := TangentCone(rc, F, R);
```

$$cases := \{[[\_z - 1,\ 3\ \_x^2 - \_y^2],\ regular\_chain]\}$$

Summary

Theorem

Consider a one-dimensional regular chain R_1 solving the system $f_1(x_1, \ldots, x_n) = \cdots = f_{n-1}(x_1, \ldots, x_n) = 0$ at a point $p(y_1, \ldots, y_n)$ given by a zero-dimensional T such that $V(T) \subseteq V(f_1, \ldots, f_n)$. W.o.l.g. $x_1 - y_1$ is regular modulo $\mathrm{sat}(R_i)$. Then, each line of $TC_p(\mathcal{C})$ not contained in the hyperplane $x_1 = y_1$ has his slopes $s_2, \ldots, s_n$ obtained by $\lim(W(S))$ where S is the regular chain (for $y_1 < \cdots < y_n < x_1 < \cdots < x_n < s_2 < \cdots < s_n$)

$$S = R_1 \cup \{s_2(x_1 - y_1) - (x_2 - y_2), \ldots, s_n(x_1 - y_1) - (x_n - y_n)\}$$

Remarks

Additional computations are needed to capture the lines contained in $x_1 = y_1$: There are essentially two options:

1. Perform a random linear change of the coordinates so as to assume that, generically, $y_1 = x_1$ contains no lines of $TC_p(\mathcal{C})$.
2. Compute in turn the lines not contained in the hyperplane $y_i = x_i$ for all $i = 0, \ldots, n$ and remove the duplicates; indeed no lines of the tangent cone can simultaneously satisfy $y_i = x_i$ for all $i = 0, \ldots, n$.

Concluding remarks

Theorem (Ssame as before)

Consider a one-dimensional regular chain R_1 solving the system $f_1(x_1, \ldots, x_n) = \cdots = f_{n-1}(x_1, \ldots, x_n) = 0$ at a point $p(y_1, \ldots, y_n)$ given by a zero-dimensional T such that $V(T) \subseteq V(f_1, \ldots, f_n)$. W.o.l.g. $x_1 - y_1$ is regular modulo $\mathrm{sat}(R_i)$. Then, each line of $TC_p(\mathcal{C})$ not contained in the hyperplane $x_1 = y_1$ has his slopes $s_2, \ldots, s_n$ obtained by $\lim(W(S))$ where S is the regular chain (for $y_1 < \cdots < y_n < x_1 < \cdots < x_n < s_2 < \cdots < s_n$)

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Remarks

- The proposed method reduces tangent cone computation to that of limits of rational functions.
- Thanks to the size estimates on R_1 (X. Dahan, A. Kadri & E. Schost; 2012; and run time estimates on Puiseux series calculation (P. G. Walsh; 2000) the proposed method is singly exponential in the size of the input system $f_1, \ldots, f_n$.
- Relaxing the one-dimensional constraint is work in progress.