Adapted Coordinates in Two Dimensions and a Proof of Puiseux's Theorem

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1. Introduction

Let $\mathbf{C}\{x_1, ..., x_n\}$ denote the ring of power series whose coefficients increase slowly enough so that the series converges in a neighborhood of the origin in $\mathbf{C}^n$. Suppose $f(x,y) \in \mathbf{C}\{x,y\}$ with $f(0,0) = 0$. Then one version of Puiseux's theorem is the statement that there exists a factorization

$$f(x,y) = u(x,y)x^c \prod_{i=1}^{m}(y - g_i(x)) \tag{1.1}$$

Here for some natural number n, each $g_i \in \mathbf{C}\{x^{\frac{1}{n}}\}$ with $g_i(0) = 0$, and $u(x,y) \in \mathbf{C}\{x^{\frac{1}{n}}, y^{\frac{1}{n}}\}$ with $u(0,0) \neq 0$. Hence the zeroes of $f(x,y)$ are parameterized by analytic functions of one variable. (With a little more effort one can show $u(x,y) \in \mathbf{C}\{x,y\}$). One method to prove the factorization (1.1) goes back to Isaac Newton himself. Newton's method produces the terms of the $g_i(x)$ through an infinite recursion; in modern treatments one then shows the resulting power series converges in a neighborhood of the origin. The latter is normally done by invoking a topological argument involving Riemann surfaces (see [BK]). Alternatively, one may carefully examine the properties of Newton's algorithm as one proceeds and then directly prove that the resulting $g_i(x)$ are in some $\mathbf{C}\{x^{\frac{1}{n}}\}$; this is done in [Ca] and [Ch] (Puiseux's original proof was somewhat different).

The purpose of this paper is to provide an argument based on Newton's method and some ideas from resolution of singularities that gives a quick proof of the factorization (1.1) (including the convergence of the $g_i(x)$). It is then shown that similar ideas can be used to give a short proof of the existence of smooth adapted coordinates in two dimensions (Theorem 1.2 below). This result was first proved in the real-analytic case by Varchenko [V] and then recently for the general smooth case by Ikromov-Müller [IM]. These proofs use detailed information about the zero set of $S(x,y)$.

The arguments of this paper will use only the two-dimensional implicit function theorem and some basic properties of Newton polygons; they are however inspired by more modern resolution of singularities ideas as will be discussed at the end of section 2.

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It should also be pointed out that if one is willing to assume the Weierstrass preparation theorem and Hensel's lemma, there exist short and rather different elementary proofs of Puiseux's theorem of a more algebraic nature. We refer to [N] for more information.

We make extensive use of the following object, essentially used in Newton's letter.

Definition 1.2. Let $\sum_{a,b} s_{ab}x^a y^b$ be a power series in $x^{\frac{1}{n}}$ and $y^{\frac{1}{n}}$ for some positive integer n, and assume that at least one s_{ab} is nonzero. For any (a, b) for which $s_{ab} \neq 0$, let Q_{ab} be the quadrant $\{(x, y) \in \mathbf{R}^2 : x \geq a, y \geq b\}$. Then the *Newton polygon* $N(S)$ is defined to be the convex hull of all Q_{ab}.

The boundary of a Newton polygon consists of finitely many (possibly zero) bounded edges of negative slope as well as an unbounded vertical ray and an unbounded horizontal ray. We also will make use the following.

Definition 1.3. Let $S(x, y)$ be as in Definition 1.2. The *Newton distance* $d(S)$ is defined by $d(S) = \inf\{d : (d, d) \in N(S)\}$.

Definition 1.4. Suppose e is a compact edge of $N(S)$. Define $S_e(x, y)$ by $S_e(x, y) = \sum_{(a,b)\in e} s_{a,b}x^a y^b$. In other words $S_e(x, y)$ is the sum of the terms of the Taylor expansion of S over all $(a, b) \in e$.

In the study of oscillatory integrals in two dimensions, the notion of adapted coordinates plays an important role.

Definition 1.5. A coordinate system is said to be *adapted* if $d(S) = \sup_\alpha d(S \circ \alpha)$, where the supremum is taken over all smooth coordinate changes α defined in a neighborhood of $(0, 0)$ such that $\alpha(0, 0) = (0, 0)$.

The significance of adapted coordinate systems is the following. Consider the oscillatory integral

$$J_\lambda = \int_{\mathbf{R}^2} e^{i\lambda S(x,y)}\phi(x, y)\, dx\, dy \tag{1.2}$$

Assume $S(0, 0) = 0$ and that S has a critical point at the origin; that is, $\nabla S(0, 0) = 0$. The function $\phi(x, y)$ is a cutoff function in a neighborhood of the origin and λ denotes a real parameter which one assumes to be large. Then the best (supremal) ϵ for which one has the estimate $|J_\lambda| < C \ln(2 + |\lambda|)|\lambda|^{-\epsilon}$ for all λ and all $\phi(x, y)$ supported in a sufficiently small neighborhood of the origin has the nice form $\epsilon = \frac{1}{d(S)}$ if and only if $S(x, y)$ is in an adapted coordinate system. This was proved by Ikromov, Kempe, and Müller in [IKM]. For the real-analytic case, where one considers real-analytic phase and takes the supremum of (1.2) over real-analytic coordinate changes, the corresponding result was earlier proven by Varchenko in [V].

Theorem 1.1. Suppose S has nonvanishing Taylor expansion at $(0, 0)$ and $S(0, 0) = 0$.

A coordinate system is adapted if any of the following three cases hold.

Case 1. The line $y = x$ intersects $N(S)$ in the interior of a bounded edge e and any real zero r of $S_e(1, y)$ or $S_e(-1, y)$ with $r \neq 0$ has order less than $d(S)$.

Case 2. The line $y = x$ intersects $N(S)$ at a vertex (d, d).

Case 3. The line $y = x$ intersects $N(S)$ in the interior of one of the unbounded edges.

Proof: By the main theorem of [G], if U is a small enough neighborhood of the origin, and ϵ_0 denotes the supremum of the numbers ϵ for which $\int_U |f|^{-\epsilon}$ is finite, then $d(S) \leq \frac{1}{\epsilon_0}$, with $d(S) = \frac{1}{\epsilon_0}$ in cases 1, 2, and 3. Hence if one is in cases 1, 2, or 3, one is in adapted coordinates.

Theorem 1.2. Suppose S has nonvanishing Taylor expansion at $(0, 0)$ and $S(0, 0) = 0$. Then there exists some coordinate system in a neighborhood of the origin such that one of Cases 1, 2, or 3 hold. Hence there exists an adapted coordinate system for $S(x, y)$. The associated coordinate change can always be taken to be of the form $(x, y) \to (x, y - \psi(x))$ or $(x, y) \to (x - \psi(y), y)$ for a smooth ψ.

In [IM], a slightly weaker version of Theorem 1.2 is proven which also shows that for any smooth phase there exists a smooth adapted coordinate system. The arguments of [IM] use Puiseux's theorem and do a careful analysis of the different $g_i(x)$. The proof of Theorem 1.2 is in section 3. It should be also pointed out that Theorem 1.1 follows from Theorem 3.3 of [IM].

2. Proof of Puiseux's Theorem.

Suppose $f(x, y) \in \mathbf{C}\{x, y\}$. After factoring out the largest possible power of x out of $f(x, y)$, we can write $f(x, y) = x^c g(x, y)$, where $\partial_y^e g(0, 0) \neq 0$ for some e. Since (1.1) trivially holds if e is zero, we can assume $e > 0$. Assuming e to be chosen minimal, we have that $(0, e)$ is on the Newton polygon $N(g)$. We will prove Puiseux's theorem by proving the following theorem:

Theorem 2.1. Suppose that $h(x, y) = \sum_{a,b} h_{ab} x^a y^b \in \mathbf{C}\{x^{\frac{1}{n}}, y\}$ such that $h(0, 0) = 0$ and such that $(0, E) \in N(h)$ for some $E > 0$. Then one has a factorization $h(x, y) = H(x, y)(y - g(x))$ where for some natural number N, $H(x, y) \in \mathbf{C}\{x^{\frac{1}{N}}, y\}$ and $g \in \mathbf{C}\{x^{\frac{1}{N}}\}$ with $g(0) = 0$.

Puiseux's theorem follows by applying Theorem 2.1 repeatedly; $(0, E-1) \in N(H)$ and thus starting with $g(x, y)$, after e iterations one has (1.1).

Proof of Theorem 2.1. If y divides $h(x, y)$ we are done, so we may assume that there is some point $(D, 0)$ on the Newton polygon $N(h)$ with $D > 0$. Let (p, q) denote the vertex of $N(h)$ with $q > 0$ such that q is minimal. Thus the segment e connecting (p, q) to $(D, 0)$ is an edge of $N(h)$. Let $h_e(x, y)$ denote the sum of the terms $h_{ab}x^a y^b$ of the series $h(x, y) = \sum_{a,b} h_{ab}x^a y^b$ such that (a, b) is on e. Thus $h_e(x, y)$ is a polynomial in $x^{\frac{1}{n}}$ and y. Write the equation of the edge e as $x + my = \alpha$. Hence if $h_{ab}x^a y^b$ appears in $h_e(x, y)$ then $a + mb = \alpha$. We factor out x^α, writing $h_e(x, y) = x^\alpha h'_e(x, y)$. Each term of $h'_e(x, y)$ is now of the form $h_{ab}x^{a-\alpha}y^b$ and $(a - \alpha) + mb = 0$ or $(a - \alpha) = -mb$. Thus we have

$$h_{ab}x^{a-\alpha}y^b = h_{ab}(\frac{y}{x^m})^b \tag{2.1}$$

Conequently for a polynomial $P(z)$, we can write

$$h_e(x, y) = x^\alpha P(\frac{y}{x^m}) \tag{2.2}$$

The proof of Theorem 2.1 will now proceed by an inductive process. At each stage we will perform a coordinate change of the form $(x, y) \to (x, y + a(x))$ for some $a(x) \in \mathbf{C}\{x^{\frac{1}{N}}\}$ with $a(0) = 0$. The resulting function $h(x, y + a(x))$ will fall into one of the following two (not mutually exclusive) cases.

Case 1: y divides $h(x, y + a(x))$.

Case 2: $h(x, y+a(x))$ satisfies the hypotheses of Theorem 2.1 and the second-lowest vertex (p'', q'') of the Newton polygon of $h(x, y + a(x))$ satisfies $q'' < q$.

In the first case, one transfers back to the original coordinates and we have the conclusions of Theorem 2.1. In the second case, one is back under the assumptions of Theorem 2.1 and thus can repeat the upcoming argument, finding the next $a(x)$. Since $q'' < q$, after at most q iterations one will have to be in the first case and we will be done.

So our task is to show that under the assumptions of Theorem 2.1 we can always find an $a(x)$ such that one of the two cases holds. Suppose first that the polynomial P has a (complex) root r of order $q' < q$. Then the function $P(z+r)$ has a root at $z = 0$ of order q'. Hence there is a term of $h_e(x, y+rx^m) = x^\alpha P(\frac{y}{x^m}+r)$ with y appearing to the q'th power, but no terms with y appearing to a lower power than q'. Define $H(x, y) = h(x, y + rx^m)$. Note that $H_e(x, y) = h_e(x, y + rx^m)$. Thus a segment of the line $x + my = \alpha$ is an edge of the Newton polygon $N(H)$ of H, as was the case for h. However, instead of going down to $(D, 0)$, for H the segment terminates at (p', q') for some p'. Hence either (p', q') is the lowest vertex of $N(H)$, in which case one is in Case 1 with $a(x) = rx^m$, or the second-lowest vertex (p'', q'') of $N(H)$ (which could be (p', q')) satisfies $q'' \leq q' < q$. Therefore if we let $a(x) = rx^m$, $h(x, y)$ is in case 2 and we are done.

Thus it remains to analyze the situation where $P(z)$ has a single complex root r of order q. Here $P(z) = c(z-r)^q$ for some c. This is the situation where Newton's method gives an infinite iteration; here we will do something different. We look at the function $h(x, x^m y)$. Since $x + my = \alpha$ is a supporting line for $N(h)$, the terms of $h_e(x, x^m y)$ are the terms of $h(x,x^m y)$ with with the lowest power of x appearing. Since $h_e(x,x^m y) = x^\alpha P(\frac{x^m y}{x^m}) = cx^\alpha (y-r)^q$, for some $\epsilon > 0$ we may write

$$h(x,x^m y) = cx^\alpha (y-r)^q + x^{\alpha+\epsilon} l(x,y) \tag{2.3}$$

By (2.3), the function $h'(x,y) = \frac{h(x,x^m y)}{x^\alpha}$ is in $\mathbf{C}\{x^{\frac{1}{N}}, y\}$ for some N and we have

$$h'(x,y) = c(y-r)^q + x^\epsilon l(x,y) \tag{2.4}$$

The trick is now as follows. The function $\frac{\partial^{q-1} h'}{\partial y^{q-1}}$ has a zero at $(0,r)$, but has non-vanishing y derivative there. Hence by applying the 2-dimensional implicit function theorem (technically to $\frac{\partial^{q-1} h'}{\partial y^{q-1}}(x^N, y))$, one has that there is some function $k(x) \in \mathbf{C}\{x^{\frac{1}{N}}\}$ with $k(0) = r$ such that $\frac{\partial^{q-1} h'}{\partial y^{q-1}}(x, k(x)) = 0$ near the origin. One now defines $H(x,y) = h(x, y + x^m k(x))$. The fact that allows the induction to proceed is that

$$\frac{\partial^{q-1} H}{\partial y^{q-1}}(x,0) = \frac{\partial^{q-1} h}{\partial y^{q-1}}(x, x^m k(x)) = x^{\alpha-(q-1)m} \frac{\partial^{q-1} h'}{\partial y^{q-1}}(x, k(x)) = 0 \tag{2.5}$$

Like before, the coordinate change is such that $x + my = \alpha$ is still a supporting line for $N(H)$. This time it intersects $N(H)$ in the single vertex (p,q). This may be easiest to see from (2.3) using the fact that in the coordinates of (2.3) the coordinate change is of the form $(x,y) \to (x, y + r + \tilde{k}(x))$ where $\tilde{k}(0) = 0$.

If y divides H we are back in case 1 and we are done. So we may assume there is some vertex $(d', 0)$ on $N(H)$ with $d' > 0$. If (p,q) is anything other than the second-lowest vertex, we are in case 2 and thus we'd be done again. Hence we can assume that the segment e' connecting (p,q) to $(d',0)$ is an edge of $N(H)$. The condition (2.5) ensures that $H_{e'}(x,y)$ cannot have a single complex root of order q; for if this were to happen like above $H_{e'}(x,y)$ would be of the form $cx^{\alpha'}(\frac{y}{x^{m'}} - r')^q$. But this expression has a nonvanishing y^{q-1} term; this contradicts (2.5) which implies that for every a the Taylor series coefficient $H_{a\,q-1}$ is zero. Hence $H_{e'}(x,y)$ must have a root of order less than q. We dealt with this situation above; a further coordinate change of the correct form puts us in case 1 or 2. This completes the proof of Theorem 2.1.

Those familiar with resolution of singularities algorithms can recognize this idea of taking the zero set of the $(q-1)$st derivative of a function and making it a hyperplane, so that an inductive procedure may continue. So essentially what is happening here is that an argument of this type is being incorporated into Newton's method to construct a process that terminates after finitely many applications of the implicit function theorem rather than an infinite iteration.

3. Proof of Theorem 1.2.

We now assume that $S(x,y)$ is a smooth function defined in a neighborhood of the origin with $S(0,0) = 0$ and having a nonvanishing Taylor expansion at $(0,0)$. Let

$N(S)$ denote the Newton polygon of this Taylor expansion. Assume we are not in any of the three cases of Theorem 1.2. Thus the line $y = x$ intersects the Newton polygon $N(S)$ in the interior of a compact edge e, and $S_e(1, y)$ or $S_e(-1, y)$ has a real zero $r \neq 0$ of order $k \geq d(S)$. Replacing x by $-x$ and/or y by $-y$ if necessary, we may assume $r > 0$ is a zero of $S_e(1, y)$ of order k. The goal is to do a coordinate change of the proper form that puts us into one of these three cases. Denote the equation of the line containing e by $x + my = \alpha$. Exactly as (2.2), there is some polynomial $Q(y)$ such that for $x > 0$ we have

$$S_e(x, y) = x^\alpha Q(\frac{y}{x^m})$$

Plugging in $x = 1$, we see that $Q(y) = S_e(1, y)$. Hence $S_e(x, y)$ has zeroes of order k on the curve $y = rx^m$. This implies that $S_e(x, 1)$ has a zero of order k at $x = r^{-\frac{1}{m}}$. As a result, we may switch the roles of the x and y axes if necessary and assume that $m \geq 1$; this makes our subsequent arguments somewhat easier.

Next, we show that m must in fact be an integer. To see this, note that if m were not an integer, then the degrees of the powers of y appearing in $S_e(1, y)$ would have to be separated by at least 2. Hence $S_e(1, y)$ would have to be of the form $y^\beta R(y^c)$ for some $\beta \geq 0$, $c \geq 2$, where R is a polynomial. Next, since $(d(S), d(S))$ is on $N(S)$, we have $\alpha = (1+m)d(S)$. Since $m > 1$ when $m \geq 1$ is not an integer, the maximum possible value of y on the line $x+my = (1+m)d(S)$ for $x, y \geq 0$ is $\frac{m+1}{m}d(S) < 2d(S)$. Thus the degree of $y^\beta R(y^c)$ is less than $2d(S)$, and hence the degree of $R(y)$ is less than $\frac{2d(S)}{c} \leq d(S)$. Hence the zeroes of $R(y)$ are of order less than $d(S)$, implying the zeroes of of $S_e(1, y) = y^\beta R(y^c)$ other than $y = 0$ are of order less than $d(S)$. This contradicts our assumption that $S_e(1, y)$ has a zero of order $k \geq d(S)$ and we conclude that m is an integer.

Note that if m is even, then $S_e(1, y) = \pm S_e(-1, y)$, while if m is odd one has $S_e(1, y) = \pm S_e(-1, -y)$. Hence both $S_e(1, y)$ and $S_e(-1, y)$ have a zero of order k; we never really had to replace x by $-x$ in the above. The preceding arguments thus imply:

Fact: If one is not in adapted coordinates and $m \geq 1$, then m is an integer and $Q(y) = S_e(1, y)$ has a zero of order $k \geq d(S)$.

We now come to the main argument; we will prove the existence of a coordinate change of the form $(x, y) \to (x, y + a(x))$, $a(x)$ smooth, that puts us into one of the three cases. (The coordinate change $(x, y) \to (x + a(y), y)$ corresponds to $m \leq 1$). We proceed as follows. Let (p, q) denote the upper vertex of the edge e; necessarily $q > d(S)$. We will find a smooth function $a(x)$ such that $S'(x, y) = S(x, y + a(x))$ is in one of the following two (not mutually exclusive) categories.

Category 1: $S'(x, y)$ is in one of the three cases of adapted coordinates.

Category 2: The line $y = x$ intersects the interior of an edge e' of $N(S')$ with equation $x + m'y = \alpha'$, $m' \geq 1$, such that the upper vertex (p', q') of e' satisfies $q' < q$.

Theorem 1.2 will then follow; there can be at most q iterations of category 2.

The arguments now resemble those of section 2. We first consider the case where the order k of the zero r of $Q(y) = S_e(1, y)$ satisfies $k < q$. Here we let $a(x) = rx^m$, and thus $S'(x, y) = S(x, y + rx^m)$. Then $x + my = \alpha$ is a supporting line of $N(S')$ as it was for $N(S)$, and like in section 2 there is an edge E of $N(S')$ on this line whose upper vertex is (p, q). Note that $S'_E(x, y) = S_e(x, y + rx^m) = x^\alpha Q(\frac{y}{x^m} + r)$. Since Q has a zero of order k at r, the lowest power of y appearing in $S'_E(x, y)$ is y^k, and therefore E's lower vertex is at a point (j, k) for some j. Since both vertices of E have y-coordinates at least $d(S)$, they are both in the portion of $x + my = \alpha$ on or above $(d(S), d(S))$. Thus the edge E lies wholly on or above the line $y = x$. If the line $y = x$ intersects $N(S')$ at a vertex or inside the horizontal or vertical rays, one is in Category 1. Otherwise, it must intersect $N(S')$ in the interior of an edge e' whose upper vertex is either (j, k) or a lower vertex. And because $x + my = \alpha$ is a supporting line for $N(S')$ and e' lies below E, e' will have equation $x + m'y = \alpha'$ for some $m' > m \geq 1$. Thus we are in category 2. Hence when $k < q$ we are in either Category 1 or 2 and we are done.

It remains to consider the situation where r is a zero of $Q(y)$ of order q. In this case we have $Q(y) = c(y - r)^q$ for some c. For a large integer n we expand $S(x, y)$ as

$$S(x, y) = cx^\alpha(\frac{y}{x^m} - r)^q + T_n(x, y) + E_n(x, y) \tag{3.1}$$

Here the polynomial $T_n(x, y)$ are the terms of S's Taylor expansion with exponents less than n. For all $0 \leq \beta, \gamma < n$ one has

$$|\frac{\partial^{\beta+\gamma} E_n}{\partial x^\beta \partial y^\gamma}(x, y)| < C(|x|^{n-\beta} + |y|^{n-\gamma}) \tag{3.2}$$

Analogous to (2.3), one has

$$S(x, x^m y) = cx^\alpha(y - r)^q + x^{\alpha+1}T'_n(x, y) + E_n(x, x^m y) \tag{3.3}$$

Here $T'_n(x, y)$ is also a polynomial. Analogous to (2.4) we define $s(x, y) = \frac{S(x, x^m y)}{x^\alpha}$, so that

$$s(x, y) = c(y - r)^q + xT'_n(x, y) + x^{-\alpha}E_n(x, x^m y) \tag{3.4}$$

We claim that the function $s(x, y)$ is smooth on a neighborhood of $(0, r)$. Off the y-axis smoothness holds because $S(x, y)$ is smooth. One can show that a given derivative of $s(x, y)$ exists when $x = 0$ and equals that of $c(y - r)^q + xT'_n(x, y)$ for large enough n by examining the difference quotient of a one-lower order derivative of (3.4), inductively assuming this lower-order derivative exists and has the right value when $x = 0$. Equation (3.2) ensures that the difference quotient of the lower derivative of $x^{-\alpha}E_n(x, x^m y)$ tends to zero as x goes to zero. We conclude that $s(x, y)$ is smooth on a neighborhood of $(0, r)$.

Analogous to after (2.4), we next use the smooth implicit function theorem on $\frac{\partial^{q-1}s}{\partial y^{q-1}}$ and find a smooth function $k(x)$ defined in a neighborhood of $x = 0$ such that $k(0) = r$ and $\frac{\partial^{q-1}s}{\partial y^{q-1}}(x, k(x)) = 0$. Transferring this back to $S(x, y)$, as in (2.5) we have

$$\frac{\partial^{q-1}S}{\partial y^{q-1}}(x, x^m k(x)) = 0 \tag{3.5}$$

Thus if we let $a(x) = x^m k(x)$ and $S'(x, y) = S(x, y + x^m k(x))$, for all x we consequently have

$$\frac{\partial^{q-1}S'}{\partial y^{q-1}}(x, 0) = 0 \tag{3.6}$$

Thus for every a the Taylor series coefficient $S'_{a\,q-1}$ is zero.

Next, since $x + my = \alpha$ is a supporting line for $N(S)$, analogous to after (2.5) this line is also a supporting line for $N(S')$ and intersects $N(S')$ at the single vertex (p, q). If $S'(x, y)$ is in adapted coordinates we are in Category 1 and have nothing to prove, so we may assume the coordinates are not adapted. Let e' denote the edge of $N(S')$ intersecting the line $y = x$ and denote its equation by $x + m'y = \alpha'$. If the upper vertex (p', q') of $N(S')$ satisfies $q' < q$, one is in Category 2 and we are done. So we assume this upper vertex is (p, q) itself. Also, since e' lies within the set $x + my \geq \alpha$ and is no higher than the vertex (p, q) of $N(S')$ that is on the supporting line $x + my = \alpha$, we have $m' > m \geq 1$.

If $S'_{e'}(1, y)$ has a real zero $r' \neq 0$ of order less than q, one is in the situation above (3.1); there is a smooth $b(x)$ such that $S'(x, y + b(x)) = S(x, y + a(x) + b(x))$ is in Category 1 or 2 as needed. The only other possibility is that $S'_{e'}(1, y)$ has a single zero $r' \neq 0$ of order q. But like at the end of section 2, this cannot happen. For this would imply $S'_{e'}(x, y) = c'x^{\alpha'}(\frac{y}{x^{m'}} - r')^q$ has a nonvanishing y^{q-1} term. Consequently, for some a the Taylor series coefficient $S'_{a\,q-1}$ would be nonzero, contradicting (3.6). Thus the case where $S'_{e'}(1, y)$ has a single zero of order q does not occur, and we are done.

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