Triangular decomposition of semi-algebraic systems

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(CDMMXX) RealTriangularize Algebra Seminar 1 / 62
1. Solving systems of polynomial equations
2. Triangular decomposition of algebraic systems
3. Triangular decomposition of a semi-algebraic system
4. Algorithm
5. Complexity analysis
6. Benchmarks
7. Concluding remarks
8. Applications
9. Cylindrical algebraic decomposition: basic ideas
Solving polynomial systems? What does this mean?

The algebra text book says:

For $F \subset \mathbb{k}[x_1, \ldots, x_n]$ this is simply

- a primary decomposition of $\langle F \rangle$ or
- the irreducible decomposition of $V(F)$ (the zero set of $F$ in $\bar{\mathbb{k}}^n$).

The computer algebra system does well:

For $F \subset \mathbb{k}[x_1, \ldots, x_n]$, with $\mathbb{k} = \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{k} = \mathbb{Q}$,

- computing a Gröbner basis of $\langle F \rangle$ or
- computing a triangular decomposition of $V(F)$.

But most scientists and engineers need:

- For $F \subset \mathbb{Q}[x_1, \ldots, x_n]$, a useful description of the points of $V(F)$ whose coordinates are real.
- For $F \subset \mathbb{Q}[u_1, \ldots, u_d][x_1, \ldots, x_n]$, the real $(x_1, \ldots, x_n)$-solutions as a function of the real parameter $(u_1, \ldots, u_d)$. 
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Solving for the real solutions: classical techniques

**In dimension zero over \( \mathbb{Q} \):**

For \( F \subset \mathbb{Q}[x_1, \ldots, x_n] \), if \( V(F) \) is finite, many standard and efficient techniques apply to identify the real solutions.

**In (generic) dimension zero over \( \mathbb{Q}[u_1, \ldots, u_d] \):**

For \( F \subset \mathbb{Q}[u_1, \ldots, u_d][x_1, \ldots, x_n] \) and an integer \( r \) one can determine “generic” conditions on \( u_1, \ldots, u_d \) for \( F \) to admit exactly \( r \) real \((x_1, \ldots, x_n)\)-solutions.

**For arbitrary systems:**

For \( F \subset \mathbb{Q}[x_1, \ldots, x_n] \), one can partition \( \mathbb{R}^n \) into cylindrical cells where the sign of each \( f \in F \) does not change.
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For arbitrary systems:
For $F \subset \mathbb{Q}[x_1, \ldots, x_n]$, one can partition $\mathbb{R}^n$ into *cylindrical cells* where the sign of each $f \in F$ does not change.
Real root isolation for zero-dimensional systems

\[ R := \text{PolynomialRing}([x, y, z]); \quad F := [x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1]; \]

\[ \text{polynomial\_ring} \]

\[ [x^2 + y + z - 1, y^2 + x + z - 1, x + y + z^2 - 1] \]

\[ \text{dec} := \text{Triangularize}(F, R); \quad \text{map}(\text{Display}, \text{dec}, R); \]

\[ [\text{regular\_chain, regular\_chain, regular\_chain, regular\_chain}] \]

\[
\begin{align*}
&\begin{cases} 
  x - z = 0 \\
  y - z = 0 \\
  z^2 + 2 z - 1 = 0
\end{cases},
\begin{cases} 
  x = 0 \\
  y - 1 = 0 \\
  z - 1 = 0
\end{cases},
\begin{cases} 
  x - 1 = 0 \\
  y = 0 \\
  z = 0
\end{cases},
\begin{cases} 
  x = 0 \\
  y = 0 \\
  z = 0
\end{cases}
\end{align*}
\]

\[ \text{boxes} := \left[ \text{seq}\left( \text{op}(\text{RealRootIsolate}(\text{rc}, R, '\text{rer}'=\frac{1}{2^9})), \text{rc} = \text{dec} \right) \right]; \quad \text{map}(\text{Display}, \text{boxes}, R) \]

\[
\begin{align*}
&\begin{cases} 
  x = \left[ \frac{3393}{8192}, \frac{6791}{16384} \right] \\
  y = \left[ \frac{3393}{8192}, \frac{6791}{16384} \right] \\
  z = \left[ \frac{217167}{524288}, \frac{868669}{2097152} \right]
\end{cases},
\begin{cases} 
  x = \left[ \frac{-4947}{2048}, \frac{-2471}{1024} \right] \\
  y = \left[ \frac{-4947}{2048}, \frac{-2471}{1024} \right] \\
  z = \left[ \frac{-79109}{32768}, \frac{316435}{131072} \right]
\end{cases},
\begin{cases} 
  x = 0 \\
  y = 0 \\
  z = 1
\end{cases},
\begin{cases} 
  x = 0 \\
  y = 1 \\
  z = 0
\end{cases},
\begin{cases} 
  x = 1 \\
  y = 0 \\
  z = 0
\end{cases}
\end{align*}
\]
Real root classification: generically 0-dimensional systems

with(RegularChains): with(SemiAlgebraicSetTools): with(ParametricSystemTools):

\[ \text{real system: } \begin{align*}
R &:= \text{PolynomialRing}([x, y, z, \epsilon]) ; \quad F := [x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1 + \epsilon] ; \\
&= \text{polynomial_ring} \\
&\begin{bmatrix} x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1 + \epsilon \end{bmatrix}
\]

\text{dec} := \text{Triangularize}(F, R); \quad \text{map} (\text{Equations}, \text{dec}, R);

\begin{align*}
\text{[regular_chain, regular_chain]} &
\begin{bmatrix} 2x + z^2 + \epsilon - 1, 2y + z^2 + \epsilon - 1, z^4 + (2\epsilon - 4)z^2 + 4z - 4\epsilon - 1 + \epsilon^2 \end{bmatrix}, \quad [x + y - 1, y^2 - y + z, z^2 + \epsilon] \\
\end{align*}

For which values of epsilon does F have 2 solutions each of which has a positive x-coordinate?
\text{rrc} := \text{RealRootClassification}(F, [ ], [x], [ ], 1, 2, R); \quad \text{Display}(\text{rrc}[1][1], R); \quad \text{Display}(\text{rrc}[2], R)

\begin{align*}
\text{[[regular_semi_algebraic_set], border_polynomial]} &
\begin{cases}
\epsilon < 0 \quad \text{and} \quad 16\epsilon < -1 \quad \text{and} \quad 5\epsilon - 1 \neq 0 \quad \text{and} \quad 16\epsilon^2 - 71\epsilon + 2 \neq 0 \\
\text{or} \quad \epsilon > 0 \quad \text{and} \quad 16\epsilon + 1 \neq 0 \quad \text{and} \quad 5\epsilon < 1 \quad \text{and} \quad 16\epsilon^2 - 71\epsilon + 2 > 0
\end{cases} \\
[\epsilon, \epsilon - \frac{1}{5}, \epsilon + \frac{1}{16}, \epsilon^2 - \frac{71}{16}\epsilon + \frac{1}{8}]
\end{align*}

For which values of epsilon does F have 5 solutions?
\text{rrc} := \text{RealRootClassification}(F, [ ], [ ], [ ], 1, 5, R); \quad \text{Display}(\text{rrc}[2], R)

\begin{align*}
\text{[[ ], border_polynomial]} &
[\epsilon, \epsilon + \frac{1}{16}, \epsilon^2 - \frac{71}{16}\epsilon + \frac{1}{8}]
\end{align*}
The cylindrical algebraic decomposition of \( \{ax^2 + bx + c\} \) is given by the tree above, where \( t = bx + c \), \( q = 2ax + b \), and \( r = 4ac - b^2 \). This is the best possible output for that method, leading to **27 cells**!
Can a computer program be as good as a high-school student?

For the equation $ax^2 + bx + c = 0$, can a computer program produce?

\[
\begin{cases}
ax^2 + bx + c = 0 \\
a \neq 0 \land b^2 - 4ac > 0
\end{cases}
\]

\[
\begin{cases}
bx + c = 0 \\
a = 0 \\
b \neq 0
\end{cases}
\]

\[
\begin{cases}
2ax + b = 0 \\
4ac - b^2 = 0 \\
a \neq 0
\end{cases}
\]

\[
\begin{cases}
c = 0 \\
b = 0 \\
a = 0
\end{cases}
\]
Yes, our new algorithm RealTriangularize can do that!

```maple
with(RegularChains) : with(SemiAlgebraicSetTools) : with(ParametricSystemTools):

R := PolynomialRing([x, c, b, a]); F := [a*x^2 + b*x + c];

Solving for the real solutions:
RealTriangularize(F, R, output = record);

\[
\begin{align*}
ax^2 + bx + c &= 0 \\
-4c a + b^2 &= 0 \\
a &\neq 0
\end{align*}
\]

Solving for the complex solutions:
\[
\begin{align*}
\begin{cases} 
ax^2 + bx + c &= 0 \\
a &\neq 0 
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases} 
bx + c &= 0 \\
b &\neq 0 
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases} 
c &= 0 \\
a &= 0
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases} 
2ax + b &= 0 \\
a &= 0 \\
2ac - b^2 &= 0 \\
a &\neq 0
\end{cases}
\end{align*}
\]
\[
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\begin{align*}
\begin{cases} 
c &= 0 \\
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a &= 0
\end{cases}
\end{align*}
\]
```
RealTriangularize applied to the *Eve* surface (1/2)
RealTriangularize applied to the *Eve* surface (2/2)

\[
R := \text{PolynomialRing}([x, y, z]); F := [5 \cdot x^2 + 2 \cdot x \cdot z^2 + 5 \cdot y^6 + 5 \cdot y^5 \cdot z^2 - 15 \cdot y^4 - 5 \cdot y^5 - 5 \cdot y^3] \\
\text{RealTriangularize}(F, R, \text{output = record});
\]

\[
\begin{align*}
5 x^2 + 2 z^2 x + 5 y^6 + 15 y^4 - 5 y^3 - 15 y^5 + 5 z^2 &= 0 \\
25 y^6 - 75 y^5 + 75 y^4 - 25 y^3 + 25 z^2 &< 0
\end{align*}
\]

\[
\begin{align*}
5 x + z^2 &= 0 \\
25 y^6 - 75 y^5 + 75 y^4 - 25 y^3 + 25 z^2 &= 0 \\
64 z^4 - 1600 z^2 + 25 &> 0 \\
z - 5 &\neq 0 \\
z + 5 &\neq 0 \\
\end{align*}
\]

\[
\begin{align*}
x + 5 &= 0, & y - 1 &= 0, & z + 5 &= 0, & 5 x + z^2 &= 0 \\
y &= 0, & y - 1 &= 0, & z + 5 &= 0, & 2 y - 1 &= 0 \\
z - 5 &= 0, & z &= 0, & z - 5 &= 0, & 64 z^4 - 1600 z^2 + 25 &= 0
\end{align*}
\]
Plan

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Triangular Set

Definition

$T \subset k[x_n > \cdots > x_1]$ is a *triangular set* if $T \cap k = \emptyset$ and $\text{mvar}(p) \neq \text{mvar}(q)$ for all $p, q \in T$ with $p \neq q$.

Theorem (J.F. Ritt, 1932)

Let $V \subset K^n$ be an irreducible variety and $F \subset k[x_1, \cdots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.

$$(\forall g \in \langle F \rangle) \; \text{prem}(g, T) = 0.$$  

Theorem (W.T. Wu, 1987)

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Regular chain

Definition

Let $T \subset k[x_n > \cdots > x_1]$ be a triangular set. For all $t \in T$ write $\text{init}(t) := \text{lc}(t, \text{mvar}(t))$ and $h_T := \prod_{t \in T} \text{init}(t)$. The quasi-component and saturated ideal of $T$ are:

$$W(T) := V(T) \setminus V(h_T) \quad \text{and} \quad \text{sat}(T) = \langle T \rangle : h_T^\infty$$

Theorem (F. Boulier, F. Lemaire and M.M.M. 2006)

We have: $\overline{W(T)} = V(\text{sat}(T))$. Moreover, if $\text{sat}(T) \neq \langle 1 \rangle$ then $\text{sat}(T)$ is strongly equi-dimensional.

Definition (M. Kalkbrner, 1991 - L. Yang, J. Zhang 1991)

$T$ is a regular chain if $T = \emptyset$ or $T := T' \cup \{t\}$ with $\text{mvar}(t)$ maximum s.t.

- $T'$ is a regular chain,
- $\text{init}(t)$ is regular modulo $\text{sat}(T')$
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Regular chain: alternative definition
Regular chain: algorithmic properties

Definition

Let $T \subset k[x_n > \cdots > x_1]$ be a triangular set and $p \in k[x_n > \cdots > x_1]$. If $T$ is empty then, the \textit{iterated resultant} of $p$ w.r.t. $T$ is $\text{res}(T, p) = p$. Otherwise, writing $T = T_{<w} \cup T_w$

\[
\text{res}(T, p) = \begin{cases} 
p  
\text{res}(T_{<w}, \text{res}(T_w, p, w)) & \text{if } \deg(p, w) = 0 
\end{cases}
\]

\text{Theorem (P. Aubry, D. Lazard, M.M.M.)}

$T$ is a regular chain iff

$$\{ p \mid \text{prem}(p, T) = 0 \} = \text{sat}(T)$$

\text{Theorem (L. Yang, J. Zhang 1991)}

$p$ is regular modulo $\text{sat}(T)$ iff

$$\text{res}(T, p) \neq 0$$
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Regular chain: algorithmic properties

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Triangular decomposition of an algebraic variety

**Kalkbrener triangular decomposition**

Let \( F \subset k[x] \). A family of regular chains \( T_1, \ldots, T_e \) of \( k[x] \) is called a Kalkbrener triangular decomposition of \( V(F) \) if

\[
V(F) = \bigcup_{i=1}^{e} V(\text{sat}(T_i)).
\]

**Wu-Lazard triangular decomposition**

Let \( F \subset k[x] \). A family of regular chains \( T_1, \ldots, T_e \) of \( k[x] \) is called a Wu-Lazard triangular decomposition of \( V(F) \) if

\[
V(F) = \bigcup_{i=1}^{e} W(T_i).
\]
Triangular decomposition of an algebraic variety

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Triangularize applied to sofa and cylinder \((1/2)\)

\[ x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0 \]
Triangularize applied to sofa and cylinder (2/2)

> R := PolynomialRing([z, y, x]): F := [x^2+y^3+z^5, x^4+z^2-1]: dec :=
   Triangularize(F, R): map(Display, dec, R);

\[
\begin{align*}
(-2 x^4 + x^8 + 1) z + x^2 + y^3 &= 0 \\
y^6 + 2 x^2 y^3 + 10 x^{12} - 10 x^8 + x^{20} - 5 x^{16} + 6 x^4 - 1 &= 0 \\
-2 x^4 + x^8 + 1 &
eq 0
\end{align*}
\]

> dec := Triangularize(F, R, output=lazard): map(Display, dec, R);

\[
\begin{align*}
(-2 x^4 + x^8 + 1) z + x^2 + y^3 &= 0 \\
y^6 + 2 x^2 y^3 + 10 x^{12} - 10 x^8 + x^{20} - 5 x^{16} + 6 x^4 - 1 &= 0 \\
-2 x^4 + x^8 + 1 &
eq 0
\end{align*}
\]

\[
\begin{align*}
z &= 0 \\
y - 1 &= 0 \\
x^2 + 1 &= 0
\end{align*}
\]

\[
\begin{align*}
z &= 0 \\
y^2 - y + 1 &= 0 \\
x + 1 &= 0
\end{align*}
\]

\[
\begin{align*}
z &= 0 \\
y^2 - y + 1 &= 0 \\
x - 1 &= 0
\end{align*}
\]

\[
\begin{align*}
z &= 0 \\
y + 1 &= 0 \\
x + 1 &= 0
\end{align*}
\]
Plan

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Regular chain: specialization properties

Notation

Let $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ be a regular chain with $y := \{\text{mvar}(t) \mid t \in T\}$ and $u := x \setminus y = u_1, \ldots, u_d$. Hence $\text{sat}(T)$ has dimension $d$.

- Recall that $h_T$ is the product of the $\text{init}(t)$, for $t \in T$.
- Denote by $s_T$ the product of the $\text{discrim}(t, \text{mvar}(t))$.

Definition

We say that $T$ specializes well at a point $u \in \mathbb{R}^d$ if $h_T(u) \neq 0$ and the triangular set $T(u)$ is a regular chain generating a radical ideal.

Theorem (X. Hou, B. Xia, L. Yang, 2001)

Define $BP_T := \text{res}(T, h_T) \text{res}(T, s_T)$, the border polynomial of $T$. Then

- $T$ specializes well at $u \in \mathbb{R}^d$ if and only if $BP_T(u) \neq 0$.
- For each connected component $C$ of $BP_T(u) \neq 0$, the number of real solutions of $T(u)$ is constant for $u \in C$. 

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Regular chain: specialization properties

Notation

Let \( T \subset \mathbb{Q}[x_1 < \ldots < x_n] \) be a regular chain with \( y := \{ \text{mvar}(t) \mid t \in T \} \) and \( u := x \setminus y = u_1, \ldots, u_d \). Hence \( \text{sat}(T) \) has dimension \( d \).

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Definition

We say that \( T \) specializes well at a point \( u \in \mathbb{R}^d \) if \( h_T(u) \neq 0 \) and the triangular set \( T(u) \) is a regular chain generating a radical ideal.

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- For each connected component \( C \) of \( BP_T(u) \neq 0 \), the number of real solutions of \( T(u) \) is constant for \( u \in C \).
Border polynomial and specialization

Example (bad specialization of a regular chain)

\[
T := \begin{cases} 
  x_4 x_5^2 + 2x_5 + 1 \\
  (x_1 + x_2)x_3^2 + x_3 + 1 \\
  x_1^2 - 1. 
\end{cases}
\]

\[T_{x_2, x_4 = -1, 1} := \begin{cases} 
  x_5^2 + 2x_5 + 1 \\
  (x_1 - 1)x_3^2 + x_3 + 1 \\
  x_1^2 - 1. 
\end{cases}\]

Example (border polynomial)

\[\text{res}(\text{dis}(t_2), t_1) \text{res}(\text{dis}(t_3), t_2), t_1) \cdot \text{res}(\text{init}(t_2), t_1) \text{res}(\text{init}(t_3), t_2), t_1).\]

For the above regular chain, it is

\[(4x_2 + 3)(4x_2 - 5)(x_2^2 - 1)(x_4 - 1)x_4\]
Border polynomial and specialization

Example (bad specialization of a regular chain)

\[ T := \begin{cases} 
  x_4 x_5^2 + 2x_5 + 1 \\
  (x_1 + x_2)x_3^2 + x_3 + 1 \\
  x_1^2 - 1.
\end{cases} \]

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  (x_1 - 1)x_3^2 + x_3 + 1 \\
  x_1^2 - 1.
\end{cases} \]

Example (border polynomial)

\[ \text{res}(\text{dis}(t_2), t_1) \text{res}(\text{res}(\text{dis}(t_3), t_2), t_1). \]

For the above regular chain, it is

\[ (4x_2 + 3)(4x_2 - 5)(x_2^2 - 1)(x_4 - 1)x_4 \]
Regular semi-algebraic system

Notation

- Let $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ be a regular chain with $\mathbf{y} := \{ \text{mvar}(t) \mid t \in T \}$ and $\mathbf{u} := \mathbf{x} \setminus \mathbf{y} = u_1, \ldots, u_d$.
- Let $P$ be a finite set of polynomials, s.t. every $f \in P$ is regular modulo $\text{sat}(T)$.
- Let $Q$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{u}]$.

Definition

We say that $R := [Q, T, P]$ is a regular semi-algebraic system if:

(i) $Q$ defines a non-empty open semi-algebraic set $S$ in $\mathbb{R}^d$,
(ii) the regular system $[T, P]$ specializes well at every point $u$ of $S$,
(iii) at each point $u$ of $S$, the specialized system $[T(u), P(u)]$ has at least one real solution.

Define

$$Z_\mathbb{R}(R) = \{(u, y) \mid Q(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}. $$
Regular semi-algebraic system

Notation

- Let $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ be a regular chain with $y := \{ \text{mvar}(t) \mid t \in T \}$ and $u := x \setminus y = u_1, \ldots, u_d$.
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(iii) at each point $u$ of $S$, the specialized system $[T(u), P(u)]$ has at least one real solution.

Define

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid Q(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$
Example

The system $[Q, T, P>]$, where

\[ Q := a > 0, \quad T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases}, \quad P> := \{y > 0\} \]

is a regular semi-algebraic system.
Proposition

Let $R := [Q, T, P]$ be a regular semi-algebraic system of $\mathbb{Q}[u_1, \ldots, u_d, y]$. Then the zero set of $R$ is a nonempty semi-algebraic set of dimension $d$.

Theorem

Every semi-algebraic system $S$ can be decomposed as a finite union of regular semi-algebraic systems such that the union of their zero sets is the zero set of $S$. We call it a (full) triangular decomposition of $S$.
Notation

Let $S = [F, N_{\geq}, P_>, H\neq]$ be a semi-algebraic system of $\mathbb{Q}[x]$. Let $c$ be the dimension of the constructible set of $\mathbb{C}^n$ corresponding to $S$.

Definition

A finite set of regular semi-algebraic systems $R_i$ is called a lazy triangular decomposition of $S$ if

- for each $i$, $Z_{\mathbb{R}}(R_i) \subseteq Z_{\mathbb{R}}(S)$ holds, and
- there exists $G \subset \mathbb{Q}[x]$ such that

$$Z_{\mathbb{R}}(S) \setminus (\bigcup_{i=1}^t Z_{\mathbb{R}}(R_i)) \subseteq Z_{\mathbb{R}}(G),$$

where the complex zero set $V(G)$ has dimension less than $c$. 

(CDMMXX)
A detailed example

Original problem
Consider the following question (Brown, McCallum, ISSAC’05): when does \( p(z) = z^3 + az + b \) have a non-real root \( x + iy \) satisfying \( xy < 1 \).

The equivalent quantifier elimination problem
\[
(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[f = g = 0 \land y \neq 0 \land xy - 1 < 0],
\]
where
- \( f = \text{Re}(p(x + iy)) = x^3 - 3xy^2 + ax + b \)
- \( g = \text{Im}(p(x + i)) / y = 3x^2 - y^2 + a \)

The semi-algebraic system to solve
\[
S := \begin{cases}
    f = 0, \\
    g = 0, \\
    y \neq 0, \\
    xy - 1 < 0
\end{cases}
\]
A lazy triangular decomposition

The command \texttt{LazyRealTriangularize}([\(f, g, y \neq 0, xy - 1 < 0\), [\(y, x, b, a\)]) returns the following:

\[
\begin{align*}
\text{\%LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, h_1 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & \quad h_1 > 0, h_2 \neq 0 \\
\text{\%LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, h_2 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & \quad h_2 = 0 \\
\text{[ ]} & \quad \text{otherwise}
\end{align*}
\]

where

\[
\begin{align*}
t_1 &= 8x^3 + 2ax - b, \\
h_1 &= 4a^3 + 27b^2, \\
h_2 &= -4a^3b^2 - 27b^4 + 16a^4 + 512a^2 + 4096.
\end{align*}
\]
A full triangular decomposition

Evaluate the output with the \texttt{value} command, which yields

\[
\begin{cases}
[t_1 = 0, t_2 = 0, 1 - xy > 0] & h_1 > 0, h_2 \neq 0 \\
[ ] & h_1 = 0 \\
[t_3 = 0, t_4 = 0, h_2 = 0] & h_2 = 0 \\
[ ] & \text{otherwise}
\end{cases}
\]

where

\[
t_3 = (2a^3 + 32a + 18b^2)x - a^2b - 48b
\]

\[
t_4 = xy + 1
\]

\[
h_1 = 4a^3 + 27b^2,
\]

\[
h_2 = -4a^3b^2 - 27b^4 + 16a^4 + 512a^2 + 4096
\]
Plan

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Outline of the algorithm

Definition
Let \([T, P]\) be as before and \(B \subset \mathbb{Q}[u]\). We say that \([B \neq, T, P >]\) is a \textit{pre-regular semi-algebraic system} of \(\mathbb{Q}[u, y]\) if \([T, P]\) specializes well at every point of \(B(u) \neq 0\).

Computation in complex space
\[
Z_\mathbb{R}(F, N_\geq, P >, H_\neq) \\
\downarrow \\
\bigcup Z_\mathbb{R}(B \neq, T, P >)
\]

Computation in real space
\[
[B \neq, T, P >] \\
\downarrow \\
Q := \exists y \ (B(u) \neq 0, T(u, y) = 0, P(u, y) > 0) \\
\downarrow \\
\text{output} \ [Q, T, P >], \ \text{where} \ Q \neq \text{false}
\]
Fingerprint polynomial set

**Definition**

Let \( R := [B_{\neq}, T, P_>] \). Let \( D \subset \mathbb{Q}[u] \). Let \( dp \) and \( b \) be the product of \( D \) and \( B \). We call \( D \) a **fingerprint polynomial set** (FPS) of \( R \) if:

(i) for all \( \alpha \in \mathbb{R}^d, b \in B: dp(\alpha) \neq 0 \Rightarrow b(\alpha) \neq 0 \),

(ii) for all \( \alpha, \beta \in \mathbb{R}^d \) with \( \alpha \neq \beta \) and \( dp(\alpha) \neq 0, dp(\beta) \neq 0 \), if for \( p \in D \), \( \text{sign}(p(\alpha)) = \text{sign}(p(\beta)) \), then \( R(\alpha) \) has real solutions iff \( R(\beta) \) does.

Open projection operator (Brown-McCalumn operator)

Let \( A \) be a squarefree basis in \( \mathbb{Q}[u_1 < \cdots < u_d] \). Define

\[
\text{oproj}(A, u_d) := \bigcup_{f \in A} \text{lcr}(f, u_d) \cup \bigcup_{f \in A} \text{discrim}(f, u_d) \cup \bigcup_{f, g \in A} \text{res}(f, g, u_d).
\]

**Theorem**

For \( A \subset \mathbb{Q}[u_1, \ldots, u_d] \), let \( \text{oaf}(A) = \text{der}(A, u_d) \cup \text{oaf}(\text{oproj}(\text{der}(A, u_d), u_{d-1})) \).

If \( R := [B_{\neq}, T, P_>] \) is a PRSAS, then, \( \text{oaf}(B) \) is a fingerprint polynomial set.
Fingerprint polynomial set

Definition

Let \( R := [B \neq, T, P >] \). Let \( D \subset \mathbb{Q}[u] \). Let \( dp \) and \( b \) be the product of \( D \) and \( B \). We call \( D \) a fingerprint polynomial set (FPS) of \( R \) if:

(i) for all \( \alpha \in \mathbb{R}^d \), \( b \in B \): \( dp(\alpha) \neq 0 \Rightarrow b(\alpha) \neq 0 \),

(ii) for all \( \alpha, \beta \in \mathbb{R}^d \) with \( \alpha \neq \beta \) and \( dp(\alpha) \neq 0 \), \( dp(\beta) \neq 0 \), if for \( p \in D \), \( \text{sign}(p(\alpha)) = \text{sign}(p(\beta)) \), then \( R(\alpha) \) has real solutions iff \( R(\beta) \) does.

Open projection operator (Brown-McCalumn operator)

Let \( A \) be a squarefree basis in \( \mathbb{Q}[u_1 < \cdots < u_d] \). Define

\[
\text{oproj}(A, u_d) := \bigcup_{f \in A} \text{lc}(f, u_d) \cup \bigcup_{f \in A} \text{discrim}(f, u_d) \cup \bigcup_{f, g \in A} \text{res}(f, g, u_d).
\]

Theorem

For \( A \subset \mathbb{Q}[u_1, \ldots, u_d] \), let \( \text{oaf}(A) = \text{der}(A, u_d) \cup \text{oaf}(\text{oproj}(\text{der}(A, u_d), u_{d-1})) \). If \( R := [B \neq, T, P >] \) is a PRSAS, then, \( \text{oaf}(B) \) is a fingerprint polynomial set.
A detailed example (1/3)

RealTriangularize, RefineBox, RefineListBox, RemoveRedundantComponents, RepresentingBox, RepresentingChain, RepresentingQuantifierFreeFormula, RepresentingRootIndex, VariableOrdering

> F := [x^2+y+a, y^2+b*x+a];
  \[ F := \left[y + x^2 + a, y^2 + b \cdot x + a\right] \]

> R := PolynomialRing([y, x, b, a]);
  \[ R := \text{polynomial\_ring} \]

> dec := Triangularize(F, R);
  \[ \text{dec := \{regular\_chain}\} \]
  \[ \text{rc := regular\_chain} \]

> Display(rc, R);
  \[ \begin{align*}
  y + x^2 + a &= 0 \\
  x^4 + 2a x^2 + b x + a^2 + a &= 0
  \end{align*} \]

> rceqs := Info(rc, R);
  \[ \text{rceqs := \{y + x^2 + a, x^4 + 2 a x^2 + b x + a^2 + a\}} \]

> bp := discriminant(rceqs[2], x);
  \[ \text{bp := 256 \cdot b^2 \cdot a^3 - 27 \cdot b^4 + 288 \cdot b^2 \cdot a^2 + 256 \cdot a^4 + 256 \cdot a^3} \]

> with(plots);
A detailed example (2/3)

4 real roots

2 real roots

No real roots
A detailed example (3/3)

\[
\begin{align*}
\text{polynomial\_ring} & \quad [x^2 + y + a, y^2 + bx + a] \\
\text{dec := Triangularize}(F, R) : \text{rc := dec[1]} : \text{Display}(rc, R); \\
\begin{cases} 
\quad y + x^2 + a = 0 \\
\quad x^4 + 2ax^2 + bx + a^2 + a = 0 \\
\end{cases} \\
\quad y + x^2 + a = 0
\end{align*}
\]

\[
\begin{align*}
\text{LazyRealTriangularize}(F, R, \text{output = record}); \\
\begin{cases} 
\quad y + x^2 + a = 0 \\
\quad x^4 + 2ax^2 + bx + a^2 + a = 0 \\
\quad 256b^2a^3 - 27b^4 + 288b^2a^2 + 256a^4 + 256a^3 < 0 \\
\quad a \neq 0 \\
\end{cases} \\
\quad y + x^2 + a = 0
\end{align*}
\]

\[
\begin{align*}
\quad x^4 + 2ax^2 + bx + a^2 + a = 0 \\
\quad 256b^2a^3 - 27b^4 + 288b^2a^2 + 256a^4 + 256a^3 > 0 \\
\quad a < 0 \\
\end{align*}
\]

\[
= 0, y^2 + bx + a = 0, x^4 + 2ax^2 + bx + a^2 + a = 0, 256b^2a^3 - 27b^4 + 288b^2a^2 + 256a^4 + 256a^3 = 0, x^4 + 2ax^2 + bx + a^2 + a = 0, \text{polynomial\_ring, output = record})
\]

\[
%\text{LazyRealTriangularize}([y + x^2 + a = 0, y^2 + bx + a = 0, 256b^2a^3 - 27b^4 + 288b^2a^2 + 256a^4 + 256a^3 = 0, x^4 + 2ax^2 + bx + a^2 + a = 0], \text{polynomial\_ring, output = record})
\]
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LazyRealTriangularize for a system of equations

Algorithm 1: LazyRealTriangularize($S$)

**Input:** a semi-algebraic system $S = [F, \emptyset, \emptyset, \emptyset]$

**Output:** a lazy triangular decomposition of $S$

$\mathcal{T} := \text{Triangularize}(F)$

for $T_i \in \mathcal{T}$ do

$Bp_i := \text{BorderPolynomial}(T_i, \emptyset)$

solve $\exists y (Bp_i(u) \neq 0, T_i(u, y) = 0)$,

and let $Q_i$ be the resulting quantifier-free formula

if $Q_i \neq false$ then output $[Q_i, T_i, \emptyset]$
Complexity results (1/2)

Assumptions

(\(H_0\)) \(V(F)\) is equidimensional of dimension \(d\),

(\(H_1\)) \(x_1, \ldots, x_d\) are algebraically independent modulo each associated prime ideal of the ideal generated by \(F\) in \(\mathbb{Q}[x]\),

(\(H_2\)) \(F\) consists of \(m := n - d\) polynomials, \(f_1, \ldots, f_m\).

Geometrical formulation

Hypotheses (\(H_0\)) and (\(H_1\)) are equivalent to the existence of regular chains \(T_1, \ldots, T_e\) of \(\mathbb{Q}[x_1, \ldots, x_n]\) such that

- \(x_1, \ldots, x_d\) are free w.r.t. each \(T_i\)
- \(V(F) = V(\text{sat}(T_1)) \cup \ldots \cup V(\text{sat}(T_e))\).
Complexity results (2/2)

Notation
Let $n$, $m$, $\delta$, $\bar{\ell}$ be respectively the number of variables, number of polynomials, maximum total degree and height of polynomials in $F$.

Proposition
Within $m^{O(1)}(\delta^{O(n^2)})^{d+1} + \delta^{O(m^4)}O(n)$ operations in $\mathbb{Q}$, one can compute a Kalkbrener triangular decomposition $E_1, \ldots, E_e$ of $V(F)$, where each polynomial of each $E_i$

- has total degree upper bounded by $O(\delta^{2m})$,
- has height upper bounded by $O(\delta^{2m}(m\bar{\ell} + dm\log(\delta) + n\log(n)))$.

From which, a lazy triangular decomposition of $F$ can be computed in $(\delta^{n^2}n4^\delta)^{O(n^2)}\bar{\ell}^{O(1)}$ bit operations.
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### Table 1 Notions for Tables 2 and 3

<table>
<thead>
<tr>
<th>symbol</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>#e</td>
<td>number of equations in the system</td>
</tr>
<tr>
<td>#v</td>
<td>number of variables in the equations</td>
</tr>
<tr>
<td>d</td>
<td>max total degree of the equations</td>
</tr>
<tr>
<td>G</td>
<td>Groebner:-Basis (with plex order) in <strong>MAPLE 13</strong></td>
</tr>
<tr>
<td>T</td>
<td>Triangularize in <strong>RegularChains</strong> library of <strong>MAPLE</strong></td>
</tr>
<tr>
<td>LR</td>
<td>lazy <strong>RealTriangularize</strong> implemented in <strong>MAPLE</strong></td>
</tr>
<tr>
<td>R</td>
<td>complete <strong>RealTriangularize</strong> implemented in <strong>MAPLE</strong></td>
</tr>
<tr>
<td>Q</td>
<td><strong>QePCAD b</strong></td>
</tr>
<tr>
<td>&gt; 1h</td>
<td>the examples cannot be solved in 1 hour</td>
</tr>
<tr>
<td>FAIL</td>
<td><strong>QePCAD b</strong> failed due to prime list exhausted</td>
</tr>
</tbody>
</table>
### Table 2: Timings for algebraic varieties

<table>
<thead>
<tr>
<th>system</th>
<th>#v/#e/d</th>
<th>G</th>
<th>T</th>
<th>LR</th>
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</thead>
<tbody>
<tr>
<td>Hairer-2-BGK</td>
<td>13/11/4</td>
<td>25</td>
<td>1.924</td>
<td>2.396</td>
</tr>
<tr>
<td>Collins-jsc02</td>
<td>5/4/3</td>
<td>876</td>
<td>0.296</td>
<td>0.820</td>
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<tr>
<td>Leykin-1</td>
<td>8/6/4</td>
<td>103</td>
<td>3.684</td>
<td>3.924</td>
</tr>
<tr>
<td>8-3-config-Li</td>
<td>12/7/2</td>
<td>109</td>
<td>5.440</td>
<td>6.360</td>
</tr>
<tr>
<td>Lichtblau</td>
<td>3/2/11</td>
<td>126</td>
<td>1.548</td>
<td>11</td>
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<td>60</td>
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# Timings for semi-algebraic systems

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<tr>
<th>system</th>
<th>#v/#e/d</th>
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<th>LR</th>
<th>R</th>
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Plan

1. Solving systems of polynomial equations
2. Triangular decomposition of algebraic systems
3. Triangular decomposition of a semi-algebraic system
4. Algorithm
5. Complexity analysis
6. Benchmarks
7. Concluding remarks
8. Applications
9. Cylindrical algebraic decomposition: basic ideas
Conclusion

- We have proposed adaptations of the notions of regular chains and triangular decompositions in order to solve semi-algebraic systems symbolically.

- We have shown that any such system can be decomposed into finitely many *regular semi-algebraic systems*.

- We propose two specifications of such a decomposition and present corresponding algorithms:

  - Under some assumptions, one type of decomposition (LazyRealTriangularize) can be computed in *singly exponential time* w.r.t. the number of variables.

- We have implemented both types of decompositions and reported on comparative benchmarks.

- Our experimental results suggest that these approaches are promising.
Work in progress

- We have obtained geometrical invariants for the notion of border polynomial.
- We have improved the performances of our algorithms by avoiding unnecessary recursive calls.
- We have developed an incremental algorithms for decomposing semi-algebraic systems.
- We have procedures for performing set theoretical operations on semi-algebraic sets.
- As a consequence we can produce decomposition free of redundant components.
Thank you!
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(CDMMXX) RealTriangularize
Laurent’s model for the mad cow disease (1/4)

The dynamical system ruling the transformation

The normal form $PrP^C$ is harmless, while the infectious form $PrP^S_c$ catalyzes a transformation from the normal form to the infectious one.

\[
\begin{align*}
\frac{dx}{dt} &= k_1 - k_2 x - ax \frac{(1+by^n)}{1+cy^n} \\
\frac{dy}{dt} &= ax \frac{(1+by^n)}{1+cy^n} - k_4 y
\end{align*}
\]

where $x = [PrP^C], y = [PrP^S_c]$ and where $b, c, n, a, k_4, k_1$ are biological constants which can be set as follows:

\[b = 2, \quad c = 1/20, \quad n = 4, \quad a = 1/10, \quad k_4 = 50 \quad \text{and} \quad k_1 = 800.\]

The dynamical system to study

\[
\begin{align*}
\frac{dx}{dt} &= \frac{16000+800y^4-20k_2x-k_2xy^4-2x-4xy^4}{20+y^4} \\
\frac{dy}{dt} &= \frac{2(x+2xy^4-500y-25y^5)}{20+y^4}
\end{align*}
\]
Laurent’s model for the mad cow disease (2/4)

The semi-algebraic system to be solved

\[ S := \begin{cases} 
16000 + 800y^4 - 20k_2x - k_2xy^4 - 2x - 4xy^4 & = 0 \\
2(x + 2xy^4 - 500y - 25y^5) & = 0 \\
k_2 & > 0
\end{cases} \]

Computations (1/5)

LazyRealTriangularize to this system, yields the following regular semi-algebraic system (and unevaluated recursive calls)

\[ \begin{cases} 
(2y^4 + 1)x - 500y - 25y^5 = 0 \\
(k_2 + 4)y^5 - 64y^4 + (20k_2 + 2)y - 32 = 0 \\
(k_2 > 0) \land (R_1 \neq 0)
\end{cases} \]

where

\[ R_1 = 100000k_2^8 + 1250000k_2^7 + 5410000k_2^6 + 8921000k_2^5 - 9161219950k_2^4 \\
- 5038824999k_2^3 - 1665203348k_2^2 - 882897744k_2 + 1099528405056. \]
Laurent’s model for the mad cow disease (3/4)

Computations (2/5)

Through the computation of sample points, we easily obtain the following observation. Whenever $R_1 > 0$ holds, the system has 1 equilibrium, while $R_2 < 0$ implies that the system has 3 equilibria.

Computations (3/5)

Now we study the stability of those equilibria. To this end, we consider the two Hurwitz determinants. Adding to $S$ the constraints \{\[\Delta_1 > 0\text{, }a_2 > 0\}\}

\[
\Delta_1 = 54y^8 + 40k_2y^4 + 2082y^4 - 312xy^3 + 20040 + k_2y^8 + 400k_2,
\]

\[
a_2 = 20000k_2 + 2000 + 50k_2y^8 + 200y^8 + 2000k_2y^4 - 312k_2xy^3 + 4100y^4.
\]

we obtain a new semi-algebraic system $S'$. 
Laurent’s model for the mad cow disease (4/4)

Computations (4/5)

Applying LazyRealTriangularize to $S'$ in conjunction with sample point computations brings the following conclusion. If $R_1 > 0$, then the system has 1 asymptotically stable hyperbolic equilibria.

Computations (5/5)

If $R_1 < 0$ and $R_2 \neq 0$, then System has 2 asymptotically equilibria, where $R_2$ is given by:

\[
R_2 = 10004737927168k_2^9 + 624166300700672k_2^8 + 7000539052537600k_2^7
+ 45135589467012800k_2^6 - 840351411856453750k_2^5 - 50098004352248446875k_2^4
- 273881689894550000000k_2^3 - 86752092666960000000k_2^2
+ 1029609173568000000000k_2 + 5932546064102400000000.
\]

To further investigate the number of asymptotically stable hyperbolic equilibria on the hypersurface $R_2 = 0$ and the equilibria when $R_1 = 0$, one can apply SamplePoints on $S'$, which produces 14 points.
Program verification: an example from Lafferriere (1/4)

Reachability computation

This problem reduces to determine the set

\[
\{(y_1, y_2) \in \mathbb{R}^2 \mid (\exists a \in \mathbb{R})(\exists z \in \mathbb{R}) (0 \leq a) \land (z \geq 1) \land (h_1 = 0) \land (h_2 = 0)\}
\]

where

\[
h_1 = 3y_1 - 2a(-z^4 + z) \quad \text{and} \quad h_2 = 2y_2z^2 - a(z^4 - 1).
\]

The semi-algebraic system to be solved

One wishes to compute the projection of the semi-algebraic set defined by

\[
(0 \leq a) \land (z \geq 1) \land (h_1 = 0) \land (h_2 = 0)
\]

onto the \((y_1, y_2)\)-plane.

For the variable ordering \(a > z > y_1 > y_2\). we obtain the five following regular semi-algebraic systems \(R_1\) to \(R_5\)
The triangular decomposition (1/3)

\[ R_2^T = \begin{cases} \frac{a}{y_1} \\ y_2 \end{cases} \quad R_3^T = \begin{cases} z - 1 \\ y_1 \\ y_2 \end{cases} \quad R_4^T = \begin{cases} a \\ z - 1 \\ y_1 \\ y_2 \end{cases} \]

The projection on the \((y_1, y_2)\)-plane of \(Z_R(R_2) \cup Z_R(R_3) \cup Z_R(R_4)\) is clearly equal to the \((y_1, y_2) = (0, 0)\) point.
The triangular decomposition (2/3)

\[ R_1^T = \begin{cases} 
4y_2z^5 + 4y_2z^4 + (3y_1 + 4y_2)z^3 + 3y_1z^2 + 3y_1z + 3y_1 \\
(z^4 - 1)a - 2z^2y_2 
\end{cases} \]

\[ R_1^Q = \begin{cases} 
3y_1^5 - 6y_2y_1^4 - 63y_2^2y_1^3 + 192y_2^3y_1^2 + 112y_2^4y_1 + 16y_2^5 
\neq 0 \\
(y_1 + y_2 < 0) \land (y_1 < 0) \land (0 < y_2) 
\end{cases} \]

\[ R_1^P = \{ z > 1 \} \]

The projection on the \((y_1, y_2)\)-plane of \(Z_\mathbb{R}(R_1)\) is given by \(Z_\mathbb{R}(R_1^Q)\).
The triangular decomposition (3/3)

\[
R^T_5 = \begin{cases} 
(z^4 - 1) a - 2 z^2 y_2 \\
3 y_1^5 - 6 y_2 y_1^4 - 63 y_2^2 y_1^3 + 192 y_2^3 y_1^2 + 112 y_2^4 y_1 + 16 y_2^5 
\end{cases}
\]

where \( t_z \) is a large polynomial of degree 4 in \( z \).

The polynomial with main variable \( y_1 \), say \( t_{y_1} \) is delineable above \( 0 < y_2 \).

Using a sample point we check that \( t_{y_1} \) admits a single real root.

Conclusion

It follows that the projection on the \((y_1, y_2)\)-plane of \( Z_\mathbb{R}(R_5) \) is given by:

\[(0 < y_2) \wedge (3 y_1^5 - 6 y_2 y_1^4 - 63 y_2^2 y_1^3 + 192 y_2^3 y_1^2 + 112 y_2^4 y_1 + 16 y_2^5).\]
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Recall: cylindrical algebraic decomposition of \( \{ax^2 + bx + c\} \)

The cylindrical algebraic decomposition of \( \{ax^2 + bx + c\} \) is given by the tree above, where \( t = bx + c \), \( q = 2ax + b \), and \( r = 4ac - b^2 \). This is the best possible output for that method.
Cylindrical algebraic decomposition of $\mathbb{R}^n$ (1/2)

**Definition**

A CAD of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$, where

- all the cells are **cylindrically** arranged, that is, for all $1 \leq j < n$ the projections on the first $j$ coordinates $(x_1, \ldots, x_j)$ of any two cells are either identical or disjoint.
- each cell is a **connected semi-algebraic** subset, called a region

**Complexity of CAD**

Unfortunately the number of cells can be **doubly exponential** in $n$.

**Case of $n = 1$**

This is a finite partition of the real line into points and open intervals.
Cylindrical algebraic decomposition of $\mathbb{R}^n$ (2/2)

Case of $n > 1$

From a CAD $D'$ of $\mathbb{R}^{n-1}$, one builds a CAD $D$ of $\mathbb{R}^n$. Above each $R \in D'$:

- consider finitely many disjoint graphs (called *sections*) of continuous real-valued algebraic functions,
- decomposing the cylinder $R \times \mathbb{R}^1$, into sections and *sectors* (located between two consecutive sections), which form a stack over $R$,
- then all the sections and sectors are the elements of $D$.

![Diagram](https://via.placeholder.com/150)
A Cylindrical Algebraic Decomposition of $\mathbb{R}^2$ Induced by the Tacnode Curve

Tacnode curve: $y^4 - 2y^3 + y^2 - 3x^2y + 2x^4 = 0$. 
RealTriangularize applied to the Tacnode Curve

> R := PolynomialRing([x,y]);
> F := [y^4-2*y^3+y^2-3*x^2*y+2*x^4];
> RealTriangularize(F, R, output=record);
{ 4 2 4 3 2
 { 2 x - 3 y x + y - 2 y + y = 0
 { 0 < y { x = 0
 { y - 1 <> 0 { y = 0
 { 2
 { 8 y - 16 y < 1

 { x = 0 { 2
 { 32 y x - 48 y - 3 = 0
 { 2
 { 8 y - 16 y - 1 = 0

(CDMMXX)