Quantifier Elimination via Triangular Decomposition

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Quantifier Elimination (QE)

Let \( F(x_1, \ldots, x_n) \) be a quantifier-free formula in disjunctive normal form. Let \( P := (Q_1(x_1, \ldots, x_n), \ldots, Q_m(x_1, \ldots, x_n)) \) be a prenex formula, where each \( Q_i \) is either the existential quantifier (\( \exists \)) or the universal quantifier (\( \forall \)). The goal of Quantifier Elimination (QE) is to compute a quantifier-free formula \( S(x_1, \ldots, x_n) \) such that \( SF \iff PF \) holds for all \( x_1, \ldots, x_n \).

As an example, consider
\[
(3x)(\forall y)(ax^2 + bx + c) - (ay^2 + by + c) \geq 0
\]
for which QE yields
\[
(a < 0) \lor (a = b = 0).
\]

In many research articles and computer software (QEPCAD, Mathematica, Reduce) QE is performed via cylindrical algebraic decomposition.

Cylindrical Algebraic Decomposition

A cylindrical algebraic decomposition (CAD) of \( \mathbb{R}^2 \) such that \( y^2 + y - x \) is sign invariant in each of the 9 cells is shown below.

Motivation

In 1973, Collins introduced the first algorithm for computing CAD via a projection and lifting scheme. The projection phase relies on a global projection operator, which is a function defined independently of the input system, to eliminate variables. A strong projection operator usually produces much more polynomials than needed while a weak projection operator may fail for non-generic cases.

At ISSAC’09, the authors (with B. Xia and L. Yang) proposed a new scheme for computing CAD based on triangular decomposition (TCAD), which replaces the projection phase by computing a complex cylindrical tree. It is a natural question to ask how QE can be done via TCAD.

Main Contribution

We propose an operation which refines a given complex cylindrical tree with a new polynomial. This yields an incremental algorithm for computing the cylindrical decomposition of the complex space. An existing CAD can be refined through this operation. Moreover, QE is achieved based on refined CADs. This operation can fully take advantage of equational constraints.

Refining a Complex Cylindrical Tree

Fig 1 shows a \( y^2 + x \)-invariant cylindrical decomposition of \( \mathbb{C}^2 \), which is represented by a tree. This tree can be refined by the polynomial \( y^2 + y \) into a \( (y^2 + x, y + y + y^2 + x) \)-invariant cylindrical decomposition of \( \mathbb{C}^2 \), see Fig. 2.

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Experimentation and Application

Results with a preliminary implementation in the MAPLE’s RegularChains library.

<table>
<thead>
<tr>
<th>System</th>
<th>TCAD(#)</th>
<th>CAD(#)</th>
<th>QCAD(#)</th>
<th>QCAD(#)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maple</td>
<td>1087</td>
<td>1017</td>
<td>9.945</td>
<td>10.448</td>
</tr>
<tr>
<td>Maple</td>
<td>11.881</td>
<td>11.236</td>
<td>11.881</td>
<td>11.236</td>
</tr>
</tbody>
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Consider a dynamical system with a control parameter \( a \). We’d like to know if there exists a control parameter function of the form \( a = \alpha + \beta \). We’ll examine the function of the form \( a = \alpha + \beta \).

The input control system.

\[
S := \{ \text{diff}(x(t), t) = -\alpha x(t), \text{diff}(y(t), t) = -\beta y(t) \} \]

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