

# **Semi-algebraic description of the equilibria of dynamical systems**

Changbo Chen and Marc Moreno Maza

ORCCA, University of Western Ontario, Canada

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## Outline

- 1 Motivation: a biochemical network
- 2 A new tool for solving parametric polynomial systems
- 3 Study the equilibria of dynamical systems symbolically

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## Mad cow disease

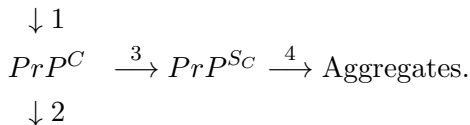


[http://x-medic.net/infections/  
bovine-spongiform-encephalopathy/attachment/mad-cow-disease](http://x-medic.net/infections/bovine-spongiform-encephalopathy/attachment/mad-cow-disease)

## A mad cow disease model (M. Laurent, 1996)

**Hypothesis:** the mad cow disease is spread by prion proteins.

The kinetic scheme

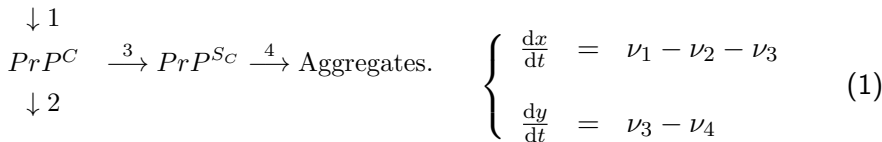


- $PrP^C$  (resp.  $PrP^{Sc}$ ) is the normal (resp. infectious) form of prions
- Step 1 (resp. 2) : the synthesis (resp. degradation) of native  $PrP^C$
- Step 3 : the transformation from  $PrP^C$  to  $PrP^{Sc}$
- Step 4 : the formation of aggregates

**Question:** Can a *small amount of  $PrP^{Sc}$*  cause prion disease?

## The dynamical system governing the reaction network

- Let  $x$  and  $y$  be respectively the concentrations of  $PrP^C$  and  $PrP^{Sc}$ .
- Let  $\nu_i$  be the rate of Step  $i$  for  $i = 1, \dots, 4$ .
- $\nu_1 = k_1$  for some constant  $k_1$ .
- $\nu_2 = k_2x$  and  $\nu_4 = k_4y$ .
- $\nu_3 = ax \frac{(1+by^n)}{1+cy^n}$ .



## The simplified dynamical system by experimental values

Experiments (M. Laurent 96) suggest to set  $b = 2$ ,  $c = 1/20$ ,  $n = 4$ ,  $a = 1/10$ ,  $k_4 = 50$  and  $k_1 = 800$ . Now we have:

$$\begin{cases} \frac{dx}{dt} = f_1 \\ \frac{dy}{dt} = f_2 \end{cases} \quad \text{with} \quad \begin{cases} f_1 = \frac{16000 + 800y^4 - 20k_2x - k_2xy^4 - 2x - 4xy^4}{20 + y^4} \\ f_2 = \frac{2(x + 2xy^4 - 500y - 25y^5)}{20 + y^4} \end{cases} \quad (2)$$

- $x$  and  $y$  are unknowns and  $k_2$  is the only parameter.
- A constant solution  $(x_0, y_0)$  of system (2) is called an **equilibrium**.
- $(x_0, y_0)$  is called **asymptotically stable** if the solutions of system (2) starting out close to  $(x_0, y_0)$  become arbitrary close to it.
- $(x_0, y_0)$  is called **hyperbolic** if all the eigenvalues of  $\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} \end{pmatrix}$  have nonzero real parts at  $(x_0, y_0)$ .

## The polynomial system to solve (CASC 2011)

### Theorem: Routh-Hurwitz criterion

A hyperbolic equilibrium  $(x_0, y_0)$  is asymptotically stable if and only if

$$\Delta_1(x_0, y_0) := -\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) > 0 \quad \text{and} \quad \Delta_2(x_0, y_0) := \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial x} > 0.$$

### The semi-algebraic systems encoding the equilibria

- Let  $p_1$  (resp.  $p_2$ ) be the numerator of  $f_1$  (resp.  $f_2$ ).
- The system  $\mathcal{S}_1 : \{p_1 = p_2 = 0, x > 0, y > 0, k_2 > 0\}$  encodes the equilibria of (2).
- The system  $\mathcal{S}_2 : \{p_1 = p_2 = 0, x > 0, y > 0, k_2 > 0, \Delta_1 > 0, \Delta_2 > 0\}$  encodes the asymptotically stable hyperbolic equilibria of (2).

### The corresponding constructible systems

- $\mathcal{C}_1 := \{p_1 = 0, p_2 = 0, x \neq 0, y \neq 0, k_2 \neq 0\}$  in  $\mathbb{C}^3$ .

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## Objectives

For a parametric polynomial system  $F \subset \mathbf{k}[\mathbf{u}][\mathbf{x}]$ , the following problems are of interest:

- 1 compute the values  $u$  of the parameters for which  $F(u)$  has solutions, or has finitely many solutions.
- 2 compute the solutions of  $F$  as continuous functions of the parameters.
- 3 provide an automatic case analysis for the number (dimension) of solutions depending on the parameter values.

## Related work

- **(Comprehensive) Gröbner bases:** (V. Weispfenning, 92, 02), (D. Kapur 93), (A. Montes, 02), (M. Manubens & A. Montes, 02), (A. Suzuki & Y. Sato, 03, 06), (D. Lazard & F. Rouillier, 07), (Y. Sun, D. Kapur & D. Wang, 10) and others.
- **Triangular decompositions:** (S.C. Chou & X.S. Gao 92), (X.S. Gao & D.K. Wang 03), (D. Kapur 93), (D.M. Wang 05), (L. Yang, X.R. Hou & B.C. Xia, 01), (R. Xiao, 09) and others.
- **Cylindrical algebraic decompositions:** (G.E. Collins 75), (H. Hong 90), (G.E. Collins, H. Hong 91), (S. McCallum 98), (A. Strzeboński 00), (C.W. Brown 01) and others.

## A CASC story

### At CASC 2007

- We investigated the **specialization** property of regular chains
- We introduced the concept of **comprehensive triangular decomposition** (CTD) of an algebraic variety.

### At CASC 2011

- CTD of a parametric **constructible set** with application to **complex root classification**
- CTD of a parametric **semi-algebraic system** with application to **real root classification**

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## Specialization

### Definition

A (squarefree) regular chain  $T$  of  $\mathbf{k}[\mathbf{u}, \mathbf{y}]$  **specializes well** at  $u \in \mathbf{K}^d$  if  $T(u)$  is a (squarefree) regular chain of  $\mathbf{K}[\mathbf{y}]$  and  $\text{init}(T)(u) \neq 0$ .

### Example

$$T = \begin{cases} (s+1)z \\ (x+1)y + s \\ x^2 + x + s \end{cases} \quad \text{with } s < x < y < z$$

does **not** specialize well at  $s = 0$  or  $s = -1$

$$T(0) = \begin{cases} z \\ (x+1)y \\ (x+1)x \end{cases} \quad T(-1) = \begin{cases} 0z \\ (x+1)y - 1 \\ x^2 + x - 1 \end{cases}$$

## Comprehensive Triangular Decomposition (CTD)

### Definition

Let  $F \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$ . A CTD of  $V(F)$  is given by :

- a finite **partition**  $\mathcal{C}$  of the parameter space into constructible sets,
- above each  $C \in \mathcal{C}$ , there is a set of regular chains  $\mathcal{T}_C$  such that
  - each regular chain  $T \in \mathcal{T}_C$  specializes well at any  $u \in C$  and
  - for any  $u \in C$ , we have  $V(F(u)) = \bigcup_{T \in \mathcal{T}_C} W(T(u))$ .

### Example

A CTD of  $F := \{x^2(1+y) - s, y^2(1+x) - s\}$  is as follows:

- ①  $s \neq 0 \longrightarrow \{T_1, T_2\}$
- ②  $s = 0 \longrightarrow \{T_2, T_3\}$

where

$$T_1 = \begin{cases} x^2y + x^2 - s \\ x^3 + x^2 - s \end{cases} \quad T_2 = \begin{cases} (x+1)y + x \\ x^2 - sx - s \end{cases} \quad T_3 = \begin{cases} y + 1 \\ x + 1 \\ s \end{cases}$$

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## Disjoint squarefree comprehensive triangular decomposition (DSCTD)

## Definition

Let  $F \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$ . A DSCTD of  $V(F)$  is given by :

- a finite partition  $\mathcal{C}$  of the parameter space,
- each cell  $C \in \mathcal{C}$  is associated with a set of **squarefree** regular chains  $\mathcal{T}_C$  such that
  - each squarefree regular chain  $T \in \mathcal{T}_C$  specializes well at any  $u \in C$  and
  - for any  $u \in C$ ,  $V(F(u)) = \cup_{T \in \mathcal{T}_C} W(T(u))$ . ( $\cup$  denotes **disjoint** union)

## Example

- 1  $s \neq 0, s \neq 4/27$  and  $s \neq -4 \rightarrow \{T_1, T_2\}$
- 2  $s = -4 \rightarrow \{T_1\}$
- 3  $s = 0 \rightarrow \{T_3, T_4\}$
- 4  $s = 4/27 \rightarrow \{T_2, T_5, T_6\}$

$$T_4 = \begin{cases} y \\ x \\ s \end{cases} \quad T_5 = \begin{cases} 3y - 1 \\ 3x - 1 \\ 27s - 4 \end{cases} \quad T_6 = \begin{cases} 3y + 2 \\ 3x + 2 \\ 27s - 4 \end{cases}$$

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## Properties of CTD

Above each cell,

- ① either there are no solutions
- ② or finitely many solutions and the solutions are **continuous functions** of parameters
- ③ or infinitely many solutions, but the **dimension** is **invariant**.

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## Additional properties of DSCTD

Above each cell, where the system has finitely many solutions

- ① the graphs of functions are disjoint
- ② the number of distinct complex solutions is constant

### Example

- ①  $s \neq 0, s \neq 4/27$  and  $s \neq -4 \rightarrow \{T_1, T_2\}$
- ②  $s = -4 \rightarrow \{T_1\}$
- ③  $s = 0 \rightarrow \{T_3, T_4\}$
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## Comprehensive triangular decomposition of semi-algebraic systems?

### Related concepts

- Cylindrical algebraic decomposition (CAD by G.E. Collins 75)
- Border polynomial (BP by L. Yang, X.R. Hou & B.C. Xia, 01)
- Discriminant variety (DV by D. Lazard & F. Rouillier, 07)

### Why we want more?

- CAD does too much work when used for the purpose of solving semi-algebraic systems.
- BP and DV are only about the parameter space.
- Algorithm based on BP or DV focus on the components of maximal dimension in the parameter space.

# Comprehensive triangular decomposition of semi-algebraic systems

## Input

A parametric semi-algebraic system  $S \subset \mathbb{Q}[\mathbf{u}][\mathbf{y}]$ .

## Output

- A **partition** of the **whole parameter space** into **connected cells**, such that above each cell
  - ① either the corresponding constructible system of  $S$  has **infinitely many complex solutions**,
  - ② or  $S$  has no real solutions
  - ③ or  $S$  has finitely many real solutions which are continuous functions of parameters with disjoint graphs
- A **description** of the solutions of  $S$  as functions of parameters by **triangular systems** in case of finitely many complex solutions.

## How to compute a RCTD?

### Specifications

- Input: a parametric semi-algebraic system  $S$
- Output: a RCTD of  $S$ , that is, parameter space partition + triangular systems.

### Algorithm

For simplicity, we assume  $S$  consists of only equations.

- (1) Compute a **DSCTD**  $(\mathcal{C}, (\mathcal{T}_C, C \in \mathcal{C}))$  of  $S$ .
- (2) Refine each constructible set cell  $C \in \mathcal{C}$  into **connected** semi-algebraic sets by CAD.
- (3) Let  $C$  be a connected cell above which  $S$  has finitely many complex solutions.  
 Compute the number of real solutions of  $T \in \mathcal{T}_C$  at a **sample point**  $u$  of  $C$ .  
 Remove those  $T$ s which have no real solutions at  $u$ .

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## Equilibria of mad cow disease model

Recall the dynamical system

$$\begin{cases} \frac{dx}{dt} = f_1 \\ \frac{dy}{dt} = f_2 \end{cases} \quad \text{with} \quad \begin{cases} f_1 = \frac{16000 + 800y^4 - 20k_2x - k_2xy^4 - 2x - 4xy^4}{20 + y^4} \\ f_2 = \frac{2(x + 2xy^4 - 500y - 25y^5)}{20 + y^4} \end{cases} .$$

Let  $p_1$  (resp.  $p_2$ ) be the numerator of  $f_1$  (resp.  $f_2$ ).

$$\begin{aligned} p_1 &:= (-20k_2 - k_2y^4 - 2 - 4y^4)x + 16000 + 800y^4 \\ p_2 &:= (2y^4 + 1)x - 500y - 25y^5 \end{aligned}$$

The system  $\mathcal{S}_1 : \{p_1 = p_2 = 0, x > 0, y > 0, k_2 > 0\}$  encode the equilibria.

RCTD of  $S_1$ 

Let  $0 < \alpha_1 < \alpha_2$  be the two positive real roots of the following polynomial

$$r := 100000k_2^8 + 1250000k_2^7 + 5410000k_2^6 + 8921000k_2^5 - 9161219950k_2^4 - 5038824999k_2^3 - 1665203348k_2^2 - 882897744k_2 + 1099528405056.$$

The isolating intervals for  $\alpha_1$  and  $\alpha_2$  are respectively  $[3.175933838, 3.175941467]$  and  $[14.49724579, 14.49725342]$ .

A RCTD of  $S_1$  is as follows.

$$\left\{ \begin{array}{ll} \{ \} & k_2 \leq 0 \\ \{B_1\} & 0 < k_2 < \alpha_1 \\ \{B_2\} & k_2 = \alpha_1 \\ \{B_1\} & \alpha_1 < k_2 < \alpha_2 \\ \{B_2\} & k_2 = \alpha_2 \\ \{B_1\} & k_2 > \alpha_2 \end{array} \right. \quad \left\{ \begin{array}{ll} 0 & k_2 \leq 0 \\ 1 & 0 < k_2 < \alpha_1 \\ 2 & k_2 = \alpha_1 \\ 3 & \alpha_1 < k_2 < \alpha_2 \\ 2 & k_2 = \alpha_2 \\ 1 & k_2 > \alpha_2 \end{array} \right.$$

## Theorem

If  $0 < k_2 < \alpha_1$  or  $k_2 > \alpha_2$ , then the dynamical system has 1 equilibrium;  
 if  $k_2 = \alpha_1$  or  $k_2 = \alpha_2$ , then the dynamical system has 2 equilibria;  
 if  $\alpha_1 < k_2 < \alpha_2$ , then dynamical system has 3 equilibria.

## Hurwitz determinants and hyperbolicity

- Let  $(x, y)$  be an equilibrium of the dynamical system
- Let  $J$  be the Jacobian matrix of the dynamical system at  $(x, y)$
- Then the characteristic polynomial of  $J$  is  $\lambda^2 + \Delta_1\lambda + \Delta_2$ .
- Let  $\lambda_1$  and  $\lambda_2$  be the two eigenvalues of  $J$
- Then we have  $\lambda_1 + \lambda_2 = -\Delta_1$  and  $\lambda_1\lambda_2 = \Delta_2$

Thus

- $S_1 := \{p_1 = p_2 = 0, x > 0, y > 0, k_2 > 0\}$  encodes the equilibria.
- $S_2 := \{S_1, \Delta_1 = \Delta_2 = 0\}$  encodes the nonhyperbolic equilibria with zero as eigenvalue of multiplicity two.
- $S_3 := \{S_1, \Delta_1 \neq 0, \Delta_2 = 0\}$  encodes the nonhyperbolic equilibria with zero as eigenvalue of multiplicity one.
- $S_4 := \{S_1, \Delta_1 = 0, \Delta_2 > 0\}$  encodes the nonhyperbolic equilibria with a pair of pure imaginary eigenvalues, that is, a Hopf bifurcation.
- $S_5 := \{S_1, \Delta_1 > 0, \Delta_2 > 0\}$  encodes the asymptotically stable hyperbolic equilibria.

## Stability and bifurcation analysis (I)

- RCTD( $S_1$ ) shows that the system has
  - one equilibrium if and only if  $k_2 < \alpha_1$  or  $k_2 > \alpha_2$ ;
  - two equilibria if and only if  $k_2 = \alpha_1$  or  $k_2 = \alpha_2$ ;
  - three equilibria if and only if  $k_2 > \alpha_1$  and  $k_2 < \alpha_2$ .
- RCTD( $S_2$ ) and RCTD( $S_4$ ) show that neither  $S_2$  nor  $S_4$  have real solutions.
- RCTD( $S_3$ ) show that the system has
  - one nonhyperbolic equilibria with zero eigenvalue of multiplicity one if and only if  $k_2 = \alpha_1$  or  $k_2 = \alpha_2$ .
- RCTD( $S_5$ ) show that the system has
  - one asymptotically stable hyperbolic equilibria if and only if  $k_2 \leq \alpha_1$  or  $k_2 \geq \alpha_2$ ;
  - two asymptotically stable hyperbolic equilibria if and only if  $k_2 > \alpha_1$  and  $k_2 < \alpha_2$ .

## Stability and bifurcation analysis

### Combining several RCTDs

- $\text{RCTD}(\mathcal{S}_1)$  : equilibria.
- $\text{RCTD}(\mathcal{S}_1, \Delta_1 = \Delta_2 = 0)$ ,  $\text{RCTD}(\mathcal{S}_1, \Delta_1 \neq 0, \Delta_2 = 0)$ , and  $\text{RCTD}(\mathcal{S}_1, \Delta_1 = 0, \Delta_2 > 0)$ : nonhyperbolic equilibria.
- $\text{RCTD}(\mathcal{S}_1, \Delta_1 > 0, \Delta_2 > 0)$  : asymptotically stable hyperbolic equilibria.

### Theorem

- $0 < k_2 < \alpha_1$  or  $k_2 > \alpha_2 \longrightarrow$  the system has 1 equilibrium, which is hyperbolic and asymptotically stable
- $k_2 = \alpha_1$  or  $k_2 = \alpha_2 \longrightarrow$  the system has 2 equilibria, one is nonhyperbolic, another one is hyperbolic and asymptotically stable
- $\alpha_1 < k_2 < \alpha_2 \longrightarrow$  the system has 3 equilibria, two are hyperbolic and asymptotically stable, one is hyperbolic and non-stable.
- the system experiences a bifurcation at  $k_2 = \alpha_1$  or  $k_2 = \alpha_2$

# Can a small amount of $PrP^{Sc}$ cause prion disease? (I)

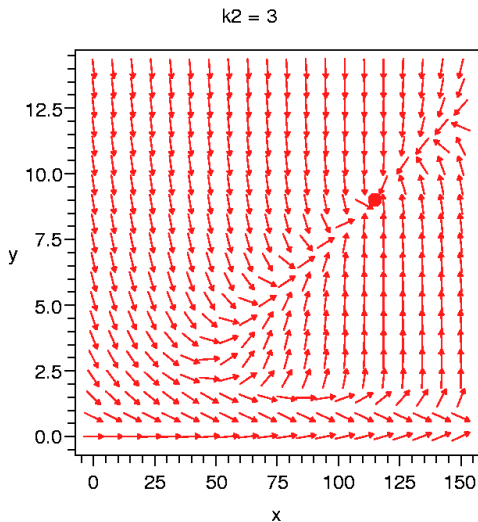


Figure: Vector field for  $k_2 = 3$  ( $x : PrP^C$ ,  $y : PrP^{Sc}$ )

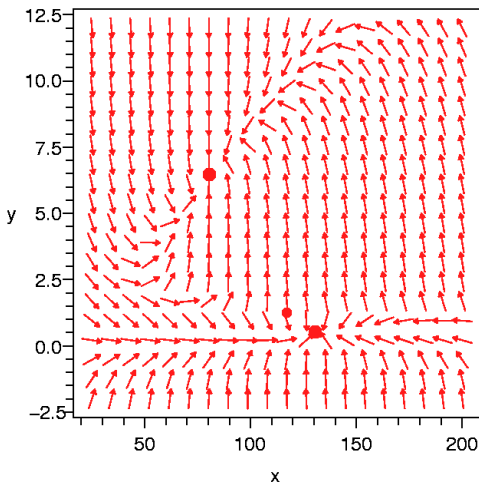
Can a small amount of  $PrP^{Sc}$  cause prion disease? (II) $k_2 = 8$ 

Figure: Vector field for  $k_2 = 8$  ( $x : PrP^C$ ,  $y : PrP^{Sc}$ )

## Can a small amount of $PrP^{Sc}$ cause prion disease? (III)

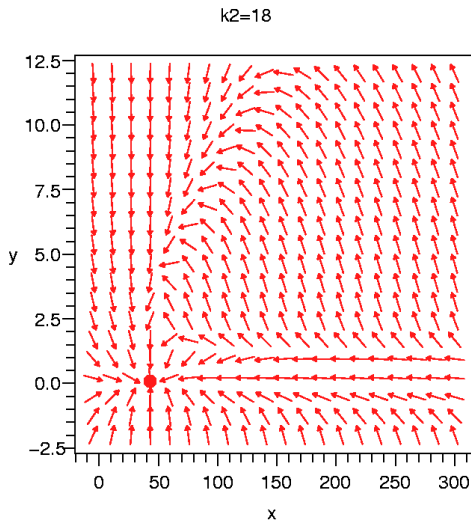


Figure: Vector field for  $k_2 = 18$  ( $x : PrP^C$ ,  $y : PrP^{Sc}$ )