Overview

Many experimentations reported in the literature show that, for dense univariate polynomials, FFT-based multiplication provides better running times than other techniques such as the plain algorithm and the ones based on the tricks of Toom-Cook and Karatsuba. This remains true on the latest hardware architectures. However, implementation techniques and thus thresholds are certainly changing. In addition, if coefficient-oblivious algorithms cannot outperform FFT-based methods for sufficiently large input, these former algorithms are by nature generic and easier to implement.

Therefore, it is important to understand how to implement them efficiently on today’s parallel hardware architectures.

In this poster, we focus on two algorithms which are independent of the coefficient ring: the plain and the Toom-Cook univariate multiplications (for the latter, 6 is assumed to be a unit in the base ring). We analyze their cache complexity and report on experimentation with parallel implementations in Cilk++.

Plain univariate multiplication

```c
void multPoly1(int *A, int n, int *B, int m, int *C) {
    for (i=0; i<n+1; i++) C[i] = 0;
    for (i=0; i<n; i++)
        for (j=0; j<m; j++) C[i+j] += A[i] * B[j];
}
void multPoly2(int *A, int i, int j, int *B, int k, int *C) {
    int da = j-i; int db = l-k;
    if (da == 0) {
        for (int x=k; x<i; ++x) C[i+x] += A[i] * B[x];
    } else if (db == 0) {
        for (int y=i; y<k; ++y) C[y+k] += A[y] * B[k];
    } else if (da > db) {
        int m = (da+1)/2;
        multPoly2(A, i, i + m - 1, B, k, C);
        multPoly2(A, i + m, j, B, k, C);
    } else {
        int m = (da + 1) / 2;
        multPoly2(A, i, j, B, k, C + m - 1);
        multPoly2(A, i + m, j, B, k, C);
    }
}
```

The above `multPoly1` is a naive pseudo-implementation of the plain multiplication, where the input coefficient arrays `A` and `B` have size `n` and `m`. For an ideal cache with `Z` words and cache line size `L`, this algorithm incurs $O((\log_2 n + 1) + n(2 + 1/L))$ cache misses. The divide-and-conquer `multPoly2` reduces this cache complexity to $O(nm/LZ)$.

In practice, even with a threshold between its iterative and recursive modes in the order of `512`, `multPoly2` brings a speedup factor limited to 2, due to the overhead of the recursive calls. The same algorithm can be parallellized in a natural manner with a span of $\Theta(n)$ (assuming $n = m$ for simplicity) and a space complexity of $\Theta(n \log n)$.

Toom-Cook univariate multiplications

Our initial goal was to obtain a parallel version of Toom-Cook multiplication for which both span and space would be in $\Theta(n)$. The procedure below sketches the principle of a solution for input polynomials `A` and `B` of degree less than `n`.

Indeed one needs to perform more linear combinations of the segments $A_0, A_1, A_2, B_0, B_1, B_2$ than with a straightforward parallelization of Toom-Cook Algorithm. However, the latter incurs a space consumption within $\Theta(n^{k-3})$ (in bytes), that is, in the order of the work.

The table below reports timings in milliseconds for multiplying two dense univariate polynomials of degree less than $n = 2^k$. We use a desktop machine (Intel Core 2 Quad @ 2.66GHz) with a L2 cache of standard size (3 Mb). The base size is `512` for both serial codes `multPoly2` and `multPoly2.p`. The speedup of the parallel version `multPoly2.p` is around 3 comparing to `multPoly2`.

<table>
<thead>
<tr>
<th>$\log_2(n)$</th>
<th><code>multPoly1</code></th>
<th><code>multPoly2</code></th>
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</table>

We have benchmarked the above version Toom-Cook Algorithm against a straightforward parallel implementation (span $\Theta(n)$ and space $O(n^{k-3})$) on a 16 cores (each core is an Intel Xeon @ 2.40GHz with 4096 KB of cache). In the figure below, the degree of the input polynomials is given on a logarithmic scale while the vertical coordinate is the speedup factor. Finally, we have proved that the cache complexity of both implementations is within $O((n/2)^{k-3})(1 + 1/L)$.

In conclusion, coefficient-oblivious univariate multiplication algorithms offer opportunities for improvement in terms of cache and space complexity. However, turning those improvements into practical benefits is challenging. In particular, space saving implementations may reduce parallelism in a dramatic manner.

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