Motivation

Stability analysis of (polynomial) dynamical systems leads to manipulate the solution sets of systems of equations, inequations or inequalities, so-called semi-algebraic sets. We generalize comprehensive triangular decomposition (CTD) to these sets and apply this new tool to a concrete example.

Laurent Model for Prion Diseases

Mad cow disease is a transmissible disease of the central nervous system, thought to be caused by prion proteins. Prion proteins exist in normal and pathogenic forms; the normal form causes prion diseases. However, compounds that inhibit addition of PrPSc can be seen as a possible therapy against prion diseases. However, compounds that increase the turnover rate $k_2$ would be the best therapeutic strategy against prion diseases.

Basic Definitions

- A semi-algebraic set of $\mathbb{R}^n$ is a finite union of sets of form:
  \[
  \{ (u, x) \in \mathbb{R}^n \mid (\forall f \in F) f(u, x) = 0 \wedge (\forall g \in G) g(u, x) > 0 \},
  \]
  where $F$ and $G$ are any finite polynomial sets of $\mathbb{R}[U, X]$.
- A pair $[T, G+]$ is called a regular semi-algebraic system if
  \- $T$ is a squarefree regular chain of $\mathbb{R}[U, X]$,
  \- $G+$ is a finite set of strict polynomial inequalities \( \{ g > 0 \} \),
  \- each $g$ is regular w.r.t the saturated ideal of \( T \).
- A regular semi-algebraic system $[T, G+]$ separates well at \( u \in \mathbb{R}^d \) if \( [T(u), G(u)+] \) is a regular semi-algebraic system of $\mathbb{R}[X]$ after specialization and no initials of polynomials in $T$ vanish during the specialization.

CTD of Semi-algebraic Sets

Let $S$ be a parametric semi-algebraic set of $\mathbb{R}[U, X]$. A comprehensive triangular decomposition of $S$ is given by:

- a finite partition $C$ of the parameter space $\mathbb{R}^d$ into connected semi-algebraic sets,
- for each $C \in C$, an associated sample point $s \in C$,
- for each $C \in C$ a set of regular semi-algebraic systems $A_C$ of $\mathbb{R}[U, X]$ such that for each $u \in C$: each $A \in A_C$ separates well at $u$: the solutions sets of the systems $(A(u), v)$, for $A \in A_C$, are pairwise disjoint and their union is exactly the set of points of $S$ whose $U-$coordinates are equal to $u$.

In system (1), let $p_1$ and $p_2$ be respectively the numerators of $f_1$ and $f_2$. The parametric semi-algebraic set

\[ S : \{ (u, x) \in \mathbb{R}^n \mid (\forall f \in F) f(u, x) = 0 \wedge (\forall g \in G) g(u, x) > 0 \} \]

encodes exactly the asymptotically hyperbolic equilibria of system (1). A comprehensive triangular decomposition of $S$ is illustrated as follows:

Equilibria Analysis

With CTD at hand, we can count the number of (asymptotically stable) equilibria of (1) depending on parameters. Let

\[
R_1 = 1000000k_1^5 + 1250000k_2^3 + 5410000k_2^3 + 8921000k_2^5 - 9161219950k_2^3 - 5038824999k_2^5 - 1665203348k_2^7 - 88297744k_2^9 + 109528460656.
\]

If $R_1 > 0$ (Figures 1 and 3), then the system has one equilibrium, which is asymptotically stable. If $R_1 < 0$ (Figure 2), then the system has three equilibria, two of which are asymptotically stable. If $R_1 = 0$, the system experiences a bifurcation.

From these figures, we also observe that: In Figure 1, the concentration of PrPSc (y-coordinate) finally becomes low and thus the system enters a harmless state. Conversely, in Figure 3 the concentration of PrPSc goes high and thus the system enters a pathogenic state. In Figure 2, the system exhibits bistability, the initial concentrations of PrPSc determines whether the final state pathogenic or not. We thus deduce the following facts:

- The turnover rate $k_2$ determines whether it is possible for a pathogenic state to occur.
- As an answer to our question, a small amount of PrPSc does not lead to a pathogenic state when $k_2$ is large.
- Compounds that inhibit addition of PrPSc can be seen as a possible therapy against prion diseases. However, compounds that increase the turnover rate $k_2$ would be the best therapeutic strategy against prion diseases.