# Solving Parametric Polynomial Systems with the RegularChains Library in Maple

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# What does "solving parametric systems" mean?

For a parametric polynomial system $F \subset \mathbf{k}[\mathbf{u}][\mathbf{x}]$, the following problems are of interest:

1. compute the values $u$ of the parameters for which $F(u)$ has solutions, or has finitely many solutions.
2. compute the solutions of $F$ as continuous functions of the parameters.
3. provide an automatic case analysis for the number (dimension) of solutions depending on the parameter values.

# Related work ($C^3$)

- (Comprehensive) Gröbner bases (CGB): (V. Weispfenning, 92, 02), (D. Kapur 93), (A. Montes, 02), (A. Suzuki & Y. Sato, 03, 06), (D. Lazard & F. Rouillier, 07), (Guillaume Moroz, 06) and others.

- Cylindrical algebraic decompositions (CAD): (G.E. Collins 75), (G.E. Collins, H. Hong 91), (H. Hong 92), (S. McCallum 98), (A. Strzeboński 00), (C.W. Brown 01) and others.

- (Comprehensive) triangular decompositions (CTD): (S.C. Chou & X.S. Gao 92), (X.S. Gao & D.K. Wang 03), (D. Kapur 93), (D.M. Wang 05), (L. Yang, X.R. Hou & B.C. Xia, 01), (C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza & W. Pan, 07) and others.

# Outline

- comprehensive triangular decomposition of parametric constructible sets
- complex root classification
- real root classification
- border polynomial and related notions
- Maple demo

# Triangular decompositions of a constructible set

A pair $R = [T, h]$ is called a regular system if $T$ is a regular chain, and $h$ is a polynomial which is regular w.r.t $\mathrm{sat}(T)$.

### Theorem (CGLMP, CASC2007)

*Every constructible set can be written as a finite union of the zero sets of regular systems.*

The constructible set

$$\left\{ \begin{array}{rcl} x(1+y)-s & = & 0 \\ y(1+x)-s & = & 0 \\ x+y-1 & \neq & 0 \end{array} \right. \tag{1}$$

can be represented by two regular systems

$$R_1 : \left| \begin{array}{l} T_1 = \left\{ \begin{array}{l} (y+1)x - s \\ y^2 + y - s \end{array} \right. \\ h_1 = y - 2s + 1 \end{array} \right. \qquad R_2 : \left| \begin{array}{l} T_2 = \left\{ \begin{array}{l} x+1 \\ y+1 \\ s \end{array} \right. \\ h_2 = 1 \end{array} \right.$$

# Specialization

## Definition

A regular system $R := [T, h]$ specializes well at $u \in \mathbf{K}^d$ if $[T(u), h(u)]$ is a regular system of $\mathbf{K}[\mathbf{x}]$ after specialization and no initials of polynomials in $T$ vanish during the specialization.

## Example

$$R_1 : \left| \begin{array}{l} T_1 = \left\{ \begin{array}{l} (y+1)x - s \\ y^2 + y - s \end{array} \right. \\ h_1 = y - 2s + 1 \end{array} \right.$$

does not specializes well at $s = 0$ or at $s = \frac{3}{4}$

$$R_1(0) : \left| \begin{array}{l} T_1(0) = \left\{ \begin{array}{l} (y+1)x \\ (y+1)y \end{array} \right. \\ h_1(0) = y + 1 \end{array} \right. \qquad R_1(\frac{3}{4}) : \left| \begin{array}{l} T_1(\frac{3}{4}) = \left\{ \begin{array}{l} (y+1)x - \frac{3}{4} \\ (y - \frac{1}{2})(y + \frac{3}{2}) \end{array} \right. \\ h_1(\frac{3}{4}) = y - \frac{1}{2} \end{array} \right.$$

# Comprehensive Triangular Decomposition (CTD)

### Definition

Let $CS$ be a constructible set of $\mathbf{k}[\mathbf{u}, \mathbf{x}]$. A *comprehensive triangular decomposition* of $CS$ is given by :

1. a finite partition $\mathcal{C}$ of the parameter space $\mathbf{K}^d$,
2. for each $C \in \mathcal{C}$ a set of regular systems $\mathcal{R}_C$ s.t. for $u \in C$
   - 2.1 each of the regular systems $R \in \mathcal{R}_C$ specializes well at $u$
   - 2.2 and we have
$$CS(u) = \bigcup_{R \in \mathcal{R}_C} Z(R(u)).$$

Let $F := \{x(1+y) - s = 0, y(1+x) - s = 0, x + y - s \neq 0\}$.

A CTD of F is as follows:

1. $s \neq 0$ and $s \neq \frac{3}{4} \longrightarrow \{R_1\}$
2. $s = 0 \longrightarrow \{R_2, R_3\}$
3. $s = \frac{3}{4} \longrightarrow \{R_4\}$

$$R_1 : \left| \begin{array}{l} T_1 = \left\{ \begin{array}{l} (y+1)x - s \\ y^2 + y - s \end{array} \right. \\ h_1 = y - 2s + 1 \end{array} \right. \qquad R_2 : \left| \begin{array}{l} T_2 = \left\{ \begin{array}{l} x+1 \\ y+1 \\ s \end{array} \right. \\ h_2 = 1 \end{array} \right.$$

$$R_3 : \left| \begin{array}{l} T_3 = \left\{ \begin{array}{l} x \\ y \\ s \end{array} \right. \\ h_3 = 1 \end{array} \right. \qquad R_4 : \left| \begin{array}{l} T_4 = \left\{ \begin{array}{l} 2x+3 \\ 2y+3 \\ 4s-3 \end{array} \right. \\ h_4 = 1 \end{array} \right.$$

# Separation

## Definition

A squarefree regular system $R := [T, h]$ separates well at $u \in \mathbf{K}^d$ if: $R$ specializes well at $u$ and $R(u)$ is a squarefree regular system of $\mathbf{K}[\mathbf{x}]$.

$$R_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ h_1 = y - 2s + 1 \end{array} \right.$$

specializes well but not separates well at $s = -\frac{1}{4}$:

$$R_1(-\frac{1}{4}) : \left| \begin{array}{l} T_1(-\frac{1}{4}) = \begin{cases} (y+1)x + \frac{1}{4} \\ (y + \frac{1}{2})^2 \end{cases} \\ h_1(-\frac{1}{4}) = y + \frac{3}{2} \end{array} \right.$$

where the second polynomial of $T_1(-\frac{1}{4})$ is not squarefree.

# Disjoint Squarefree Comprehensive Triangular Decomposition (DSCTD)

## Definition

A disjoint squarefree comprehensive triangular decomposition of a constructible set $CS$ is given by:

1. a finite partition $\mathcal{C}$ of the parameter space $\mathbf{K}^d$,
2. for each $C \in \mathcal{C}$ a set of squarefree regular systems $\mathcal{R}_C$ such that for each $u \in C$:
    2.1 each $R \in \mathcal{R}_C$ separates well at $u$,
    2.2 the zero sets $Z(R(u))$, for $R \in \mathcal{R}_C$, are pairwise disjoint and

$$CS(u) = \bigcup_{R \in \mathcal{R}_C} Z(R(u)).$$

# DSCTD and complex root counting

Let $F := \{x(1+y) - s = 0, y(1+x) - s = 0, x + y - s \neq 0\}$. A DSCTD of $F$ is as follows:

1. $s \neq 0,\ s \neq -\frac{1}{4},\ s \neq \frac{3}{4} \longrightarrow \{R_1\}$
2. $s = 0 \longrightarrow \{R_2, R_3\}$
3. $s = \frac{3}{4} \longrightarrow \{R_4\}$
4. $s = -\frac{1}{4} \longrightarrow \{R_5\}$

where

$$R_1 : \left| \begin{array}{l} T_1 = \left\{ \begin{array}{l} (y+1)x - s \\ y^2 + y - s \end{array} \right. \\ h_1 = y - 2s + 1 \end{array} \right. \quad R_2 : \left| \begin{array}{l} T_2 = \left\{ \begin{array}{l} x+1 \\ y+1 \\ s \end{array} \right. \\ h_2 = 1 \end{array} \right. \quad R_3 : \left| \begin{array}{l} T_3 = \left\{ \begin{array}{l} x \\ y \\ s \end{array} \right. \\ h_3 = 1 \end{array} \right.$$

$$R_4 : \left| \begin{array}{l} T_4 = \left\{ \begin{array}{l} 2x+3 \\ 2y+3 \\ 4s-3 \end{array} \right. \\ h_4 = 1 \end{array} \right. \quad R_5 : \left| \begin{array}{l} T_5 = \left\{ \begin{array}{l} 2x+1 \\ 2y+1 \\ 4s+1 \end{array} \right. \\ h_5 = 1 \end{array} \right.$$

Therefore, we conclude that: if $(s + \frac{1}{4})(s - \frac{3}{4}) = 0$, system (1) has 1 complex root; otherwise system (1) has 2 complex roots.

# Real root classification of semi-algebraic system

A parametric semi-algebraic system (SAS) is of the following form:

$$\begin{cases} p_1(\mathbf{u},\mathbf{x}) = 0, ..., p_r(\mathbf{u},\mathbf{x}) = 0, \\ g_1(\mathbf{u},\mathbf{x}) \geq 0, ..., g_k(\mathbf{u},\mathbf{x}) \geq 0, \\ g_{k+1}(\mathbf{u},\mathbf{x}) > 0, ..., g_t(\mathbf{u},\mathbf{x}) > 0, \\ h_1(\mathbf{u},\mathbf{x}) \neq 0, ..., h_s(\mathbf{u},\mathbf{x}) \neq 0, \end{cases} \tag{2}$$

Here all polynomials are with rational number coefficients. The system is denoted by a quadruple $[F, N, P, H]$, where

- $F = [p_1, \ldots, p_r]$
- $N = [g_1, \ldots, g_k]$, $P = [g_{k+1}, \ldots, g_t]$, $H = [h_1, \ldots, h_s]$

For an integer n, the problem of real root classification is to provide conditions on parameters s.t. the system has exactly n distinct real solutions.

# Delayed computation and border polynomial

In (L. Yang, X.R. Hou & B.C. Xia, 01), the author proposed the idea of delayed computation for solving semi-algebraic systems:

- Decompose original system into a family of triangular systems
- Solve each sub-system forgetting degenerate parametric values
- Wrap all the degenerate values as a border polynomial
- Add the border polynomial into the input system and start over

### Definition

Let $S(\mathbf{u}, \mathbf{x})$ be a parametric semi-algebraic system. A polynomial $R(\mathbf{u})$ is called a border polynomial of $S$ if

(a) $S(u)$ has only finitely many real solutions when $R(u) \neq 0$

(b) the number of distinct real solutions of $S$ is constant in each connected component of $R(\mathbf{u}) \neq 0$ in $\mathbb{R}^d$

# Output of RealRootClassification

The output of RealRootClassification $[F, N, P, H, n]$ is a pair $[\Phi(\mathbf{u}), bp(\mathbf{u})]$ interpreted by the following theorem.

### Theorem

*Provided the border polynomial $bp(\mathbf{u})$ does not vanish, the system has $n$ distinct real solutions if and only if a logic formula $\Phi(\mathbf{u})$ holds.*

Here the logic formula $\Phi(\mathbf{u})$ can be

- a quantifier free formula, like

$$(a > 0 \wedge a + b < 0) \vee (a - b^2 < 0)$$

- a quantifier free formula plus a description of which root of a regular chain, like $a > 0$ and $(b, c)$ is the first root of the regular chain $[c - b, b^2 - 1]$.

Such a description can be encoded as a regular semi-algebraic set.

# Regular semi-algebraic set

Let $[Q, T, L]$ be a triple of $\mathbb{Q}[\mathbf{w}, \mathbf{y}]$ where

- $Q(\mathbf{w})$ is a *quantifier-free formula* defining a nonempty set $S$,
- $T = \{t_1, \ldots, t_m\}$ is a *squarefree regular chain*, with main variables $\mathbf{y} = y_1, \cdots, y_m$ and free variables $\mathbf{w} = w_1, \ldots, w_s$
- for each point $\alpha$ of $S$, $T$ separates well at $\alpha$ and $T(\alpha)$ has at least one real solution,
- $L$ is a list of root indices of $T$

The zero set of $[Q, T, L]$ is defined as the set of $(\alpha, \beta) \in \mathbb{R}^{s+m}$ s.t.

- $\alpha$ satisfies $Q$ and $\beta = (\beta_1, \ldots, \beta_m)$ is a real solution of $T(\alpha, \mathbf{y})$
- if $L$ is empty, then $\beta$ is any real solution of $T(\alpha, \mathbf{y})$; otherwise there exists an $L_j$ in $L$ s.t. each $\beta_i$ is the $L_{j,i}$-th real solution of $t_i(\alpha, \beta_1, ..., \beta_{i-1}, y_i)$ w.r.t $y_i$.

The zero set of $[Q, T, L]$ as defined above is called a regular semi-algebraic set.

**Theorem**

*A regular semi-algebraic set is not empty and every semi-algebraic set can be decomposed as a finite union of regular semi-algebraic sets.*

**Algorithm:** RealRootClassification

**Input:** A parametric semi-algebraic system $S$ and a solution number to query

**Output:** Necessary and sufficient conditions on the parameters for the system to have a given number of solutions provided its border polynomial does not vanish. The conditions are encoded by a list of regular semi-algebraic sets.

# Example

Given a system

$$F := \begin{cases} x(y+1) - s & = & 0 \\ y(x+1) - s & = & 0 \\ x + y - 1 & > & 0 \end{cases}$$

By `RealRootClassification`, we get the following theorem

**Theorem**

*Provided the border polynomial* $bp = s(4s+1)(4s-3) \neq 0$*, the system* $F$ *has real solutions if and only if* $4s - 3 > 0$*.*

Adding $bp = 0$ into $F$, one finds that the system has no real solutions. So the final conclusion is that

**Corollary**

*The system* $F$ *has real solutions if and only if* $4s - 3 > 0$*.*

# Border polynomial and related notions (I)

Let $R = [T, h]$ be a squarefree regular system of $\mathbb{Q}[\mathbf{u}, \mathbf{x}]$, where $\mathbf{x} = \mathrm{mvar}(T)$. Denote

$$bp := \mathrm{sqrres}(\mathrm{sep}(T)h, T).$$

**Theorem**

*Considering the real solutions of $R$, then the polynomial $bp$ is a border polynomial of the semi-algebraic system $R$.*

**Theorem**

*Considering the complex solutions of $R$, then the variety of $bp$ is the minimal discriminant variety of the constructible set $Z(R)$.*

# Border polynomial and related notions (II)

## Theorem

*Let CS be a parametric constructible set of* $\mathbb{Q}[\mathbf{u}, \mathbf{x}]$. *Let d be the number of parameters. Assume that for almost all parameter values u,* $CS(u)$ *has finitely many complex solutions. Then one could compute a disjoint squarefree CTD of CS s.t.*

- *there exists one and only one cell C, whose complement in* $\mathbb{C}^d$ *is a hypersurface*
- *the hypersurface is a discriminant variety of CS*
- *let* $bp(\mathbf{u})$ *be the squarefree polynomial defining the hypersurface, then bp is a border polynomial of the semi-algebraic set* $CS \cap \mathbb{R}^d$