Computing Cylindrical Algebraic Decomposition via Triangular Decomposition

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Overview

Cylindrical algebraic decomposition (CAD) is a fundamental tool in real algebraic geometry. For $F_n \subset \mathbb{R}[y_1, \ldots, y_n]$ an $F_n$-invariant CAD of $\mathbb{R}^n$ is a partition $C_1, \ldots, C_n$ of $\mathbb{R}^n$ together with one sample point $x_i \in C_i$ for all $1 \leq i \leq n$, such that the sign of each $f \in F_n$ does not change in $C_i$ and can be determined at $x_i$.

The original algorithm of Collins and its subsequent ameliorations are based on a projection and lifting scheme which computes from $F_n$ a set $F_{n-1} \subset \mathbb{R}[y_1, \ldots, y_{n-1}]$ such that an $F_{n-1}$-invariant CAD of $\mathbb{R}^{n-1}$ results from an $F_{n-1}$-invariant CAD of $\mathbb{R}^{n-1}$. This construction and the case $n = 1$ rely on real root isolation of univariate polynomials. We propose a different approach which proceeds by transforming successive partitions of the complex $n$-dimensional space. Our future goal is to investigate whether fast polynomial arithmetic and modular methods available for triangular decomposition could improve the practical efficiency of CAD implementation.

Our Method

Initial Partition: we decompose $\mathbb{R}^n$ into disjoint constructible sets $C_1, \ldots, C_n$ such that for all $1 \leq i \leq n$, each $f \in F_n$ either $f$ is identical zero in $C_i$ or $f$ vanishes at no points of $C_i$.

Make Cylindrical: we transform the initial partition and obtain another partition of $\mathbb{R}^n$ into disjoint constructible sets such that this second decomposition is cylindrical in the following sense: for all $1 \leq j < n$ the projections on the first $j$ coordinates $(y_1, \ldots, y_j)$ of any two constructible sets are either identical or disjoint.

Make Semi-Algebraic: from the previous decomposition we produce an $F_n$-invariant CAD of $\mathbb{R}^n$.

Zero Separation

We describe our core routine: SeparateZeros. Let $rs = [T, h]$ be a regular system of $k[y_1 < \cdots < y_n]$. We regard $rs$ as a parametric system in $u = y_1, \ldots, y_{n-1}$, that we solve via comprehensive triangular decomposition. As a result, we obtain a partition of the projection onto the $u$-space of zero-set $Z(rs) = W(T) \setminus V(h)$ that, above each cell $R$ of the partition, $Z(rs)$ equals the union of the zero sets of polynomials $p_1, \ldots, p_n \in k[y_1, \ldots, y_n]$, where

- the initial of each $p_i$ does not vanish on $R$.
- the $p_i$'s are squarefree and pairwise coprime at any point of $R$.

Make Cylindrical

We apply SeparateZeros to the regular systems in the output of InitialPartition recursively. Finally we produce a cylindrical decomposition of $\mathbb{R}^n$, that is, a finite partition of $\mathbb{R}^n$ into constructible sets, called cells and satisfying the following:

- $n = 1$. A cylindrical decomposition of $\mathbb{R}^1$ is either $\mathbb{R}^1$ itself or of the form $p_1 = 0, \ldots, p_n = 0$, $p_1 \cdots p_n \neq 0$ where $p_1, \ldots, p_n$ are nonconstant squarefree and pairwise coprime polynomials of $k[y_i]$.
- $n > 1$. From a cylindrical decomposition $D'$ of $\mathbb{R}^{n-1}$, one builds a cylindrical decomposition $D$ of $\mathbb{R}^n$ as follows. For each cell $D_i$ of $D'$:

  1. either $D_i \times x$ is an element of $D$, or
  2. there exist polynomials $p_1, \ldots, p_n \in k[y_1, \ldots, y_n]$ such that (a) the initial of each $p_i$ does not vanish on $D_i$ and,
  3. the $p_i$'s are squarefree and pairwise coprime at all $u \in D_i$.

Make SemiAlgebraic

Let $p \in k[y_1, \ldots, y_n]$ be nonconstant and $R$ be a region of $\mathbb{R}^{n-1}$. If $\text{gcd}(p, y_i)$ does not vanish on $R$ and the number of distinct complex roots of $p$ is invariant on $R$, then $p$ is %reducible on $R$, that is, $V(p)$ is the union of finitely many disjoint graphs of continuous functions over $R$.

From this theorem of Collins, we derive a CAD of $\mathbb{R}^n$ from a cylindrical decomposition of $\mathbb{R}^n$, by means of real root isolation of zero-dimensional regular chains.

Example

Consider the polynomial $p = ax^2 + bx + c$, where $x > c > b > a$.

The first step, InitialPartition, decomposes $c^4$ into four pairwise disjoint sets, each of which is the zero set of a regular system.

$$r_1 := \begin{cases} c = 0, & b = 0 \quad r_2 := \begin{cases} b + c = 0, & b \neq 0, \quad r_3 := \begin{cases} ax^2 + bx + c = 0, & a = 0 \quad r_4 := \begin{cases} a \neq 0. \end{cases} \end{cases} \end{cases}$$

and $r_4 := \{ax^2 + bx + c \neq 0\}$. The algorithm MakeCylindrical takes $r_1, r_2, r_3$ and $r_4$ as input and outputs a cylindrical decomposition of $c^4$. Let $t = bx+c$, $q = 2ax+b$, and $r = 4ac - b^2$, the decomposition can be described by a tree.

Concluding Remarks

- Our preliminary implementation, realized with the RegularChains library, involves only high-level MAPLE interpreted code. In the table, $N_e$ denotes the number of elements in our CAD.
- Our experimental results show that our method can already process well-known test examples from the literature, see our ISSAC'09 paper for details. Our data also show that polynomial GCDs and resultants modulo regular chains are the dominant cost.
- This suggests that the modular methods and efficient implementation techniques being developed in RegularChains library have a large potential for improving our current implementation.

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