Math 506: Complex Variables

James D. Lewis

`Winter Term`

Detailed Syllabus

Part I: *Single Variables: Review and extensions.* CR equations and analyticity; Cauchy-Goursat theorem and Cauchy integral formula, Louiville's theorem; Morera's theorem; maximum modulus principle; Laurent series and singularities; Riemann extension theorem; residues; Schwartz's lemma; open mapping theorem; analytic continuation and the dilogarithm; linear fractional transformations; spaces of analytic functions: Normal families and Montel's theorem; Riemann mapping theorem; Picard theorems.

Part II: *Several variables.* Complex linearity and holomorphicity in several variables; Hartog's theorem; Weierstrass preparation theorem; Riemann extension theorem; Weierstrass division theorem; Applications: $\mathcal{O}_n$ a UFD, analytic Nullstellensatz; implicit and inverse function theorems; complex manifolds and analytic subvarieties; meromorphic maps.

Part III: *Applications of sheaf theory to complex analysis.* Introduction to sheaves, coherent sheaves, and motivation for sheaf cohomology theory: First and second Cousin problems [these include the classical Mittag-Leffler and Weierstrass theorems]; Stein manifold theory and the holomorphic de Rham complex; Dolbeault cohomology and applications; Riemann surface theory.

Reference Texts: 1. John B. Conway, Functions of One Complex variable; 2. Serge Lang, Complex Analysis.

`Notes will be provided in this course.`

Grading: Midterm Exam (40%), Final Exam (60%).

MATH 506: COMPLEX ANALYSIS

James D. Lewis

§1. Review

Notation

$\mathbb{C} = \mathbb{R} \oplus \mathrm{i}\,\mathbb{R}$, $\mathrm{i}^2 = -1$, $z = x + \mathrm{i}\,y$, $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $z = r(\cos\theta + \mathrm{i}\sin\theta) = re^{\mathrm{i}\,\theta}$ (polar form), $\theta = \tan^{-1}(\frac{y}{x}) = \arg(z)$, $r = |z| = \sqrt{z\overline{z}}$. Note the real isomorphism:

$$\begin{aligned} \mathbb{C} &\simeq \mathbb{R}^2 \\ z &\mapsto (x, y) \end{aligned}$$

Any map $f : \mathbb{C} \to \mathbb{C}$ can be written in the form $f = \mu + \mathrm{i}\,\nu$, $\mu(x,y)$, $\nu(x,y) : \mathbb{R}^2 \to \mathbb{R}$. f of type $C^1 \Leftrightarrow \mu$, ν of type C^1, i.e. $\mu_x, \mu_y, \nu_x, \nu_y$ are continuous. A domain $D \subset \mathbb{C}$ is an open connected set.

Complex differentiation

We will assume f is of type C^1. We introduce the following derivative operators:

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left\{\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y}\right\} \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left\{\frac{\partial}{\partial x} + \mathrm{i}\,\frac{\partial}{\partial y}\right\} \end{aligned}$$

One also has differentials:

$$\begin{aligned} dz &= dx + \mathrm{i}\,dy \\ d\overline{z} &= dx - \mathrm{i}\,dy \end{aligned}$$

Define $\partial f = \frac{\partial f}{\partial z} dz$, $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}$, $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$.

Claim. *$df = \partial f + \overline{\partial} f$, viz. $d = \partial + \overline{\partial}$.*

Proof. Write $f = \mu + \mathrm{i}\,\nu$. We compute: $f = \mu + \mathrm{i}\,\nu$; by definition $df = d\mu + \mathrm{i}\,d\nu$.

$$\begin{aligned} df &= \mu_x dx + \mu_y dy + \mathrm{i}\,\nu_x dx + \mathrm{i}\,\nu_y dy \\ &= (\mu_x + \mathrm{i}\,\nu_x) dx + (\mu_y + \mathrm{i}\,\nu_y) dy \\ \partial f &= \frac{1}{2}(\mu_x - \mathrm{i}\,\mu_y)(dx + \mathrm{i}\,dy) + \frac{\mathrm{i}}{2}(\nu_x - \mathrm{i}\,\nu_y)(dx + \mathrm{i}\,dy) \\ &= \left(\frac{\mu_x + \nu_y - \mathrm{i}(\mu_y - \nu_x)}{2}\right) dx + \mathrm{i}\left(\frac{\mu_x + \nu_y - \mathrm{i}(\mu_y - \nu_x)}{2}\right) dy \\ \overline{\partial} f &= \frac{1}{2}(\mu_x + \mathrm{i}\,\mu_y)(dx - \mathrm{i}\,dy) + \frac{\mathrm{i}}{2}(\nu_x + \mathrm{i}\,\nu_y)(dx - \mathrm{i}\,dy) \\ &= \left[\frac{(\mu_x - \nu_y) + \mathrm{i}(\mu_y + \nu_x)}{2}\right] dx - \mathrm{i}\left[\frac{(\mu_x - \nu_y) + \mathrm{i}(\mu_y + \nu_x)}{2}\right] dy \\ (\partial + \overline{\partial}) f &= [\mu_x + \mathrm{i}\,\nu_x] dx + [\mu_y + \mathrm{i}\,\nu_y] dy \\ &= df \end{aligned}$$

Definition 1.0. *f is complex analytic on a domain $D \subset \mathbb{C}$, if $\overline{\partial} f = 0$ on D, i.e. $\frac{\partial f}{\partial \overline{z}} = 0$.*

Remarks 1.1.

1)

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\{\mu_x + \mathrm{i}\,\mu_y\} + \frac{1}{2}\{\mathrm{i}\,\nu_x - \nu_y\} = 0$$
$$\Leftrightarrow \begin{array}{rcl} \mu_x & = & \nu_y \\ \mu_y & = & -\nu_x \end{array}$$
$$\Leftrightarrow \text{CR equations hold.}$$

2) f complex analytic on $D \Leftrightarrow f'(z)\ \exists$ on D[1], viz.,

$$\forall z \in D, \quad \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists.}$$

3) Analytic at $z \in \mathbb{C} \overset{\text{def}}{\Rightarrow}$ analytic on an open $D \ni z$.

Examples.

$$e^z = e^x(\cos y + \mathrm{i}\sin y), \quad \log z = \log|z| + \mathrm{i}\arg(z),$$

$$\sin z = \frac{e^{\mathrm{i}\,z} - e^{-\mathrm{i}\,z}}{2\,\mathrm{i}}, \quad \cos(z) = \frac{e^{\mathrm{i}\,z} + e^{-\mathrm{i}\,z}}{2}, \text{ etc.}$$

are analytic (on their respective domains).

4) Analytic on $\mathbb{C} \Leftrightarrow$ entire.

Complex Integration. Let $f : \mathbb{C} \to \mathbb{C}$ be of type C^1, $z(t) : I = [a,b] \subset \mathbb{R} \to C \subset \mathbb{C}$ a piecewise C^1 curve

$$\int_C f dz \overset{\text{def}}{=} \int_a^b f(z(t))z'(t)dt.$$

Note that

$$\left|\int_C f dz\right| \le \int_C |f||dz| \le \big(\max_C |f|\big) \bullet \big(\text{Arclength}(C)\big)$$

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[1] This is in fact the precise definition of complex analytic, where one only assumes *a priori* that f is continuous (It will later follow that f is C^∞, using f analytic $\Rightarrow f'$ analytic) . By choosing $\Delta z = \Delta x$ or $\mathrm{i}\,\Delta y$, the limit process gives $f'(z) = \mu_x + \mathrm{i}\,\nu_x = \nu_y - \mathrm{i}\,\mu_y$, hence the CR equations hold. Conversely, if f is of type C^1 on D and if the CR equations hold, then f is complex analytic in the sense of (1.1)(2). The reason is this: Using the mean value theorem, one can write $\Delta\mu = \mu(x+\Delta x, y+\Delta y) - \mu(x,y) = \mu_x(x,y)\Delta x + \mu_y(x,y)\Delta y + |\Delta z|\epsilon_1$, $\Delta\nu = \nu(x+\Delta x, y+\Delta y) - \nu(x,y) = \nu_x(x,y)\Delta x + \nu_y(x,y)\Delta y + |\Delta z|\epsilon_2$, where $\Delta z = \Delta x + \mathrm{i}\,\Delta y$ and $\lim_{\Delta z \to 0} \epsilon_j = 0$. Using the CR equations, one has $\lim_{\Delta z \to 0} \Delta f/\Delta z = \mu_x + \mathrm{i}\,\nu_x$.

Details

$$\int_C f dz = \int_a^b f(z(t))z'(t)dt = re^{\mathrm{i}\,\theta}$$

$$\begin{aligned} r = \left|\int_C f dz\right| = e^{-\mathrm{i}\,\theta}\int_C f dz &= \int_a^b e^{-\mathrm{i}\,\theta} f(z)(t))z'(t)dt \\ &= \int_a^b \mathrm{Re}\left(e^{-\mathrm{i}\,\theta} f(z(t))z'(t)\right)dt \\ &\le \int_a^b |\left(e^{-\mathrm{i}\,\theta} f(z(t))z'(t)\right)|dt \\ &= \int_a^b |f(z(t))||z'(t)|dt \\ &:= \int_C |f||dz| \end{aligned}$$

Let $z(t) : [a,b] \to C \subset \mathbb{C}$ define a simple-closed curve C in $\mathbb{C}$. Then we have $C = \partial D$ some region D.

Facts from exterior algebra:

$dz = dx = \mathrm{i}\,dy$, $d\overline{z} = dx - \mathrm{i}\,dy$, $dx \wedge dx = dy \wedge dy = 0$, $dx \wedge dy = -dy \wedge dx$. Thus $d\overline{z} \wedge dz = (dx - \mathrm{i}\,dy) \wedge (dx + \mathrm{i}\,dy) = 2\,\mathrm{i}\,dx \wedge dy$. $dz \wedge dz = d\overline{z} \wedge d\overline{z} = 0$.

$$\begin{aligned} \int_C f dz &\overset{\text{Stokes'}}{=} \iint_D df \wedge dz \\ &= \iint_D (\partial + \overline{\partial}) f \wedge dz \\ &= \iint_D \frac{\partial f}{\partial z} \underbrace{dz \wedge dz}_{=0} + \iint_D \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz \\ &= 2\,\mathrm{i} \iint_D \frac{\partial f}{\partial \overline{z}} dx \wedge dy \\ &=: 2\,\mathrm{i} \iint_D \frac{\partial f}{\partial \overline{z}} dxdy \end{aligned}$$

Upshot:

Cauchy-Goursat Theorem 1.2. *Assume given f analytic on D, viz., analytic on and inside a simple-closed curved $C \subset \mathbb{C}$. Then:*

$$\int_C f dz = 0.$$

[Alternatively, recall Greens' Theorem:

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Thus:

$$\int_C f dz = \int_C (\mu + \mathrm{i}\,\nu)(dx + \mathrm{i}\, dy) = \int_C \underbrace{\mu dx}_{Pdx} + \underbrace{(-\nu dy)}_{Qdy} \;+\; \mathrm{i}\int_C \underbrace{\nu dx}_{Pdx} + \underbrace{\mu dy}_{Qdy}$$
$$= -\iint_D \Big(\underbrace{\nu_x + \mu_y}_{=0}\Big) dA \;+\; \iint_D \Big(\underbrace{\mu_x - \nu_y}_{=0}\Big) dA = 0.]$$

Remark 1.2.1 Assuming only f continuous and that $f'(z)$ exists on D in the sense of (1.1)(2), one can still prove the Cauchy-Goursat theorem in this case. The idea is this: Firstly, the theorem holds if $f(z) = z^n$, n an integer ≥ 0 and where C is any closed curve (and is still true for n any integer $\neq -1$ if C is any closed curve not passing through 0). This uses the existence of an antiderivative $F(z) = z^{n+1}/(n+1)$. Next, for a simple-closed curve $C \subset \mathbb{C}$, let $\overline{D}$ be $C \bigcup$ region inside C. $\overline{D}$ is contained in a rectangle R which can be subdivided into subrectangles R_{ij}. Put $D_{ij} = R_{ij} \cap D$, and choose any $p_{ij} \in D_{ij}$. Then

$$\int_C f dz = \sum_{i,j} \int_{\partial D_{ij}} f dz = \sum_{i,j} \int_{\partial D_{ij}} f_{ij} dz,$$

where

$$f_{ij} = f(z) - f(p_{ij}) - \Delta z_{ij} f'(p_{ij}), \quad \Delta z_{ij} = z - p_{ij},$$

and where we use the fact that ∂D_{ij} is a closed curve. Using $\overline{D}$ compact, a sequential compactness argument gives us the following: For any given $\epsilon > 0$, there is a subdivision $R = \bigcup_{ij} R_{ij}$ for which $|f_{ij}| \leq |\Delta z_{ij}|\epsilon$. The rest of the proof uses standard estimates, and is left to the reader.

Corollary 1.3. *[Cauchy-Integral Formula (CIF)] (Same assumption as in Cauchy-Goursat) Let $p \in \mathrm{int}(D)$. Then*

$$f_(p) = \frac{1}{2\pi\,\mathrm{i}} \int_C \frac{f(z)}{(z-p)} dz,$$

where the orientation on the curve C is counterclockwise.

Proof. By the extended Cauchy-Goursat Theorem,

$$\begin{aligned}\int_C \frac{f(z)}{z-p} dz &= \lim_{\epsilon \to 0} \int_{|z-p|=\epsilon} \frac{f(z)}{z-p} dz \\ &= \lim_{\epsilon \to 0} \mathrm{i} \int_0^{2\pi} f(p + \epsilon e^{it}) dt \quad (z(t) = p + \epsilon e^{it}) \\ &= 2\pi\,\mathrm{i}\, f(p)\end{aligned}$$

Corollary 1.4. *[Cauchy-Integral Formula, Version II]*

$$f^{(n)}(p) = \frac{n!}{2\pi\,\mathrm{i}} \int_C \frac{f(z)}{(z-p)^{n+1}} dz.$$

Proof. By usual CIF & $w \in \operatorname{int}(D)$,

$$f(w) = \frac{1}{2\pi\,\mathrm{i}} \int_C \frac{f(z)}{(z-w)} dz.$$

Thus

$$f'(w) = \frac{1}{2\pi\,\mathrm{i}} \int_C \left(\frac{d}{dw} \frac{f(z)}{(z-w)} \right) dz = \frac{1}{2\pi\,\mathrm{i}} \int_C \frac{f(z)}{(z-w)^2} dz$$

The rest is induction...

Corollary 1.5. *f analytic $\Rightarrow f'$ analytic.*

Corollary 1.6. *[Louiville's Theorem] The only bounded entire functions are constants.*

Proof. Assume given entire $f(z)$ with $|f(z)| \le M$ on $\mathbb{C}$. Then $\forall p \in \mathbb{C}$,

$$f'(p) = \lim_{R\to\infty} \frac{1}{2\pi\,\mathrm{i}} \int_{|z-p|=R} \frac{f(z)}{(z-p)^2} dz.$$

Thus

$$|f'(p)| \le \lim_{R\to\infty} \frac{M}{2\pi} \frac{2\pi R}{R^2} = \lim_{R\to\infty} \frac{M}{R} = 0.$$

[Thus $f' \equiv 0$ on $\mathbb{C}$ with $f = \mu + \mathrm{i}\,\nu \Rightarrow 0 = f' = \mu_x + \mathrm{i}\,\nu_x \overset{\text{CR}}{=} \nu_y - \mathrm{i}\,\mu_y$, hence $\mu_x \equiv \mu_y \equiv \nu_x \equiv \nu_y \equiv 0$ on $\mathbb{C}$. Hence $\mu, \nu \in \mathbb{R} \Rightarrow f = \mu + \mathrm{i}\,\nu \in \mathbb{C}$.]

Corollary 1.7. *[Maximum-Modulus Principle] Let $f(z)$ be analytic on an open connected set $D \in \mathbb{C}$. Fix $p \in D$. If $|f(z)| \le |f(p)| \forall z \in D$, then $f(z)$ is constant on D.*

Proof. We have $f(p) = \frac{1}{2\pi\,\mathrm{i}} \int_{|z-p|=\epsilon} \frac{f(z)}{z-p} dz$. Thus

$$|f(p)| \le \frac{1}{2\pi} \int_{|z-p|=\epsilon} \frac{|f(z)|}{|z-p|} |dz| \le \frac{1}{2\pi} \int_{|z-p|=\epsilon} \frac{|f(p)|}{|z-p|} |dz| = |f(p)|.$$

Therefore $\forall\epsilon$-circles in D centered at p, $|f(z)| = |f(p)|$ on $|z-p| = \epsilon$, and hence $|f(z)| = |f(p)|$ is constant on an ϵ-disk $\Delta_\epsilon(p)$ centered at p. But $f(z)$, $|f(z)| = |f(p)|$ are analytic on $\Delta_\epsilon(p) \Rightarrow \overline{f(z)} = |f(p)|^2/f(z)$ is analytic on $\Delta_\epsilon(p)$ (provided non-zero), hence $f(z) \equiv C$ on $\Delta_\epsilon(p)$. Finally $\{z \in D \mid |f(z)| = |f(p)|\}$ is both open and closed in D, and hence is all of D since D is connected. □

<u>Remark 1.7.1</u>. Lets assume the setting in 1.7 above with the added assumptions that D is bounded and that f extends to a continuous function $f : \overline{D} \to \mathbb{C}$. Let $C := \overline{D} \backslash D$ be the boundary. By the Heine-Borel theorem, $|f(z)|$ attains a maximum at some point $p \in \overline{D}$. Then the maximum-modulus principle implies that if f is nonconstant on D, then $p \in C$.

<u>Application of Louiville</u>

Corollary 1.8. *[Fundamental Theorem of Algebra] Let $p(z) \in \mathbb{C}[z]$ be a polynomial of degree ≥ 1, i.e. a non-constant polynomial. then $\exists r \in \mathbb{C}$ such that $p(r) = 0$.*

Proof. Otherwise $q(z) := \frac{1}{p(z)}$ is entire on $\mathbb{C}$. But

$$\lim_{z \to \infty} q(z) = 0,$$

hence $|q(z)| \leq 1$ on $|z| \geq R$ for some R. Therefore by continuity, $q(z)$ is bounded, hence constant, therefore $p(z)$ is constant.

Corollary 1.9.

$$\deg p(z) = n \ \Rightarrow p(z) = \lambda \prod_{j=1}^{n} (z - r_j), \quad \lambda \in \mathbb{C}.$$

Proof. Euclid division and induction.

Morera's Theorem 1.10. *If f is continuous throughout a domain $D \in \mathbb{C}$ [viz. open connected set] and if $\int_C f(z)dz = 0 \forall$ closed contours C lying in D, then f is analytic throughout D.*

Proof. Fix $p \in D$ and define

$$F(z) = \int_p^z f(z)dz, \quad z \in D,$$

i.e. by choosing any path from p to z. This is well defined by the hypothesis, and using the connectedness of D. Then:

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} &= \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)dz \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_0^1 f(z + t\Delta z)\Delta z dt, \quad \begin{matrix}\text{(using} \\ z(t) = z + t\Delta z, \\ 0 \leq t \leq 1)\end{matrix} \\
&= \lim_{\Delta z \to 0} \int_0^1 f(z + t\Delta z)dt \\
&= \int_0^1 \lim_{\Delta z \to 0} f(z + t\Delta z)dt \\
&= f(z)t \Big|_0^1 = f(z)
\end{aligned}$$

Thus $F'(z)\exists \Rightarrow F$ analytic $\Rightarrow f = F'$ analytic by a previous Corollary (1.5).

Corollary 1.11. *Suppose that $f(z)$ is analytic on $r < |z - p| < R$. [Here $0 \leq r < R \leq \infty$.] Then $f(z)$ is equal to a Laurent series on this annular region, i.e.*

$$f(z) = \underbrace{\sum_{n=1}^{\infty} b_n (z - p)^{-n}}_{\text{principal part}} + \underbrace{\sum_{n=0}^{\infty} a_n (z - p)^n}_{\text{analytic part}}$$

Proof. for $r' < |z-p| < R' \qquad r' > r \qquad R' < R$,

$$f(z) \overset{\text{CIF}}{=} \frac{1}{2\pi\,\mathrm{i}} \underbrace{\int_{|w-p|=R'}}_{(I)} \frac{f(w)}{w-z}dw - \frac{1}{2\pi\,\mathrm{i}} \underbrace{\int_{|w-p|=r'}}_{(II)} \frac{f(w)}{w-z}dw$$

Term I:

$$\begin{aligned} \frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=R'} \frac{f(w)}{w-z}dw &= \frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=R'} \frac{f(w)}{(w-p)-(z-p)}dw \\ &= \frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=R'} \frac{f(w)}{(w-p)\left[1-\left(\frac{z-p}{w-p}\right)\right]}dw \end{aligned}$$

and using $|z-p| < |w-p|$:

$$= \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=R'} \frac{f(w)}{(w-p)^{n+1}}dw\right]}_{\text{call this } a_n} (z-p)^n$$

i.e., using $\left|\dfrac{z-p}{w-p}\right| < 1$, & interchanging $\int$ & $\sum$

Term II:

$$\begin{aligned} -\frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=r'} \frac{f(w)}{w-z}dw &= \frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=r'} \frac{f(w)}{(z-p)-(w-p)}dw \\ &= \frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=r'} \frac{f(w)}{(z-p)\left[1-\left(\frac{w-p}{z-p}\right)\right]}dw \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=r'} f(w)(w-p)^n dw\right] (z-p)^{-n-1} \end{aligned}$$

using $\dfrac{|w-p|}{|z-p|} < 1$

$$= \sum_{n=1}^{\infty} \underbrace{\left[\frac{1}{2\pi\,\mathrm{i}} \int_{|w-p|=r'} f(w)(w-p)^{n-1}\right]}_{b_n} (z-p)^{-n}$$

Corollary 1.12. *[Riemann Extension Theorem] Suppose that $f(z)$ is analytic and bounded on $0 < |z-p| < R$. Then $f(z)$ extends analytically to $z = p$.*

Proof. $f(z)$ bounded $\Rightarrow |f(z)| \leq M$. Thus using Cauchy-Goursat:

$$b_n = \lim_{\epsilon\to 0} \frac{1}{2\pi\,\mathrm{i}} \int_{|z-p|=\epsilon} f(z)(z-p)^{n-1}dz,$$

hence

$$|b_n| \leq \frac{1}{2\pi} \lim_{\epsilon\to 0} \int_{|z-p|=\epsilon} |f(z)||z-p|^{n-1}|dz| \leq \frac{2\pi\epsilon}{2\pi} M\epsilon^{n-1} = M\epsilon^n \xrightarrow{\epsilon\to 0} 0,$$

where $n \geq 1$. Thus $b_n = 0\ \forall n \geq 1 \ \Rightarrow$ no principal part. $\square$

Corollary 1.13. *Same hypothesis as in previous corollary. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n \text{ on } |z-p| < R.$$

Proof.

$$a_n = \frac{1}{2\pi\,\mathrm{i}} \int_{|z-p|=R'<R} \frac{f(z)}{(z-p)^{n+1}} dz \overset{\text{CIF}}{=} \frac{2\pi\,\mathrm{i}}{n!}\frac{f^{(n)}(p)}{2\pi\,\mathrm{i}} = \frac{f^{(n)}(p)}{n!} \;\forall n \geq 0, \& b_n = 0,$$

$\forall n \geq 1$.

Corollary 1.14. f *entire* $\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ *on* $\mathbb{C}$.

Notes 1.

$$|a_n| \leq \frac{1}{2\pi} \int_{|w-p|=R'} \frac{|f(w)|}{|w-p|^{n+1}} |dw| \leq \frac{M}{2\pi}\frac{2\pi R'}{(R')^{n+1}} = \frac{M}{(R')^n},$$

using $|f(w)| \leq M$ on $\underbrace{r' \leq |z-p| \leq R'}_{\text{cpt}}$ Thus for $|z-p| < R'$, $|a_n(z-p)^n| \leq M\lambda^n$, $\lambda = \frac{|z-p|}{R'} < 1$. $\Rightarrow$ abs convergence on limiting subdisks of radius $\to R$; $\Rightarrow$ uniform convergence on limiting subdisks of radius $\to R$ or $|\text{Remainder term}| \xrightarrow{n\to\infty} 0$. Thus e.g. $f_N(z) := \sum_{n=0}^{N} a_n(z-p)^n \xrightarrow{\text{Uniform}} f(z)$ on $|z-p| \leq R'$.

2. C closed in $|z-p| \leq R'$ $\Rightarrow$:

$$0 = \int_C f_N(z)dz \mapsto \int_C f(z)dz.$$

Now use Morera and $R' \mapsto R$ to deduce that $f(z)$ is analytic on on $|z-p| < R$.

Isolated Zeros/Poles/and Essential Singularities

Assume given $f(z)$ analytic on $0 < |z-p| < R$,

$$\Rightarrow f(z) = \sum_{n=1}^{\infty} b_n(z-p)^n + \sum_{n=0}^{\infty} a_n(z-p)^n.$$

(i) $f(z)$ is said to have a removeable singularity at p, if $b_n = 0 \forall n \geq 1$ [i.e. $f(z)$ extends analytically to p.]

(ii) $f(z)$ is said to have a pole at p if $b_n \neq 0$ for some $n \geq 1$ and $b_n = 0\ \forall n >> 1$, i.e. a truncated principal part.

(iii) $f(z)$ is said to have an essential singularity at p if $b_n \neq 0$ for infinitely many $n \in \mathbb{N}$.

Theorem 1.15. *Assume given the setting above. Then*

(i) *p is removeable* $\Leftrightarrow \lim_{z\to p} f(z)\exists$. *(E.g. $\frac{\sin z}{z}$.)*

(ii) *p is a pole* $\Leftrightarrow \lim_{z\to p} f(z) = \infty$. *(E.g. $\frac{\cos z}{z^2}$.)*

(iii) *p is an essential singularity* $\Leftrightarrow \lim_{z\to p} f(z) \not\exists$. *(E. g. $e^{1/z}$ has essential singularity at $z = 0$.)*

Proof. Part (iii) is a consequence of the Picard theorem (to be dicussed later), but will be proven directly later. Thus the theorem will following from showing that if p is a pole, then $\lim_{z\to p} f(z) = \infty$. Thus $f(z) = \frac{b_M}{(z-p)^M} + \cdots + \frac{b_1}{(z-p)} +$ analytic part $g(z)$, (where $b_M \neq 0$),

$$\Rightarrow |f(z)| \geq \left| \left| \frac{b_M}{(z-p)^M} \right| - \left(\left| \frac{b_{M-1}}{(z-p)^{M-1}} + \cdots + \frac{b_1}{(z-p)} + g(z) \right| \right) \right|$$

$$\underset{R=|z-p|^{-1}}{\overset{|z-p| \text{ small}}{\geq}} \frac{|b_M|}{|z-p|^M} - \left(\frac{|b_{M-1}|}{|z-p|^{M-1}} + \cdots + \frac{|b_1|}{|z-p|} + |g(z)| \right)$$

$$= |b_M| R^M - \left(|b_{M-1}| R^{M-1} + \cdots \right) \to \infty.$$

Definitions 1.15.1.

(1) In the above, p is a pole of order M, i.e. M is the largest integer for which $b_M \neq 0$ [$M = 1 \Leftrightarrow p =$ simple pole.]

(2) Suppose $f(z)$ is analytic on $|z - p| < R$, and that $f(p) = 0$ but $f \not\equiv 0$ in a neighbourhood of p. Then $f(z) = a_N(z-p)^N +$ higher order terms (h.o.t.), where $a_N \neq 0$ & $N \geq 1$. In this case p is a zero of order N. Note that

$$f(z) = (z-p)^N \underbrace{(a_N + (z-p)h(z))}_{g(z) \text{ analytic}},$$

where $g(p) \neq 0$. Hence the zeros of a nonvanishing analytic function on a connected open set are isolated (hence no limit points in the zero set).

Note that f analytic at $p \Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n$ in a neighbourhood of p, therefore $f^{(n)}(p) = 0\ \forall n \geq 0 \Rightarrow f \equiv 0$ in a neighbourhood of p. Note that the subset of the domain of f where $f \equiv 0$ is both open and closed.

Proposition 1.16. *Let $D \subset \mathbb{C}$ be a domain (i.e. an open connected set), and assume given analytic $f(z) : D \to \mathbb{C}$. Then the following are equivalent:*

1) $f \equiv 0$ on D.

2) $\exists p \in D$ such that $f^{(n)}(p) = 0\ \forall n \geq 0$.

3) $\{z \in D \mid f(z) = 0\}$ has a limit point in D.

Remark 1.16.1. Let $\mathbb{P}^1 = \mathbb{C} \cup \infty$ be the extended complex plane. Note e^z has an essential singularity at ∞, where the coordinate at infinity is given by $w = 1/z$, viz., $z = \infty \Leftrightarrow w = 0$. In contrast to this is the following: We say that f is meromorphic[2] on $\mathbb{P}^1$, if the only singularities of f on $\mathbb{P}^1$ are poles [$\Rightarrow f$ has isolated poles and zeros, if nonvanishing on $\mathbb{P}^1$, hence only a finite number, since $\mathbb{P}^1 \approx S^2$ is compact]. We claim that f is a rational function on $\mathbb{C}$, i.e. $f(z) = P(z)/Q(z)$, where $P(z)$ & $Q(z)$ are polynomials.

Proof. We can assume that $\{p_1, \dots, p_N\}$ with $n_j =$ multiplicity of p_j, are the poles of f on $\mathbb{C}$. It is obvious then that

$$f(z)\prod_{j=1}^{M}(z-p_j)^{n_j}$$

is entire, which we can write in the form:

$$f(z)\prod_{j=1}^{M}(z-p_j)^{n_j} = \sum_{n=0}^{\infty} a_n z^n,$$

on $\mathbb{C}$. But f meromorphic at $\infty \Rightarrow f(1/w)$ has a pole at $w = 0$. Thus $\sum_{n=0}^{\infty} a_n z^n$ is a polynomial (i.e. $a_n = 0$ for $n >> 0$). Hence f must be a rational function.

Casorati-Weierstrass Theorem 1.17. *Suppose f is analytic on $0 < |z-p| < \delta$, with an essential singularity at p. Then $\overline{f(0 < |z-p| < \delta)} = \mathbb{C}$.*

Proof. Assume to the contrary, then $\exists c \in \mathbb{C}$ such that $|f(z) - c| > s$, some $s > 0$, on $0 < |z-p| < \delta$, therefore $g(z) := \frac{1}{f(z)-c}$ is holomorphic on $0 < |z-p| < \delta$ and bounded, $\Rightarrow g$ is analytic on $|z-p| < \delta$ by the Riemann Extension Theorem. Therefore $\frac{1}{g(z)} = f(z) - c$ has at worst a pole at p, $\Rightarrow$ same for f.

Corollary 1.18. *p is essential $\Leftrightarrow \lim_{z\to p} f(z) \nexists$.*

Corollary 1.18.1. *The only analytic automorphisms [= biholomorphisms] of $\mathbb{C}$ are the functions of the form $f(z) = az + b,\ a, b \in \mathbb{C},\ a \neq 0$.*

Proof. Without loss of generality (by replacing $f(z)$ by $f(z) - f(0)$), $f(0) = 0$. Must show $f(z) = az$. Set $h(z) = f(\frac{1}{z}),\ z \neq 0$. We claim that h does not have an essential singularity at $z = 0$. Since f is an analytic isomorphism and $f(0) = 0$, f takes a neigbourhood of 0 onto a neighbourhood of 0 (bijectively.) Thus f an automorphism of $\mathbb{C} \Rightarrow |f(w)| > c$ for $|w| > \frac{1}{\delta}$, some $\delta, c > 0$. Thus for $w = \frac{1}{z},\ |h(z)| > c$ for $0 < |z| < \delta$. Therefore from the above theorem, $z = 0$ is not an essential singularity of h. Therefore $f(z)$ is a polynomial of $\deg N$ for some $N \in \mathbb{N}$. Therefore all roots of f are the same, $\Rightarrow f(z) = a(z - z_0)^N$, therefore $N = 1$.

Application of the Max-Mod Principle and Schwartz's Lemma

[2]Let $D \subset \mathbb{C}$ be an open set, and $f : D \to \mathbb{C}$ a map. We say that f is meromorphic on D if for any $p \in D$, there is an $\epsilon > 0$ such that $\Delta_\epsilon(p) := \{0 < |z-p| < \epsilon\} \subset D$ and that $f(z) = \sum_{n=1}^{N} b_n/(z-p)^n + \sum_{n=0}^{\infty} a_n(z-p)^n$ on $\Delta_\epsilon(p)$, i.e. a truncated principal part.

Lemma 1.19. *[Schwartz] Let $D = \{z \in \mathbb{C} \bigm| |z| < 1\}$, and assume given an analytic f on D with:*

(a) $|f(z)| \leq 1$ on D,

(b) $f(0) = 0$.

Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ on D. Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$, for some $z \neq 0$, then $\exists c$ with $|c| = 1$, such that $f(w) = cw\ \forall w \in D$, i.e. $f(w) = e^{\mathrm{i}\,t}w$ for some $t \in \mathbb{R}$ (rotation by t).

Proof. Define

$$g(z) = \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

Then $g(z)$ is analytic on D by the Riemann Extension Theorem. Fix $0 < r < 1$. Even though *a priori*, $|\frac{f(z)}{z}| \leq \frac{1}{|z|}$ has the potential of being $> r^{-1}$ for $|z| < r$, by the max-mod principle, $|g(z)| \leq r^{-1}\ \forall |z| \leq r$ and $0 < r < 1$. Thus $r \mapsto 1 \Rightarrow |g(z)| \leq 1\ \forall z \in D$, i.e. $|f(z)| \leq |z|$; and $|g(0)| \leq 1$ i.e. $|f'(0)| \leq 1$. Next, If $|f(z)| = |z|$, some $z \neq 0$ or if $|f'(0)| = 1$, then $|g|$ assumes its maximum value inside D. Therefore by max-mod, $g(z) = c$ with $|c| = 1$. Therefore $f(z) = cz$ with $|c| = 1$.

Corollary 1.20. *D as above. Let $f(z) : D \to D$ be a $1-1$ analytic map of D onto itself and let $a \in D$ be given such that $f(a) = 0$. Then $\exists$ a $c \in \mathbb{C}$, $|c| = 1$, such that $f = c\varphi_a$, where*

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

Proof. Clearly $\varphi_a(z)$ is analytic for $|z| < |a|^{-1}$, $\Rightarrow \varphi_a(z)$ analytic on an open disk $\supset \overline{D}$ (closure). Note that for $t \in \mathbb{R}$,

$$|\varphi_a(e^{\mathrm{i}\,t})| = \frac{|e^{\mathrm{i}\,t} - a|}{|1 - \overline{a}e^{\mathrm{i}\,t}|} = \frac{|e^{\mathrm{i}\,t} - a|}{|e^{-\mathrm{i}\,t} - \overline{a}|} = \frac{|\xi|}{|\overline{\xi}|} = 1, \text{ where } \xi = e^{\mathrm{i}\,t} - a.$$

Thus $\varphi_a(\partial D) = \partial D$, hence $\varphi_a(D) \subset D$ by the maximum-modulus principle. Next, for $|z| < 1$, one verifies that $\varphi_a(\varphi_{-a}(z)) = z = \varphi_{-a}(\varphi_a(z))$, hence $\varphi_a : D \to D$ is $1-1$ and onto, i.e. an analytic isomorphism. Next, assume for the moment that more generally $\alpha = f(a)$, and consider $g(z) := \varphi_\alpha \circ f \circ \varphi_{-a}$. Then $g : D \xrightarrow{\sim} D$ and $g(0) = 0$, hence by Schwartz, $|g'(0)| \leq 1$. By the chain rule,

$$\begin{aligned} g'(0) &= (\varphi_\alpha \circ f)'\big(\varphi_{-a}(0)\big)\varphi'_{-a}(0) \\ &= (\varphi_\alpha \circ f)'(a)[1 - |a|^2] \\ &= \varphi'_\alpha(\alpha)f'(a)[1 - |a|^2] \\ &= \frac{[1 - |\alpha|^2]}{[1 - |\alpha|^2]^2} f'(a)[1 - |a|^2] \\ &= \left(\frac{[1 - |a|^2]}{[1 - |\alpha|^2]}\right) f'(a). \end{aligned}$$

Next, if $|g'(0)| = 1$, then the above calculation gives

$$|f'(a)| = \frac{[1 - |\alpha|^2]}{[1 - |a|^2]}.$$

Further, if $|g'(0)| = 1$, then $g(z) = cz$ on D, where $|c| = 1$, i.e. when $\alpha = 0$, viz., $f(a) = 0$, then $f \circ \varphi_{-a}(z) = g(z) = cz$, or equivalently $f \circ \varphi_{-a} \circ \varphi_a(z) = c\varphi_a(z)$, i.e. $f = c\varphi_a(z)$. Now since $f : D \xrightarrow{\sim} D$ is an analytic isomorphism, the inverse $h : D \xrightarrow{\sim} D$ will later be shown to be analytic as well. Note by definition of inverse, $h \circ f(z) = z$, $f \circ h(z) = z$, $\forall z \in D$. Moreover since $f(a) = 0$, we have $h(0) = a$. Now by the above calculation applied to both f and h, $|f'(a)| \leq \frac{1}{1-|a|^2}$ (as $\alpha = 0$), and $|h'(0)| \leq 1 - |a|^2$ (since $h(0) = a$). But since $1 = h'(0)f'(a)$ by the chain rule, we must have $|f'(a)| = \frac{1}{1-|a|^2}$. Thus indeed $|g'(0)| = 1$, and hence $f = c\varphi_a$, for some c with $|c| = 1$.

Residues

Lets assume given $f(z)$ analytic on $0 < |z - p| < R$ and write

$$f(z) = \sum_{n=1}^{\infty} b_n (z-p)^{-n} + \sum_{n=0}^{\infty} a_n (z-p)^n.$$

Using uniform convergence on compact subannuli, it follows that $\forall$ simple-closed $C \subset \{0 < |z - p| < R\}$, oriented counterclockwise,

$$\frac{1}{2\pi\,\mathrm{i}} \int_C f(z)dz = b_1.$$

Definition 1.21. $\mathrm{Res}_p f(z) := b_1$ *is called the residue of* $f(z)$ *at* p.

Note. It is better to say $\mathrm{Res}_p f(z)dz$, i.e. as a differential (1-form).

Example 1.21.1. Consider $f(z) = \sum_{n=M}^{\infty} c_n (z-p)^n$ for some $M \in \mathbb{Z}$. On $0 < |z - p| < R$, $M =$ multiplicity of a zero or pole. Then by uniform convergence on compact subannuli,

$$f'(z) = \sum_{n=M}^{\infty} n c_n (z-p)^{n-1}.$$

In fact, we can write $f(z) = (z - p)^M g(z)$ where $g(p) \neq 0$ and $g(z)$ is analytic at p. So

$$f'(z) = M(z-p)^{M-1} g(z) + (z-p)^M g'(z).$$

Thus

$$\frac{f'(z)}{f(z)} = \frac{M}{(z-p)} + \underbrace{\left(\frac{g'(z)}{g(z)}\right)}_{\text{analytic}} \quad \Rightarrow \mathrm{Res}_p\left(\frac{f'(z)}{f(z)}\right) = M.$$

Residue Theorem I (1.22). *$f(z)$ analytic on and inside a simple-closed curve C, except for a finite number of singularities $\{p_1, \dots, p_k\}$ inside C [C oriented counterclockwise]. Then $\int_C f(z)dz = 2\pi\,\mathrm{i}\sum_{j=1}^{k} \mathrm{Res}_{p_j} f(z)$.*

Proof. Draw small circles C_j centered at p_j $(j = 1, \dots, k)$ and oriented counterclockwise, inside C. By the Cauchy-Goursat theorem:

$$\int_C f(z)dz - \sum_1^k \int_{C_j} f(z)dz = 0.$$

Thus:

$$\int_C f(z)dz = \sum_1^k \int_{C_j} f(z)dz = 2\pi\,\mathrm{i} \sum_1^k \mathrm{Res}_{p_j} f(z).$$

Notation For a residue of a function $f(z)$ at p, it is better to view this as the residue at p of the 1-form $\omega = f(z)dz$. So for example, for $p \in \mathbb{C}$, $\mathrm{Res}_p \omega = \mathrm{Res}_p f(z)$. At ∞, this works out very well:

$$\omega\left(z = \frac{1}{w}\right) = f\left(\frac{1}{w}\right) d\left(\frac{1}{w}\right) = -\frac{1}{w^2} f\left(\frac{1}{w}\right) dw.$$

We define

$$\mathrm{Res}_\infty(\omega) = \mathrm{Res}_0\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right)\right).$$

Residue Theorem II (1.23). *Suppose that ω is a meromorphic 1-form on $\mathbb{P}^1$, viz. $\omega = f(z)dz$, where $f(z)$ is meromorphic on $\mathbb{P}^1$. Then*

$$\sum_{p \in \{\mathrm{sing}(f) \cup \infty\}} \mathrm{Res}_p \omega = 0.$$

Proof. Since f is meromorphic, and the reciprocal of a pole is a zero, it follows that the poles are isolated and by compactness of $\mathbb{C} \cup \infty$, there are only a finite number of singular points. Let $p_1, \dots, p_k$ be all the (pole) singularities of f in $\mathbb{C}$, and

$$R > \max\{|p_1|, \dots, |p_k|\}$$

given. Then by the Residue Theorem I:

$$\int_{|z|=R} f(z)dz = 2\pi\,\mathrm{i} \sum_{j=1}^{k} \mathrm{Res}_{p_j} f(z) = 2\pi\,\mathrm{i} \sum_{j=1}^{k} \mathrm{Res}_{p_j}(\omega).$$

But

$$f(z) = f\left(\frac{1}{w}\right),$$

$$dz = d\left(\frac{1}{w}\right) = -\frac{dw}{w^2},$$

$$z(t) = Re^{\mathrm{i}\,t} \Rightarrow w(t) = R^{-1}e^{-\mathrm{i}\,t}.$$

Thus

$$\underbrace{\int_{|z|=R} f(z)dz}_{\text{counterclockwise}} = \underbrace{\int_{|w|=R^{-1}} f\left(\frac{1}{w}\right)\left(-\frac{dw}{w^2}\right)}_{\text{clockwise}} = \underbrace{\int_{|w|=R^{-1}} f\left(\frac{1}{w}\right)\frac{dw}{w^2}}_{\text{counterclockwise}} = -2\pi\,\mathrm{i}\,Res_{\infty}(\omega)$$

$$\Rightarrow \sum_{p\in\{\mathrm{sing}(f)\cup\infty\}} \mathrm{Res}_p(\omega) = 0$$

[Could also choose a circle not containing any singularity of ω in $\mathbb{P}^1$ to get the residue theorem.] $\square$

Remark 1.24. (i) This result is valid on any compact Riemann Surface (to be defined later), using Stokes' theorem.

(ii) The same result holds for a function f with only a *finite* number of singularities on $\mathbb{C}$.

Example

$$\int_{|z|=2} \frac{z^9 e^{1/z}}{z^{10}+2} dz = ?$$

Put $f(z) = \frac{z^9 e^{1/z}}{z^{10}+2}$. Then:

$$\int_{|z|=2} \frac{z^9 e^{1/z}}{z^{10}+2} dz = 2\pi\,\mathrm{i}\,\mathrm{Res}_{w=0}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right)\right) = 2\pi\,\mathrm{i}\,.$$

Here are the details:

$$\frac{1}{w^2} f\left(\frac{1}{w}\right) = \frac{e^w}{w^{11}\left(\frac{1}{w^{10}}+2\right)} = \frac{1}{w}\left(\frac{e^w}{1+2w^{10}}\right)$$

Thus:

$$2\pi\,\mathrm{i}\,\frac{e^w}{1+2w^{10}}\bigg|_{w=0} = 2\pi\,\mathrm{i}\,.$$

Another approach is by Laurent series:

$$\begin{aligned}
\frac{z^9 e^{1/z}}{z^{10}+2} &= \frac{z^9}{z^{10}}\frac{1}{\left(1+\frac{2}{z^{10}}\right)} e^{1/z} \\
&\overset{(|z|=2\geq {}^{10}\sqrt{2})}{=} \frac{e^{1/z}}{z}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^{10n}} = \left(\sum_{m=0}^{\infty}\frac{1}{m! z^{m+1}}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^{10n}}\right) \\
&= \frac{1}{z} + \cdots
\end{aligned}$$

Thus:

$$2\pi\,\mathrm{i} = \int_{|z|=2}\frac{dz}{z} = \int_{|z|=2} f(z)dz.$$

Argument principle and Rouche's theorem

Argument Principle 1.25. *Assume given a simple closed curve C, oriented counterclockwise, and f meromorphic in the interior of C, and analytic and non-zero on C. Then*

$$\frac{1}{2\pi}\Delta_C \arg(f(z)) = N_{\text{zero},C}(f) - N_{\text{pole},C}(f),$$

where $N_{\text{zero},C}(f)$ is the number of zeros (including multiplicity) of f inside C, $N_{\text{pole},C}(f)$ is the number of poles (including multiplicity) of f inside C, and where the LHS $\frac{1}{2\pi}\Delta_C \arg(f(z))$ is interpreted to mean the winding number of $f(C)$ (about 0). [Note $C \cup \{\text{interior of } C\}$ is compact, hence there are only a finite number of zeros and poles of f inside C.]

Proof. By local analytic continuation, the winding number $\frac{1}{2\pi}\Delta_C \arg(f(z))$ is given by:

$$\frac{1}{2\pi}\Delta_C \arg(f(z)) = \left.\frac{\log f(z)}{2\pi\,\mathrm{i}}\right|_C = \frac{1}{2\pi\,\mathrm{i}}\int_C \frac{f'(z)}{f(z)}dz$$

$$\stackrel{\text{Residue thm}}{=} \quad N_{\text{zero},C}(f) - N_{\text{pole},C}(f),$$

where the latter equality also uses example 1.21.1.

<u>Example</u>. $f(z) = z^n$, $n \in \mathbb{Z}$, and $C := \{|z| = 1\}$ (counterclockwise orientation). Then $\frac{1}{2\pi}\Delta_C \arg(f(z)) = n$.

Rouche's Theorem I (1.26). *Assume given two functions $f(z)$ and $g(z)$ analytic inside and on a simple closed curve C, and assume that $|f(z)| > |g(z)|$ on C. Then $f(z)$, and $f(z)+g(z)$ have the same number of zeros (counting multiplicities) inside C.*

Proof. Note that $f(z) \neq 0$ on C. Further, $|f(z) + g(z)| \geq \big||f(z)| - |g(z)|\big| > 0$ on C, hence $f(z) + g(z) \neq 0$ on C as well. Next, by the argument principle:

$$\underbrace{\frac{1}{2\pi}\Delta_C \arg\left(\frac{f(z) + g(z)}{f(z)}\right)}_{\text{LHS}} = N_{\text{zeros},C}(f + g) - N_{\text{zeros},C}(f).$$

But:

$$\frac{f(z) + g(z)}{f(z)} = 1 + \frac{g}{f}, \quad \text{moreover } \left|\frac{g}{f}\right| < 1 \text{ on } C.$$

Hence:

$$\left\{\frac{f(z) + g(z)}{f(z)}\right\}(C)$$

doesn't wind around 0. Thus LHS $= 0$.

Rouche's Theorem II (1.27). *Assume given f, g meromorphic in a neighbourhood of $\{z \in \mathbb{C} \mid |z - p| \leq R\}$, with no zeros or poles on $\{z \in \mathbb{C} \mid |z - p| = R\}$. Then:*

$$|f(z) + g(z)| < |f(z)| + |g(z)| \text{ on } |z - p| = R \Rightarrow \mathcal{Z}_f - \mathcal{P}_f = \mathcal{Z}_g - \mathcal{P}_g,$$

where $\mathcal{Z}$ = number of zeros (including multiplicity), and $\mathcal{P}$ = number of poles (including multiplicity).

Proof. By assumption

$$\left|\frac{f(z)}{g(z)}+1\right| = \left|\frac{f(z)+g(z)}{g(z)}\right| = \frac{|f(z)+g(z)|}{|g(z)|}$$
$$< \frac{|f(z)|+|g(z)|}{|g(z)|} = \frac{|f(z)|}{|g(z)|}+1 \text{ on } |z-p|=R.$$

If $\lambda := f(z)/g(z) \in (0,\infty)$, then from the above, we arrive at $\lambda+1<\lambda+1$, which is impossible. Hence $f(z)/g(z)$ maps $\{|z-p|=R\}$ into $\Omega := \mathbb{C}\backslash\{[0,\infty)\}$. Thus $\log(f(z)/g(z))$ is a well-defined anti-derivative of $\frac{(f(z)/g(z))'}{f(z)/g(z)}$ in a neigbourhood of $\{|z-p|=R\}$. Note that:

$$\frac{[f'g-g'f]}{g^2} \bullet \frac{g}{f} = \frac{f'}{f} - \frac{g'}{g}.$$

Thus:

$$0 = \frac{1}{2\pi\,\mathrm{i}}\int_{|z-p|=R}\frac{(f(z)/g(z))'}{f(z)/g(z)}dz = \frac{1}{2\pi\,\mathrm{i}}\int_{|z-p|=R}\left[\frac{f'}{f}-\frac{g'}{g}\right]dz$$
$$= \big(\mathcal{Z}_f - \mathcal{P}_f\big) - \big(\mathcal{Z}_g - \mathcal{P}_g\big).$$

Miscellaneous Results

Cauchy's Theorem - homotopic version

Definition 1.28. *Let $\gamma_0,\ \gamma_1 : [0,1] \to D$ = open connected subset $\subset \mathbb{C}$ be 2 closed curves. Then γ_0 is homotopic to γ_1 in D if $\exists$ a continuous function $\Gamma : [0,1]\times[0,1] \to D$ such that*

$$\Gamma(s,0) = \gamma_0(s), \quad \Gamma(s,1) = \gamma_1(s); \quad (0 \le s \le 1)$$

$$\Gamma(0,t) = \Gamma(1,t); \quad (0 \le t \le 1)$$

Taking $\partial[0,1]^2$ and applying this to Γ, we arrive at $\partial\Gamma = \gamma_1 - \gamma_0$. Note that for f analytic on D:

$$0 = \int_\Gamma (df)\wedge dz = \int_{\partial\Gamma} f dz = \int_{\gamma_1} f dz - \int_{\gamma_0} f dz$$

i.e.

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz$$

[Thus if γ is a curve in D such that $\gamma \sim 0$, i.e. γ is homotopic to the constant curve, then $\int_\gamma f dz = 0$.]

Definition 1.29. *A connected open region $D \subset \mathbb{C}$ is simply-connected, if every closed curve in D is homotopic to zero.*

Corollary 1.30. *Let $D \subset \mathbb{C}$ be simply-connected and $f : D \to \mathbb{C}$ analytic. Then $\int_\gamma f dz = 0 \; \forall$ closed curves $\gamma \subset D$.*

Corollary 1.31. *Let $D \subset \mathbb{C}$ be simply-connected and $f : D \to \mathbb{C}$ analytic. Then $\int_P^Q f dz$ is independent of the path joining P to Q in D.*

Corollary 1.32. *Let $D \subset \mathbb{C}$ be simply-connected and $f : D \to \mathbb{C}$ analytic. Then an antiderivative $F(z) = \int^z f dz$ exists on D.*

Corollary 1.33. *Let D be simply-connected and $f : D \to \mathbb{C}^\times$ analytic. Then $\exists$ $g : D \to \mathbb{C}$ such that $f(z) = e^{g(z)}$. I.e. $\log f(z)$ can be defined.*

Proof. The basic idea is this: $\frac{f'(z)}{f(z)}$ analytic on $D \Rightarrow \exists g(z)$on D such that $g'(z) = \frac{f'(z)}{f(z)}$. Up to constant, $g(z) = \log f(z)$. A more precise agument is the following: Obviously by the previous corollary, $g : D \to \mathbb{C}$ exists such that $g'(z) = \frac{f'(z)}{f(z)}$. Thus $\left[e^{g(z)}\right]' = e^{g(z)}\frac{f'(z)}{f(z)} = e^{g(z)}g'(z)$. Thus:

$$\left(\frac{f}{e^{g(z)}}\right)' = \frac{\left[f'(z) - f(z)\frac{f'(z)}{f(z)}\right]e^{g(z)}}{e^{2g(z)}} = 0.$$

Hence $\frac{f}{e^{g(z)}} = K \in \mathbb{C}^\times$, $\Rightarrow f = Ke^{g(z)} = e^{g(z)+\log K}$. Now relabel $g(z) \leftrightarrow g(z) + \log K$. $\square$

<u>Open Mapping Theorem</u>

For $R > 0$, let $B(p, R) = \{z \in \mathbb{C} \mid |z - p| < R\}$.

Lemma 1.34. *Suppose f is analytic on $B(p, R)$, with $f \not\equiv$ constant. Set $\alpha = f(p)$. If $f(z) - \alpha$ has a zero of order m at $z = p$, then $\exists$ $\epsilon > 0$ & $\delta > 0$ such that for $0 < |w - \alpha| < \delta$, the equation $w = f(z)$ has exactly m simple roots in $B(p, \epsilon)$. [<u>Note</u>: Thus $f(B(p, \epsilon)) \supset B(\alpha, \delta)$.]*

Proof. Since the zeros of an analytic function are isolated, we can choose $\epsilon > 0$ such that for $\epsilon < R/2$, $f(z) = \alpha$ has no solutions for $0 < |z - p| < 2\epsilon$, and likewise $f'(z) \neq 0$ for $0 < |z-p| < 2\epsilon$, using f nonconstant. Let $\gamma(t) = p + \epsilon e^{2\pi \mathrm{i}\, t}$, $0 \leq t \leq 1$, and put $\sigma = f \circ \gamma$. Now $\alpha \notin \{\sigma\}$; thus $\exists \delta > 0$ such that $B(\alpha, \delta) \cap \{\sigma\} = \emptyset$. Hence $B(\alpha, \delta)$ is contained in some component of $\mathbb{C}\backslash\{\sigma\}$. Thus for $\beta \in B(\alpha, \delta)$, α & β belong to the same component. We now compute:

$$\begin{aligned} m = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f'(z)}{f(z) - \alpha}dz &= \underbrace{\frac{1}{2\pi \mathrm{i}}\int_\sigma \frac{dw}{w - \alpha}}_{\text{winding } \# \text{ of } \sigma \text{ about } \alpha} \\ &= \underbrace{\frac{1}{2\pi \mathrm{i}}\int_\sigma \frac{dw}{w - \beta}}_{\text{winding } \# \text{ of } \sigma \text{ about } \beta} \\ &= \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f'(z)dz}{f(z) - \beta}. \end{aligned}$$

Finally, $f'(z) \neq 0$ on $0 < |z-p| < 2\epsilon \Rightarrow$ all roots are simple. $\square$

Corollary 1.35. *[Open Mapping theorem] Assume $D \subset \mathbb{C}$ an open set, f a non-constant analytic function on D. Then $\forall$ open sets $U \subset D$, $f(U)$ is open.*

Proof. This follows from the note in the above lemma.

Remark 1.36. The open mapping theorem illustrates a fundamental difference between complex analytic functions and real functions. For example $f(t) = t^2 : \mathbb{R} \to \mathbb{R}$ has the property that $f(-\epsilon, \epsilon) = [0, \epsilon^2)$, which is not open. Thus f is not open at $0 \in \mathbb{R}$.

Corollary 1.37. *(A variant of inverse function theorem) Assume given $f : D \to \mathbb{C}$ a $1-1$ analytic function, and put $\Omega = f(D)$. Then $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $(f^{-1})'(w) = [f'(z)]^{-1}$, where $w = f(z)$.*

Proof. The open mapping theorem $\Rightarrow f^{-1} : \Omega \to D$ is continuous, i.e. $f : D \to \Omega$ is a homeomorphism. Since $f'(z) \neq 0$ on D (by above lemma), we can do the following. Let $w = f(z)$:

$$\begin{aligned}\frac{f^{-1}(w+\Delta w) - f^{-1}(w)}{\Delta w} &= \frac{f^{-1}(f(z+\Delta z)) - f^{-1}(f(z))}{f(z+\Delta z) - f(z)} \\ &= \frac{z + \Delta z - z}{f(z+\Delta z) - f(z)} = \frac{\Delta z}{\Delta f} \mapsto \frac{1}{f'(z)},\end{aligned}$$

where we use $f(z+\Delta z) - f(z) \neq 0$ for $\Delta z \neq 0$, as f is $1-1$; together with:

$$\Delta w \to 0 \Leftrightarrow \Delta z \to 0.$$

Thus

$$\frac{d}{dw} f^{-1}(w)\big|_{w=f(z)} = \frac{1}{f'(z)}.$$

$\square$

§2. Spaces of functions

Let $D \subset \mathbb{C}$ be an open set. Recall that $\mathbb{C}$ is a complete metric space, (with metric $d(z,w) = |z-w|$, i.e. where every Cauchy sequence converges). Let

$C(D,\mathbb{C}) = \{$continuous functions from D to $\mathbb{C}\}$.

Similarly, put

$C(D,\mathbb{P}^1) = \{$continuous functions from D to $\mathbb{P}^1\}$, where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

Proposition 2.0. *$\exists$ compact sets $\{K_n\}_{n\in\mathbb{N}} \subset D$ such that:*

(i) $D = \bigcup_1^\infty K_n$

(ii) $K_n \subset int(K_{n+1})\ \forall\ n$

(iii) *$K \subset D$ compact $\Rightarrow K \subset K_n$ for some n.*

Proof. Set $K_n = \{z \mid |z| \le n\} \ \cap\ \{z \mid d(z, \mathbb{C}\backslash D) \ge \frac{1}{n}\}$. Clearly K_n is closed and bounded, hence compact. It is obvious that (i) and (ii) hold. Thus $D = \bigcup_1^\infty \operatorname{int}(K_n)$, hence (iii) holds. $\square$

Now assume $D = \bigcup_1^\infty K_n$, with K_n compact, and $K_n \subset \operatorname{int}(K_{n+1})$. Define on $C(D,\mathbb{C})$:

$$\rho_n(f,g) = \sup\{\underbrace{d(f(z),g(z))}_{|f(z)-g(z)|} \bigm| z \in K_n\},$$

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right).$$

Proposition 2.1. *$\{C(D,\mathbb{C});\ \rho\}$ is a metric space.*

Proof. We first show that

$$(*) \qquad \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \le \frac{\rho_n(f,h)}{1+\rho_n(f,h)} + \frac{\rho_n(h,g)}{1+\rho_n(h,g)}.$$

But observe that:

$$\left(\frac{t}{1+t}\right)' = \frac{1}{(1+t)^2} > 0 \quad \text{on } [0,\infty].$$

Thus $\frac{t}{1+t}$ is an increasing function in t. Therefore it is obvious that $(*)$ holds if $\rho_n(f,h) \ge \rho_n(f,g)$ or $\rho_n(h,g) \ge \rho_n(f,g)$. Thus we can assume that:

$$\rho_n(f,h) \le \rho_n(f,g) \quad \& \quad \rho_n(h,g) \le \rho_n(f,g)$$

Thus

$$\begin{aligned} \frac{\rho_n(f,g)}{1+\rho_n(f,g)} &\le \frac{\rho_n(f,h)}{1+\rho_n(f,g)} + \frac{\rho_n(h,g)}{1+\rho_n(f,g)} \\ &\le \frac{\rho_n(f,h)}{1+\rho_n(f,h)} + \frac{\rho_(h,g)}{1+\rho_n(h,g)} \end{aligned}$$

Finally, since $D = \bigcup_{n=1}^\infty K_n$, if follows that $f = g$ whenever $\rho(f,g) = 0$. $\square$

Lemma 2.2. *Let $\epsilon > 0$ be given. Then $\exists \delta > 0$ and a compact set $K \subset D$, such that for $f,\ g \in C(D, \mathbb{C})$:*

$$sup\{d(f(z), g(z)) \bigm| z \in K\} < \delta \Rightarrow \rho(f, g) < \epsilon.$$

Conversely, if $\delta > 0$ and a compact K are given, $\exists \epsilon > 0$ such that for $f,\ g \in C(D, \mathbb{C})$,

$$\rho(f, g) < \epsilon \Rightarrow \ sup\{d(f(z), g(z)) \bigm| z \in K\} < \delta.$$

Proof. Choose m such that

$$\sum_{n=m+1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{\epsilon}{2},$$

and set $K = K_m$. Further, choose $\delta > 0$ such that

$$0 \le t < \delta \Rightarrow \frac{t}{1+t} < \frac{\epsilon}{2}.$$

For $1 \le n \le m,\ K_n \subset K = K_m$, and

$$\sup\{d(f(z), g(z)) \mid z \in K\} < \delta \Rightarrow \rho_n(f, g) < \delta \quad \text{for } (1 \le n \le m),$$

$$\Rightarrow \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} < \frac{\epsilon}{2} \quad \text{for } (1 \le n \le m),$$

$$\Rightarrow \rho(f, g) < \sum_{n=1}^{m} \left(\frac{1}{2}\right)^n \left(\frac{\epsilon}{2}\right) + \sum_{n=m+1}^{\infty} \left(\frac{1}{2}\right)^n < \epsilon.$$

Conversely, let K and $\delta > 0$ be given. Since $D = \bigcup_1^\infty K_n = \bigcup_1^\infty \text{int}(K_n)$ and K is compact, it follows that $K \subset K_m$ for some $m \ge 1$. Thus $\rho_m(f, g) \ge \sup\{d(f, g)g(z) \mid z \in K\}$. Choose $\epsilon > 0$ such that

$$0 \le s \le 2^m \epsilon \Rightarrow \frac{s}{1-s} < \delta.$$

If w put $s = \frac{t}{1+t}$ then $t = \frac{s}{1-s}$, and hence:

$$\frac{t}{1+t} < 2^m \epsilon \Rightarrow t < \delta.$$

Thus:

$$\rho(f, g) < \epsilon \Rightarrow \frac{\rho_m(f, g)}{1 + \rho_m(f, g)} < 2^m \epsilon, \ \Rightarrow \rho_m(f, g) < \delta.$$

□

Proposition 2.3. (a) *$\Omega \subset \{C(D, \mathbb{C}), \rho\}$ is open $\Leftrightarrow \forall\ f \in \Omega,\ \exists$ a compact set $K \subset D$ and a $\delta > 0$ such that*

$$\Omega \supset \{g \mid d(f(z), g(z)) < \delta;\ z \in K\}.$$

(b) *A sequence $\{f_n\}$ in $\{C(D, \mathbb{C}), \rho\}$ converges to f iff $\{f_n\}$ converges to f uniformly on all compact subsets of $\mathbb{C}$.*

Proof. Obvious from previous lemma.

Corollary 2.4. *The topology of open sets on $C(D, \mathbb{C})$ is independent of the choice of K_n's. (where $D = \bigcup_1^\infty K_n$, K_n compact, and $K_n \subset int(K_{n+1})$).*

Proposition 2.5. *$C(D, \mathbb{C})$ is a complete metric space.*

Proof. $\{f_n\}$ Cauchy $\Rightarrow \lim_{z\to\infty} f_n(z)$ converges pointwise to $f : D \to \mathbb{C}$. We must show that f is continuous and that $\lim_{n\to\infty} \rho(f_n, f) = 0$. Let K be compact and fix $\delta > 0$. Choose N such that $n, m \geq N \Rightarrow \sup\{d(f_n(z), f_m(z)) : \; z \in K\} < \delta$. For fixed $z \in K$, $\exists m \geq N$ such that $d(f(z), f_m(z)) < \delta$, hence $n \geq N \Rightarrow d(f(z), f_n(z)) \leq d(f(z), f_m(z)) + d(f_m(z), f_n(z)) < 2\delta$. Since N does not depend on z, it follows that $\sup\{d(f(z), f_n(z)) \mid z \in K\} \xrightarrow{n\to\infty} 0 \Rightarrow f_n \xrightarrow{U} f$ on all compact sets, e.g. on all closed balls in D, $\Rightarrow f$ continuous on D as well. $\square$

Definition 2.6. *A subset $\mathcal{F} \subset C(D, \mathbb{C})$ is <u>normal</u>, if each sequence in $\mathcal{F}$ has a subsequence which converges to a function $f \in C(D, \mathbb{C})$.*

Corollary (to definition) 2.7. *$\mathcal{F} \subset C(D, \mathbb{C})$ is normal $\Leftrightarrow$ its closure is compact.*

Proposition 2.8. *A subset $\mathcal{F} \subset C(D, \mathbb{C})$ is normal $\Leftrightarrow \forall$ compact set $K \subset D$, and $\delta > 0$, $\exists f_1, \ldots, f_n \in \mathcal{F}$ such that for $f \in \mathcal{F}$, $\exists$ at least one k, $1 \leq k \leq n$, with $sup\{d(f(z), f_k(z)) \mid z \in K\} < \delta$.*

Proof. Assume $\mathcal{F}$ normal and K, δ given. By lemma 2.2 (part II) $\exists\epsilon > 0$ such that $\rho(f, g) < \epsilon \Rightarrow \sup(d(f(z), g(z)) \mid z \in K\} < \delta$. Since $\overline{\mathcal{F}}$ is compact, $\overline{\mathcal{F}}$ is totally bounded, i.e. $\exists\, f_1, \ldots, f_n \in \mathcal{F}$ (not on the $\partial\mathcal{F}$ by a limit/continuity argument) such that $\overline{\mathcal{F}} \subset \bigcup_{k=1}^n B(f_k; \epsilon)$, hence $\mathcal{F} \subset \bigcup_{k=1}^n B(f_k; \epsilon)$, where $B(f_k; \epsilon) = \{f \mid \rho(f, f_k) < \epsilon\}$. Thus our choice of $\epsilon \Rightarrow \mathcal{F} \subset \bigcup_{k=1}^n \{f \mid d(f(z)), f_k(z)) < \delta;\; z \in K\}$. Conversely, suppose $\mathcal{F}$ has the stated property, hence so does (the complete metric space) $\overline{\mathcal{F}}$. Thus $\overline{\mathcal{F}}$ is sequentially compact, hence is compact. [More precisely $\overline{\mathcal{F}}$ is totally bounded, hence by a pigeon hole principle, it is sequentially compact. Alternatively, by Lesbegue's covering lemma, $\overline{\mathcal{F}}$ totally bounded $\Rightarrow$ compact.[3]]

Definition 2.9. *A set $\mathcal{F} \subset C(D, \mathbb{C})$ is equicontinuous at a point z_0 in $D \Leftrightarrow \forall\epsilon > 0$, $\exists\delta > 0$ such that $|z - z_0| < \delta \Rightarrow d(f(z), f(z_0)) < \epsilon \;\forall f \in \mathcal{F}$. $\mathcal{F}$ is equicontinuous over a set $E \subset D$ if $\forall\epsilon > 0$, $\exists\delta > 0$ such that for $z, z' \in E$ and $|z - z'| < \delta$, $d(f(z), f(z')) < \epsilon \;\forall f \in \mathcal{F}$.*

Proposition 2.10. *Suppose $\mathcal{F} \subset C(D, \mathbb{C})$ is equicontinuous at each point of D. Then $\mathcal{F}$ is equicontinuous over each compact set in D.*

Proof. Cover compact K by disks $B(p_j; \delta)$, $j = 1, \ldots, N$ such that $|f(z) - f(p_j)| < \frac{\epsilon}{2}$ for $z \in B(p_j; 2\delta)$. If $|z - w| < \delta$, then $z \in B(p_j; \delta)$ some j, $\Rightarrow |w - p_j| \leq |w - z| + |z - p_j| < 2\delta, \Rightarrow w \in B(p_j; 2\delta) \Rightarrow |f(z) - f(w)| \leq |f(z) - f(p_j)| + |f(p_j) - f(w)| < \epsilon$.

[3] Lesbegue's covering lemma says the following: If a metric space (X, d) is sequentially compact and $\mathcal{U} = \{U_\alpha\})_{\alpha\in I}$ is an open cover of X, then $\exists\; \epsilon > 0$ such that if $x \in X$, $\exists U_\alpha \in \mathcal{U}$ such that $B(x; \epsilon) \subset U_\alpha$.

Arzela-Ascoli Theorem 2.11. *A set $\mathcal{F} \subset C(D, \mathbb{C})$ is normal $\Leftrightarrow$ the following two conditions are satisfied.*

(a) *For each $z \in D$, $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in $\mathbb{C}$,*

(b) *$\mathcal{F}$ is equicontinuous at each point of D.*

Proof. (Outline only) Assume $\mathcal{F}$ normal. By construction of the metric ρ, the map $C(D, \mathbb{C}) \to \mathbb{C}$ given by $f \mapsto f(p)$, ($p \in D$ given; note that we can treat $\{p\}$ as a compact set), is continuous. Since $\overline{\mathcal{F}}$ is compact, its image in $\mathbb{C}$ is likewise compact. Hence (a) follows. To show (b), fix a point $p \in D$ and let $\epsilon > 0$ be given. Choose $R > 0$ such that $K := \overline{B(p; R)} \subset D$. Thus K is compact. Thus $\exists\ f_1, \ldots, f_n \in \mathcal{F}$ such that $\forall f \in \mathcal{F}$, $\exists$ at least one f_k with $\sup\{d(f(z), f_k(z)) \mid z \in K\} < \epsilon/3$. But f_k continuous $\Rightarrow d(f_k(z) f_k(p)) < \epsilon/3$ for $|z - p| < \delta$, $k = 1, \ldots, n$. Therefore $|z - p| < \delta$ & $f \in \mathcal{F}$, and k chosen so that $\sup\{d(f(z), f_k(z)) \mid z \in K\} < \epsilon/3$, then $d(f(z) f(p)) \leq d(f(z), f_k(z)) + d(f_k(z), f_k(p)) + d(f_k(p), f(p)) < \epsilon$, i.e. $\mathcal{F}$ is equicontinuous at p, $\Rightarrow$ (b).

Conversely, suppose $\mathcal{F}$ satisfies (a) & (b). Must argue that $\mathcal{F}$ is normal. Let $\{z_n\}$ be the sequence of all points in D satisfying $(\mathrm{Re}(z_n), \mathrm{Im}(z_n)) \in \mathbb{Q}^2$. [Thus $\forall z \in D$, and $\delta > 0$, $\exists\ z_n$ with $|z - z_n| < \delta$.] For each $n \geq 1$, put:

$$X_n = \overline{\{f(z_n) \mid f \in \mathcal{F}\}} \subset \mathbb{C}.$$

From (a), $(X_n, d_n := d)$ is a compact metric space. One argues that likewise

$$X := \prod_{n=1}^{\infty} X_n,$$

is a compact metric space with metric

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \right).$$

For $f \in \mathcal{F}$, set $\tilde{f} = \{f(z_1), f(z_2), \ldots\}$. Let $\{f_k\}$ be a sequence in $\mathcal{F}$. Then $\{\tilde{f}_k\}$ is a sequence in the compact metric space X. Thus $\exists \xi \in X$ and a subsequence $\{\tilde{f}_{k_i}\}$ which converges to ξ.[4] WLOG $\lim_{k \to \infty} \tilde{f}_k = \xi$. It follows easily that $\lim_{k \to \infty} f_k(z_n) = w_n$, where $\xi = \{w_n\}$. We must show that $\{f_k\}$ is a Cauchy sequence [hence it will converge to some $f \in \overline{\mathcal{F}} \subset C(D, \mathbb{C})$]. Let K be a compact set in D and $\epsilon > 0$ be given. Then it suffices to find an integer N such that $k, j \geq N \Rightarrow \sup\{d(f_k(z), f_j(z)) \mid z \in K\} < \epsilon$. Since K is compact, we must have

[4] One can also use a diagonalization process:

$f_{k_1}(z_1), f_{k_2}(z_1), \ldots$ convergent

$f_{k_1}(z_2), f_{k_{12}}(z_2), \ldots$ convergent

$f_{k_1}(z_3), f_{k_{12}}(z_3), f_{k_{1,3}}(z_3), \ldots$ convergent

Note that we retain some terms of previous rows to get a non-empty set subsequence! We also use $\frac{\rho_m(\ )}{1+\rho_m(\ )} \to 0 \Leftrightarrow \rho_m(\ ) \to 0$.

$R := d(K, \partial D) > 0$. Let $K_1 = \{z \mid d(z, K) \leq R/2\}$. the K_1 is compact, and $K \subset \mathrm{int}(K_1) \subset K_1 \subset D$. Since $\mathcal{F}$ is equicontinuous at each point of D, it is therefore likewise equicontinuous on K_1. So choose $0 < \delta < R/2$ such that $d(f(z), f(z')) < \epsilon/3$ for all $f \in \mathcal{F}$, whenever $z, z' \in K_1$ with $|z - z'| < \delta$. Let $\Omega = \{z_n | z_n \in K_1\}$. If $z \in K$, $\exists z_n$ with $|z - z_n| < \delta$. But $\delta < R/2$ gives $d(z_n, K) < R/2$, i.e. $z_n \in K_1$. Hence $\{B(w; \delta) \mid w \in \Omega\}$ is an open cover of K. Let $w_1, \ldots, w_n \in \Omega$ be given such that $K \subset \bigcup_{i=1}^{n} B(w_i; \delta)$. Note that $\lim_{k\to\infty} f_k(w_i)\exists$ for $1 \leq i \leq n$; thus $\exists$ integer N such that $j, k \geq N \Rightarrow d(f_k(w_i), f_j(w_i)) < \epsilon/3,\ i = 1, \ldots, n$. Let z be an arbitrary point of K and w_i given such that $|w_i - z| < \delta$. If $k, j \geq N$, then:

$$d(f_k(z), f_j(z)) \leq d(f_k(z), f_k(w_i)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z)) < \epsilon.$$

□

Spaces of Analytic Functions

Set:

$$H(D) = \{f \in C(D, \mathbb{C}) \mid f \text{ analytic on } D\}.$$

Proposition 2.12. *$H(D)$ is closed in $C(D, \mathbb{C})$.*

Proof. We prove the stronger result.

Theorem 2.13. *If $\{f_n\}_{\mathbb{N}} \subset H(D)$ is a sequence, and $f \in C(D, \mathbb{C})$, and $f_n \to f$, then $f \in H(D)$; moreover $f_n^{(k)} \to f^{(k)},\ \forall\ k \geq 0$.*

Proof. Since $f_n \xrightarrow{U} f$ on compact sets $K \subset D$, we have

$$\int_C f = \lim_{n\to\infty} \int_C f_n = 0,$$

[where $C \subset D$ is simple-closed with interior to $C \subset D$]. Thus f is analytic by Morera's theorem. Next, for $p \in D$ and $|w - p| \leq R$ inside D, the CIF $\Rightarrow$

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi\,\mathrm{i}} \int_{|w-p|=R} \frac{f_n(w) - f(w)}{(w - z)^{k+1}} dw \text{ for } r := |z - p| < R.$$

But $f_n \xrightarrow{U} f$ on $|z - p| \leq R$, hence $|f_n - f| \leq M_n$ say. Thus

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{M_n k!}{2\pi} \frac{2\pi R}{(R - r)^{k+1}},$$

where

$$|z - w| = |(w - p) - (z - p)| \geq \big||w - p| - |z - p|\big| = R - r$$

$$\Rightarrow f_n^{(k)} \xrightarrow{U} f^{(k)} \text{ on } \{|z - p| \leq r\}.$$

$\forall$ compact K, $K \subset \bigcup_{j=1}^{k} B(p_j; r_j)$, $f_n^{(k)} \xrightarrow{U} f^{(k)}$ on each $\overline{B(p_j, r_j)} \Rightarrow f_n^{(k)} \xrightarrow{U} f^{(k)}$ on K. □

Corollary 2.13.1. *$\{H(D), \rho\big|_{H(D)}\}$ is a complete metric space.*

Corollary 2.13.2. *If $f_n \in H(D)$ and if $\sum_1^\infty f_n(z)$ converges uniformly on compact sets to $f(z)$, then*

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Hurwitz's Theorem 2.14. *Assume given $\{f_n\}$, $f \in H(D)$ such that $f_n \to f$. If $f \neq 0$, and $\overline{B(p;R)} \subset D$, and $f(z) \neq 0$ for $|z-p| = R$, then $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow f \& f_n$ have the same number of zeros in $B(p;R)$.*

Proof. Clearly $\delta = \inf\{|f(z)| \bigm| |z-p| = R\} > 0$. But $f_n \to f$ uniformly on $\{z \bigm| |z-p| = R\}$, hence $\exists N$ such that $n \geq N \& |z-p| = R, \Rightarrow f_n(z) \neq 0$ for $|z-p| = R$ and

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)|,$$

$\overset{\text{Rouche's (II)}}{\Rightarrow}$ $f \& f_n$ have same number of zeros in $B(p;R)$.

Corollary 2.15. *Let $\{f_n\}$, $f \in H(D)$, and suppose each f_n never vanishes on D. Then either $f \equiv 0$ or f never vanishes.*

Definition 2.16. *A set $\mathcal{F} \subset H(D)$ is locally bounded if $\forall p \in D$, $\exists$ constants $M \& r > 0$ such that $\forall f \in \mathcal{F}$, $|f(z)| \leq M$ for $|z-p| < r$.*

[Corollary to definition. $\mathcal{F} \subset H(D)$ is locally bounded $\Leftrightarrow \forall$ compact sets $K \subset D$, $\exists M$ such that $|f(z)| \leq M$ $\forall f \in \mathcal{F}$ and $z \in K$.]

Montel's Theorem 2.17. *$\mathcal{F} \subset H(D)$ is normal $\Leftrightarrow \mathcal{F}$ is locally bounded.*

Proof. Assume $\mathcal{F}$ is normal, but that $\mathcal{F}$ not locally bounded. Then $\exists$ compact $K \subset D$ such that $\sup\{|f(z)| \bigm| z \in K,\ f \in \mathcal{F}\} = \infty$. I.e., $\exists$ a sequence $\{f_n\} \subset \mathcal{F}$ such that $\sup\{|f_n(z)| \bigm| z \in K\} \geq n$. But $\mathcal{F}$ normal $\Rightarrow$ $\exists f \in H(D)$ and a convergent subsequence $\{f_{n_k}\} \to f$. But this gives $\sup\{|f_{n_k}(z) - f(z) \bigm| z \in K\} \to 0$ as $k \to \infty$. If $|f(z)| \leq M$ for $z \in K$, then

$$n_k \leq \sup\{|f_{n_k}(z) - f(z)| \bigm| z \in K\} + M.$$

But RHS converges to M, a contradiction.

Conversely, suppose that $\mathcal{F}$ is locally bounded. We refer to the Ascoli-Arzela theorem 2.11. It is obvious that theorem 2.11(a) holds; thus we must prove that $\mathcal{F}$ is equicontinuous at each point of D. Fix $p \in D$ & $\epsilon > 0$. By hypothesis,

$\exists\ r > 0$ & $M > 0$ such that $\overline{B(p,r)} \subset D$ & $|f(z)| \leq M \forall z \in \overline{B(p,r)}$ and all $f \in \mathcal{F}$. Let $|z-p| < r/2$ and $f \in \mathcal{F}$ be given. Then:

$$|f(p) - f(z)| \leq \frac{1}{2\pi}\left|\int_{|w-p|=r} \frac{f(w)(p-z)}{(w-p)(w-z)}dw\right| \leq \frac{2M|p-z|}{r^2}\frac{2\pi r}{2\pi} = \frac{2M|p-z|}{r}.$$

[Here we use:

$$\frac{f(w)}{w-p} - \frac{f(w)}{w-z} = \frac{f(w)(p-z)}{(w-p)(w-z)},$$

$$\frac{r}{2} < \big||w-p| - |z-p|\big| \leq |w-z| \leq \underbrace{|w-p|}_{=r} + \underbrace{|p-z|}_{<r/2}.]$$

Now choose $\delta < \min\{\frac{r}{2}, \frac{r\epsilon}{2M}\}$. Thus $|p-z| < \delta \Rightarrow |f(z) - f(p)| < \epsilon\ \forall\ f \in \mathcal{F}$. $\square$

Corollary 2.18. *A subset $\mathcal{F} \subset H(D)$ is compact $\Leftrightarrow$ is is closed and locally bounded.*

Remark 2.19. Philosophy of Montel's theorem. In $\big(\mathbb{R}^n, d(x,y) = |x-y|\big)$, we have:

compact $\Leftrightarrow$ sequentially compact $\Leftrightarrow$ closed and bounded.

For general metric spaces, we only have:

compact $\Leftrightarrow$ sequentially compact $\Rightarrow$ closed and bounded.

In our case, (vis-à-vis Montel)

$\mathcal{F} \subset H(D)$ compact $\Leftrightarrow$ closed and locally bounded.

Linear Fractional Transformations

Special case I: Fix $a,\ b \in \mathbb{C},\ a \neq 0$, and consider the map $w = az + b$. This is a special case of maps of the form:

$$\frac{az+b}{cz+d}; \quad ad - bc \neq 0,$$

called linear fractional transformations (LFT's). [In the case $w = az + b$, we have $c = 0$ and $d = 1$. Thus $a \neq 0 \Leftrightarrow ad - bc \neq 0$.]

Observation. $w = az + b$ maps lines to lines and circles to circles.

Reason. Lines: Any line $\ell \subset \mathbb{C}$ is given by a locus of the form $|z-p| = |z-q|$, for some fixed $p \neq q$ in $\mathbb{C}$. Then

$$z = \frac{w-b}{a} \Rightarrow \left|\frac{w-b}{a} - p\right| = \left|\frac{w-b}{a} - q\right| \Rightarrow |w - (ap+b)| = |w - (aq+p)|,$$

i.e. the corresponding line $w(\ell)$ in the w-plane is given by the locus $|w - w(p)| = |w - w(q)|$.

<u>Circles</u>: Suppose C given by $|z - p| = R$. Then

$$\left|\frac{w-b}{a} - p\right| = R, \quad \text{i.e.,} \quad |w - \underbrace{(ap+b)}_{w(p)}| = |a|R,$$

a circle in the w-plane.

Special Case II: Next, consider the map

$$w = \frac{1}{z} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\} \subset \mathbb{C}.$$

In this case $a = 0,\ b = 1,\ c = 1,\ d = 0$ and hence $ad - bc = -1 \neq 0$. Writing $z = x + \mathrm{i}\,y$ and $w = \mu + \mathrm{i}\,\nu$, we have

$$w = \frac{1}{z} = \frac{x}{x^2+y^2} - \mathrm{i}\,\frac{y}{x^2+y^2}.$$

It follows that

$$\mu = \frac{x}{x^2+y^2}$$
$$\nu = \frac{-y}{x^2+y^2}$$

Similarly,

$$z = \frac{1}{w} \Rightarrow x = \frac{\mu}{\mu^2+\nu^2}, \quad y = \frac{-\nu}{\mu^2+\nu^2}.$$

Next,

$$A(x^2+y^2) + Bx + Cy + D = 0$$

will describe any line or circle (roughly, we get a circle $\Leftrightarrow A \neq 0$, a line $\Leftrightarrow A = 0$). Thus

$$A\left(\frac{\mu^2}{(\mu^2+\nu^2)^2} + \frac{\nu^2}{(\mu^2+\nu^2)^2}\right) + B\left(\frac{\mu}{\mu^2+\nu^2}\right) - C\left(\frac{\nu}{\mu^2+\nu^2}\right) + D = 0.$$

I.e.:

$$D(\mu^2+\nu^2) + B\mu + (-C)\nu + A = 0.$$

I.e.: $w = 1/z$ takes {lines/circles} to {lines/circles}.

General case: Recall that a *linear fractional* or *bilinear* transformation is given by:

$$w = \frac{az+b}{cz+d}, \quad \text{where } \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0.$$

<u>Reason for the condition on the determinant</u>: Suppose e.g. $c \neq 0$. then we can rewrite:

$$w = \frac{\frac{a}{c}(cz+d) + b - \frac{ad}{c}}{cz+d} = \frac{a}{c} - \frac{\frac{1}{c}\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{cz+d}.$$

Thus $\det = 0 \Rightarrow w = a/c$ is constant. Note that in this form $w(z)$ factors as follows:

$$z \underset{\text{case I}}{\overset{\text{Special}}{\mapsto}} cz+d \underset{\text{case II}}{\overset{\text{Special}}{\mapsto}} \frac{1}{cz+d} \underset{+\text{ rotation}}{\overset{\text{dilation}}{\mapsto}} \frac{-\frac{1}{c}\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{cz+d} \underset{\text{by } +\frac{a}{c}}{\overset{\text{translation}}{\mapsto}} w(z),$$

and if $c = 0$, then $d \neq 0$, hence $w = \tilde{a}z + \tilde{b}$ where $\tilde{a} = a/d$ and $\tilde{b} = b/d$. In all cases $w(z)$ takes {lines / circles} to {lines / circles}.

<u>Example</u> Let $w = \frac{z-1}{z+1}$ and $D = \{z \in \mathbb{C} \mid x = \mathrm{Re}(z) \geq 0\}$. Describe $w(D)$.

<u>Solution</u>. We first describe $w(\mathrm{i}\, y\text{-axis})$. Note that $\pm\,\mathrm{i},\ 0$ belong on the $\mathrm{i}\, y$-axis. We compute:

$$\begin{aligned} w(\mathrm{i}) &= \frac{\mathrm{i}-1}{\mathrm{i}+1} = \frac{-(\mathrm{i}-1)^2}{2} = \mathrm{i}\,. \\ w(-\mathrm{i}) &= \frac{-\mathrm{i}-1}{-\mathrm{i}+1} = -\frac{(1+\mathrm{i})^2}{2} = -\,\mathrm{i}\,. \\ w(0) &= -1. \end{aligned}$$

There is a unique circle thru $-1, \mathrm{i}, -\mathrm{i}$, namely $|w| = 1$. Thus $w(z)$ takes the imaginary axis line to the circle of radius 1 in the w-plane, centered at 0. Note that $1 \in D$, and that $w(1) = 0$. Thus $w(D) = \{w \in \mathbb{C} \mid |w| \leq 1\}$. Alternatively

$$|w|^2 = \left(\frac{z-1}{z+1}\right)\left(\frac{\overline{z}-1}{\overline{z}+1}\right) = \frac{|z|^2+1\ -2\,\mathrm{Re}(z)}{|z|^2+1\ +2\,\mathrm{Re}(z)}.$$

Thus

$$|w| \leq 1 \Leftrightarrow \mathrm{Re}(z) \geq 0.$$

<u>Computing inverses</u>. If

$$w = \frac{az+b}{cz+d}, \quad \text{where} \quad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0,$$

then $z = z(w)$ can also be solved as a LFT. Namely:

$$z = \frac{dw-b}{cw-a}.$$

<u>Composites of LFT's are LFT's</u>.

If

$$T(z) = \frac{az+b}{cz+d}, \quad \text{and} \quad L(z) = \frac{kz+l}{mz+n},$$

are LFT's, then $T \circ L(z)$ is the LFT given by:

$$T \circ L(z) = \frac{a\left(\frac{kz+l}{mz+n}\right) + b}{c\left(\frac{kz+l}{mz+n}\right) + d} \bullet \left(\frac{mz+n}{mz+n}\right) = \frac{(ak+bm)z + (al+bn)}{(ck+dm)z + (cl+dn)}.$$

Notice that

$$\det \begin{bmatrix} (ak+bm) & (al+bn) \\ (ck+dm) & (cl+dn) \end{bmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \det \begin{bmatrix} k & l \\ m & n \end{bmatrix}.$$

Problem. Let $P_1, P_2, P_3 \in \mathbb{C}$ be 3 distinct points. Find a LFT $T(z)$ such that $T(P_1) = 0$, $T(P_2) = 1$, and $T(P_3) = \infty$.

Solution. Set

$$T(z) = \left(\frac{z - P_1}{z - P_3}\right)\left(\frac{P_2 - P_3}{P_2 - P_1}\right).$$

Now suppose that we do the same thing for any other set of 3 distinct points $\{Q_1, Q_2, Q_3\} \subset \mathbb{C}$, viz.,

$$L(w) = \left(\frac{w - Q_1}{w - Q_3}\right)\left(\frac{Q_2 - Q_3}{Q_2 - Q_1}\right).$$

We have the following picture:

$$\begin{array}{ccccc} P_1\bullet & & 0\ \bullet & & Q_1\bullet \\ & & & \overset{L}{\longleftarrow} & \\ P_2\bullet & \overset{T}{\longrightarrow} & 1\ \bullet & & Q_2\bullet \\ & & & \overset{L^{-1}}{\longrightarrow} & \\ P_3\bullet & & \infty\bullet & & Q_3\bullet \\ (z - \text{plane}) & & (\mathbb{C} \cup \infty) & & (w - \text{plane}) \end{array}$$

We end up with $w = L^{-1} \circ T(z)$, a LFT such that $L^{-1} \circ T(P_i) = Q_i, \ \forall i = 1, 2, 3$. I.e., $L(w) = L(L^{-1} \circ T(z)) = T(z)$, i.e.:

$$\left(\frac{z - P_1}{z - P_3}\right)\left(\frac{P_2 - P_3}{P_2 - P_1}\right) = \left(\frac{w - Q_1}{w - Q_3}\right)\left(\frac{Q_2 - Q_3}{Q_2 - Q_1}\right).$$

From this equation, we need only solve for w in terms of z.

Example. Find a LFT $T(z)$ such that $T(1) = \mathrm{i}$, $T(\mathrm{i}) = -1$, and $T(-\mathrm{i}) = 2\,\mathrm{i}$.

Solution. We must solve for w in terms of z in:

$$\left(\frac{z - 1}{z + \mathrm{i}}\right)\left(\frac{2\,\mathrm{i}}{\mathrm{i} - 1}\right) = \left(\frac{w - \mathrm{i}}{w - 2\,\mathrm{i}}\right)\left(\frac{-1 - 2\,\mathrm{i}}{-1 - \mathrm{i}}\right).$$

A brute force calculation gives:

$$w = \frac{-(2 + 3\,\mathrm{i})z + (2\,\mathrm{i} - 1)}{(2\,\mathrm{i} - 1)z + \mathrm{i}}.$$

Riemann Mapping Theorem

Definition 2.20. *Two regions $D_1,\ D_2 \subset \mathbb{C}$ are biholomorphic if $\exists\ f : D_1 \to D_2$ which is holomorphic, $1-1$, and onto. [Thus $f'(z) \neq 0\ \forall z \in D_1$, hence by the inverse function theorem, f^{-1} is holomorphic.] (Note. $\mathbb{C}$ is not biholomorphic to any bounded region by Louiville's theorem.)*

Riemann Mapping Theorem 2.21. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected region, and let $p \in \Omega$. Then $\exists$ a unique analytic function $f : \Omega \to \mathbb{C}$, such that:*

(a) $f(p) = 0\ \&\ f'(p) > 0$,

(b) f *is* $1-1$,

(c) $f(\Omega) = \{z \mid |z| < 1\} =: D$.

Proof. Uniqueness of f: Suppose g also satisfies (a) - (c). Then $f \circ g^{-1} : D \to D$ is biholomorphic. Further, $f \circ g^{-1}(0) = f(p) = 0$. By the corollary 1.20 to Schwartz's lemma, $\exists\ c \in \mathbb{C},\ |c| = 1$ and where $f \circ g^{-1}(z) = c\varphi_0(z) = cz\ \forall\ z \in D$. But $f(z) = (f \circ g^{-1}) \circ (g(z)) = cz \circ g(z) = cg(z), \Rightarrow 0 < f'(p) = cg'(p)$. But $g'(p) > 0$, hence $c = 1$, i.e. $f = g$.

Existence First, Ω simply connected and $h(z) : \Omega \to \mathbb{C}^\times$ implies [by an analytic continuation argument, to be discussed in a later section] that $\pm\sqrt{h(z)}$ exists as a function on Ω. Now set

$$\mathcal{F} = \{f \in H(\Omega) \mid f \text{ is } 1-1,\ f(p) = 0,\ f'(p) > 0\ \&\ f(\Omega) \subset D\}.$$

Note that since $f(\Omega) \subset D$, $\sup\{|f(z)| \mid z \in \Omega\} \leq 1\ \forall\ f \in \mathcal{F}$. By Montel's theorem, $\mathcal{F}$ is normal if it is non-empty. Thus we will attend to showing that $\mathcal{F} \neq \emptyset$. In fact, we will show that

$$(\star) \qquad \overline{\mathcal{F}} = \mathcal{F} \cup \{0\} \quad [\Rightarrow \mathcal{F} \neq 0].$$

Let us first assume $(\star)$, and consider the function $T : H(\Omega) \to \mathbb{C}$ given by $f \mapsto f'(p)$. T is continuous, since f' can be expressed in terms of f via the CIF, or we can use the latter statement in Theorem 2.13 on uniform convergence on compact sets; moreover $\overline{\mathcal{F}}$ compact $\Rightarrow \exists\ f \in \overline{\mathcal{F}}$ such that $f'(p) \geq g'(p)\ \forall g \in \mathcal{F}$. Since it is assumed that $\mathcal{F} \neq \emptyset$, it follows from $(\star)$ that $f \in \mathcal{F}$. It remains therefore to show that $f(\Omega) = D$. Suppose $q \in D$ is given such that $q \notin f(\Omega)$. Then the function

$$\frac{f(z) - q}{1 - \overline{q}f(z)},$$

is analytic and nowhere vanishing on Ω. Thus $\exists$ analytic $h(z) : \Omega \to \mathbb{C}$ such that

$$[h(z)]^2 = \frac{f(z) - q}{1 - \overline{q}f(z)}.$$

(Note: f 1-1 $\Rightarrow h$ 1-1.) Note that the LFT

$$T_q w = \frac{w - q}{1 - \overline{q}w},$$

maps D onto D.[5] Thus $h(\Omega) \subset D$. Define $g : \Omega \to \mathbb{C}$ by the formula.

$$g(z) = \frac{|h'(p)|(h(z) - h(p))}{h'(p)(1 - \overline{h(p)}h(z))}.$$

Again,

$$g(z) = \underbrace{\frac{|h'(p)|}{h'(p)}}_{\text{modulus 1 constant}} \times T_{h(p)}(h(z)) \Rightarrow g\ 1-1,\ g(\Omega) \subset D,\ g(p) = 0.$$

Next

$$g'(p) = \frac{|h'(p)|}{h'(p)} \left[\frac{h'(p)[1 - |h(p)|^2] + \overline{h(p)}h'(p) \cdot 0}{(1 - |h(p)|^2)^2} \right] = \frac{|h'(p)|}{(1 - |h(p)|^2)}.$$

But

$$|h(p)|^2 = |\frac{\overbrace{f(p)}^{=0} - q}{1 - \overline{q}\underbrace{f(p)}_{=0}}| = | - q| = |q|$$

and

$$\frac{d}{dz}[h(z)]^2\big|_{z=p} = 2h(p)h'(p).$$

Further,

$$\frac{d}{dz}[h(z)]^2\big|_{z=p} = \frac{f'(p)[1 - \overline{q}\overbrace{f(p)}^{=0}] + \overline{q}f'(p)[\overbrace{f(p)}^{=0} - q]}{[1 - \overline{q}f(p)]^2},$$

i.e.

$$2h(p)h'(p) = f'(p)[1 - |q|^2].$$

Note that $h(p)^2 = -q$ and $g'(p) = |h'(p)|/[1 - |h(p)|^2]$. Thus:

$$g'(p) = \frac{f'(p)(1 - |q|^2)}{2\sqrt{|q|}} \cdot \frac{1}{1 - |q|} = f'(p)\left(\frac{1 + |q|}{2\sqrt{|q|}}\right) > f'(p),$$

using $1 + |q| - 2\sqrt{|q|} = (1 - \sqrt{|q|})^2 > 0$. Thus $g \in \mathcal{F}$ & $g'(p) > f'(p)$, which violates the maximality of $f'(p) > 0$. Therefore $f(\Omega) = D$.

Proof of (⋆). Since $\Omega \neq \mathbb{C}$, choose $b \in \mathbb{C}\backslash\Omega$, and let $g(z) : \Omega \to \mathbb{C}$ be a given analytic function such that $[g(z)]^2 = z - b$. If $p_1,\ p_2 \in \Omega$ & if $g(p_1) = \pm g(p_2)$, then $p_1 = p_2$, since $p_1 - b = [g(p_1)]^2 = [g(p_2)]^2 = p_2 - b$. Therefore $g(z)$ is $1-1$. By the open mapping theorem, $g(\Omega) \supset B(g(p); r)$ for some $r > 0$. If $g(z) \in B(-g(p); r)$, then $r > |g(z) + g(p)| = |-g(z) - g(p)|$. There exists $w \in \Omega$ such that $g(w) = -g(z)$.

[5] This was proven earlier in 1.20, but here's another proof: $|T(1)| = \frac{|1-q|}{|1-\overline{q}|} = 1$; similarly $|T(\mathrm{i})| = 1$; $|T(-\mathrm{i})| = 1$; $|T(0)| = |q| < 1$.

But $g(w) = -g(z) = \pm g(z) \Rightarrow w = z$ by the above. Hence $g(z) = \frac{1}{2}(g(z) + g(w)) = 0, \Rightarrow z - b = [g(z)]^2 = 0$. Thus $b \in \Omega$, a contradiction. Hence

$$g(\Omega) \cap \{w \mid |w + g(p)| < r\} = \emptyset.$$

Let $\Delta = \{w \mid |w+g(p)| < r\} = B(-g(p); r)$. Choose a LFT T such that $T(\mathbb{P}^1 \backslash \overline{\Delta}) = D$.[6] Set $g_1 = T \circ g$; then g_1 is analytic & $g_1(\Omega) \subset D$. Set $g_2 = \varphi_{g_1(p)} \circ g_1(z)$. Then $g_2(p) = 0$, $g_2(\Omega) \subset D$, & g_2 is analytic. Choose a $c \in \mathbb{C}$, $|c| = 1$, such that $g_3(z) := cg_2(z)$ satisfies $g_3'(p) > 0$. Hence $g_3 \in \mathcal{F}, \Rightarrow \mathcal{F} \neq \emptyset$.

Finally, suppose $\{f_n\} \subset \mathcal{F}$ & $f_n \to f$ in $H(\Omega)$. Clearly, $f(p) = 0$ and since $f_n' \to f'(p)$, it follows that $f'(p) \geq 0$. Choose $q_1 \in \Omega$, and put $\xi = f(q_1)$. Let $\xi_n = f_n(q_1)$. Let $q_2 \in \Omega$, $q_2 \neq q_1$ and let $K =$ closed disk centered at q_2 such that $q_1 \notin K$. Then $f_n(z) - \xi_n$ never vanishes on K since f_n is $1-1$. But $f_n(z) - \xi_n \xrightarrow{n\to\infty} f(z) - \xi$ uniformly on K. Thus Hurwitz's theorem ensures that either $f(z) - \xi$ never vanishes on K or $f(z) \equiv \xi$. If $f(z) \equiv \xi$ on K, then f is constant on Ω, a fortiori 0 since $f(p) = 0$. Otherwise we have $f(z_2) \neq f(z_1)$ $\forall$ $z_2 \neq z_1$,, i.e. f is $1-1$. But if f is $1-1$, then $f'(z)$ can never vanish. Thus $f'(p) > 0$, and $f \in \mathcal{F}$. This proves $(\star)$.

Corollary 2.22. *Among the simply connected regions in* $\mathbb{C}$, *there are only two of them up to biholomorphism, namely* $\mathbb{C}$ *and* D.

The Picard Theorems

Lemma 2.23. *Let* $D \subset \mathbb{C}$ *be a simply connected region and suppose that* f *is an analytic function as* D *that does not assume the values* 0 *or* 1. *Then* $\exists$ *an analytic function* g *on* D *such that:*

$$f(z) = -e^{\mathrm{i}\,\pi \cosh[2g(z]} \quad \text{for } z \in D.$$

Proof. Since $f : D \to \mathbb{C}^\times$ and D simply connected, it follows that $\ell(z) := \log f$ is defined and analytic on D; viz $e^{\ell(z)} = f$. Let $F(z) = \frac{1}{2\pi\,\mathrm{i}}\ell(z)$. Then $F(z) \in \mathbb{Z} \Rightarrow f(z) = e^{2\pi\,\mathrm{i}\,F(z)} = 1$, which violates our asumptions. Therefore $F(z) \notin \mathbb{Z}$ $\forall$ $z \in D$. But $F \notin \mathbb{Z} \Rightarrow F,\ 1-F : D \to \mathbb{C}^\times$; moreover D simply connected $\Rightarrow \sqrt{F}$ & $\sqrt{F-1}$ are defined and analytic on D. Thus $H(z) := \sqrt{F(z)} - \sqrt{F(z)-1}$ is defined and analytic on D as well. Furthermore, $H(z) : D \to \mathbb{C}^\times$, hence $g := \log H(z)$ is defined;

$$\Rightarrow \cosh(2g) + 1 = \frac{1}{2}\left(e^{2g} + e^{-2g}\right) + 1 = \frac{1}{2}\left(e^g + e^{-g}\right)^2 = \frac{1}{2}\left(H + \frac{1}{H}\right)^2$$

$$= \frac{1}{2}\left(2\sqrt{F(z)}\right)^2 = 2F(z) = \frac{\ell(z)}{\pi\,\mathrm{i}}.$$

Thus

$$f(z) = e^{\ell(z)} = e^{2\pi\,\mathrm{i}\,F(z)} = e^{\pi\,\mathrm{i}[\cosh(2g)+1]} = -e^{\pi\,\mathrm{i}\cosh(2g)}.$$

□

[6] First, easy to find T_0 such that $T_0(\partial\overline{\Delta}) = \{|z| = 1\}$, namely, we pick 3 points on $\partial\overline{\Delta}$ mapping to ± 1, i say. If $T(\mathbb{P}^1 - \overline{\Delta}) = \{|z| > 1\}$, then replace T by $\frac{1}{z} \circ T = 1/T$.

Lemma 2.24. *Suppose f, g and D are given as in the lemma. Then $g(D)$ contains no disk of radius* 1.

Proof. Let $n \in \mathbb{Z}$. If $\exists\, p \in D$ such that $g(p) = \pm \log(\sqrt{n} + \sqrt{n-1}) + \frac{1}{2}\,\mathrm{i}\, m\pi$, then:

$$2\cosh[2g(p)] = e^{2g(p)} + e^{-2g(p)} = e^{\mathrm{i}\, m\pi}(\sqrt{n} + \sqrt{n-1})^{\pm 2} + e^{-\mathrm{i}\, m\pi}(\sqrt{n} + \sqrt{n-1})^{\mp 2}$$

$$= (-1)^m[2n + 2(n-1)] = (-1)^m(2(2n-1)),$$

[using $\frac{1}{\sqrt{n}+\sqrt{n-1}} = \sqrt{n} - \sqrt{n-1}$].

$$\Rightarrow \cosh(2g(p)) = (-1)^m(2n-1),$$

$$\Rightarrow f(p) = -e^{((-1)^m(2n-1)\pi\,\mathrm{i})} = 1.$$

Therefore $g(z)$ cannot assume the values:

$$\Lambda := \left\{ \pm\log(\sqrt{n} + \sqrt{n-1}) \ + \ \frac{1}{2}\,\mathrm{i}\, m\pi \ \middle|\, n = 1, 2, 3, \ldots\,,\ m = 0, \pm 1, \pm 2, \ldots \right\}.$$

The values in Λ can be regarded as the vertices of a grid of rectangles in $\mathbb{C}$, with length

$$\left|\frac{1}{2}\,\mathrm{i}(m+1)\pi - \frac{1}{2}\,\mathrm{i}\, m\pi\right| = \frac{\pi}{2} < \sqrt{3},$$

and width:

$$\log(\sqrt{n+1} + \sqrt{n}) - \log(\sqrt{n} + \sqrt{n-1}) \ (> 0)$$

$$= \log\left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n-1} + \sqrt{n}}\right) = \log\left(\frac{\sqrt{1+1/n} + 1}{\sqrt{1-1/n} + 1}\right) \ (\downarrow)$$

$$\leq \log\left(\frac{\sqrt{1+\frac{1}{1}} + 1}{\sqrt{1-\frac{1}{1}} + 1}\right) = \log(1 + \sqrt{2}) < \log e = 1.$$

Thus the diameter < 2. $\square$

Little Picard Theorem 2.25. *If f is an entire function that omits two values, then f is constant.*

Proof. If $\{a, b\} \notin f(\mathbb{C})$, $a \neq b$, set

$$h(z) = \frac{f(z) - a}{b - a}.$$

Then $\{0, 1\} \notin h(\mathbb{C})$. Therefore can assume $\{0, 1\} \notin f(\mathbb{C})$. By lemma 2, $\exists$ an entire $g(z)$ such that $f(z) = -e^{\mathrm{i}\,\pi\cosh(2g(z))}$; moreover $g(\mathbb{C})$ contains no disk of radius 1. If f is non-constant, then neither is g, hence $g'(p) \neq 0$ for some p. By translation, we can assume $g'(0) \neq 0$ (viz., replace $g(z)$ by $g(z + p)$ if necessary). If $R > 0$ is given, then according to a theorem of Bloch, $g(B(0, R))$ contains a disk of radius $LR|g'(0)|$, for some $L > 0$ independent of R. Thus for $R > \left(L|g'(0)|\right)^{-1}$, it follows that $g(\mathbb{C})$ contains a disk of radius 1, a contradiction. Therefore f must be constant.

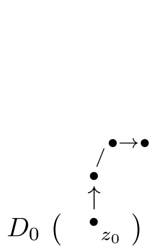
Great Picard Theorem 2.26. *Assume given an analytic function f with an essential singularity at $z = p$. Then in each neighbourhood of p, f assumes each complex number, with one possible exception, an infinite number of times.*

Example application 2.27. f $1-1$ and entire $\Rightarrow$ $f(z) = az + b$ for some $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$.

Proof. Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{C}$. Note that $f(z)$ cannot have an essential singularity at ∞, since it is $1-1$. Therefore $f(z)$ is a polynomial. Again, f $1-1 \Rightarrow$ it is a polynomial of degree 1 (i.e. one root!).

Analytic continuation along curves

Assume given a continuous curve $\gamma(t) : [a, b] \to \mathbb{C}$, with $\gamma(a) = z_0$ and $\gamma(b) = w$.



We partition the interval $[a, b]$ into:

$$a = a_0 < a_1 < a_2 < \cdots < a_{N+1} = b,$$

and further let D_i = a convex open set (e.g. a disk) containing $\gamma(a_i)$. [Definition: Recall that a set $D \subset \mathbb{C}$ is convex if $\forall$ P, $Q \in D$, the line segement $\overline{PQ} \subset D$.[7]] Note that $D_{i_1} \cap \cdots \cap D_{i_m}$ is connected, if it is non-empty.

Definition 2.28. *A sequence $\{D_0, D_1, \ldots, D_N\}$ is connected by the curve γ along the partition if $\gamma([a_i, a_{i+1}]) \subset D_i$, $\forall i$. Thus $D_i \cap D_{i+1} \ni \gamma(a_{i+1})$.*

Let f_0 be analytic on D_0. By analytic continuation of (f_0, D_0) along a connected sequence $\{D_0, \ldots, D_N\}$, we mean a sequence of pairs

$$(f_0, D_0), (f_1, D_1) \ldots, (f_N, D_N), \tag{2.29}$$

such that f_i is analytic on D_i and $f_{i+1}\big|_{D_{i+1}\cap D_i} = f_i\big|_{D_{i+1}\cap D_i}$. Thus we obtain analytic function in a neighbourhood of the end point w of the path γ, which we call the analytic continuation of (f_0, D_0) along the path γ. We denote this by f_γ. As we will see below, this will only depend on γ.

Example. $\gamma := z(t) = e^{\mathrm{i}\,t} : [0, 2\pi] \to \mathbb{C}$, $D_0 = \{|z - 1| < 1\}$. Choose a branch of $f_0 := \log z$, analytic on D_0. Then $f_\gamma(z) = \log z \;+\; 2\pi\,\mathrm{i}$.

[7] It is easy to check to check that convex $\cap$ convex = convex; moreover disks are convex [Proof: Let p_1, $p_2 \in B(p, R)$. Put $z(t) = tp_1 + (1-t)p_2$. Then $z(0) = p_2$ and $z(1) = p_1$. For $0 \le t \le 1$, we have $|z(t) - p| = |tp_1 + (1-t)p_2 - p| = |t(p_1 - p) + (1-t)(p_2 - p)| \le t|p_1 - p| + (1-t)|p_2 - p| < tr + (1-t)r = r$.

Theorem 2.30. *Let $(g_0, E_0), \ldots, (g_M E_M)$ be another analytic continuation of (g_0, E_0) along a connected sequence $\{E_0, \ldots, E_M\}$ with respect to a partition of the path γ. If $f_0 = g_0$ in some neighbourhood of z_0, then $g_M = f_N$ in some neighbourhood of $w = \gamma(b)$.*

Proof. Case 1 Same partition: Thus $M = N$. By connectivity,

$$g_0\big|_{D_0 \cap E_0} = f_0\big|_{D_0 \cap E_0}.$$

But $\gamma(a_1) \in D_0 \cap E_0 \cap D_1 \cap E_1$. Thus since $D_0 \cap E_0 \cap D_1 \cap E_1 \neq \emptyset$, hence connected, we have $f_1 = f_0 = g_0 = g_1$ on $D_0 \cap E_0 \cap D_1 \cap E_1$. Hence $f_1 = g_1$ on $D_1 \cap E_1$ (due to connectedness). Now proceed inductively.

Case 2 Change in partition: Since any two partitions have a common, refinement; to show independence of partitions it suffices to restrict to the following situation. For some k, insert a $c \in [a_k, a_{k+1}]$. Now take this connected sequence $\{D_0, \ldots, D_k, D_k, \ldots, D_N\}$. I.e. D_k repeated twice (note: $\gamma[a_k, c] \subset D_k$, and $\gamma[c, a_{k+1}] \subset D_k$). Thus $(f_0, D_0), \ldots, (f_k, D_k), (f_k, D_k), \ldots, (f_N, D_N)$ is an analytic continuation of (f_0, D_0) along this connected sequence. Thus we can reduce to the case of the same partition! $\square$

Example 2.31. $f(z) = \sqrt{z-1} := e^{\frac{1}{2}\log(z-1)}$. Let D_0 be the unit open disk centered at 2, and $\gamma := z(t) = 2e^{\mathrm{i}\,t} : [0, 2\pi] \to \mathbb{C}$. The $f_\gamma = -\sqrt{z-1}$.

Monodromy Theorem 2.32. *Let D be a domain (connected open set) and $f(z)$ analytic at $z_0 \in D$. Further, let γ, η be 2 paths joining z_0 to a point w in D. Assume*

1) γ is homotopic to η in D,

2) f can be extended analytically along any path in D. Then f_γ & f_η agree in some neighbourhod of w.

[Definition of homotopic: $\gamma \sim \eta$ means $\exists\ H : [0,1] \times [0,1] \to D$ such that $H(t,0) = \gamma(t)$, $H(t,1) = \eta(t)$, $h(0,s) = \gamma(a=0) = \eta(a=0) = z_0$, $H(1,s) = \gamma(b=1) = \eta(b=1) = w$. Here $[a,b] = [0,1]$.]

Proof. Let $s_1, s_2 \in [0,1]$, and put $\gamma_{s_j}(t) = H(t, s_j)$. If s_2 is close to s_1, then $f_{\gamma_{s_1}}$ agrees with $f_{\gamma_{s_2}}$ in a neighbourhood of w. The basic idea is that if

$$(f_0, D_0), \ldots, (f_N, D_N)$$

is a continuation along γ_{s_1}, then it will also work for γ_{s_2}, by uniform continuity of H. Hence by the previous theorem, will agree with a continuation along γ_{s_2}. Now use the compactness of $[0,1]$ to cover it by intervals $I_{\tilde{s}_1}, \ldots, I_{\tilde{s}_M}$ such that $I_{\tilde{s}_i} \bigcap I_{\tilde{s}_{i+1}} \neq \emptyset$. Thus the theorem follows from local considerations.[8]

[8]Here are more details: Fix s. Then $\gamma_s([a_i, a_{i+1}] := H([a_i, a_{i+1}], s) \subset D_i$. For any $t \in [a_i, a_{i+1}]\ \exists\ \epsilon_1,\ \epsilon_2$ such that $H\big((t-\epsilon_1, t+\epsilon_1), (s-\epsilon_2, s+\epsilon_2)\big) \subset D_i$. By compactness of $[a_i, a_{i+1}]$, we arrive at $H([a_i, a_{i+1}], (s-\epsilon, s+\epsilon)) \subset D_i$ for some $\epsilon > 0$. The partition of $\gamma_s(t)$ with connected sequence $D_0, \ldots, D_N$ deforms locally in $(s-\epsilon, s+\epsilon) =: I_\epsilon$. For any s, we can arrange such an interval I_ϵ. Thus $I = [0,1] = \bigcup_s I_{\epsilon_s} \overset{I\text{ compact}}{=} I_{\epsilon_1, s_1} \cup \cdots \cup I_{\epsilon_K, s_K}$; and where $[0,1]$ connected $\Rightarrow$ we can assume $I_{\epsilon_i, s_i} \bigcap I_{\epsilon_{i+1}, s_{i+1}} \neq \emptyset$. Analytic continuation agrees on overlaps because of common initial and end points.

For $|z| < 1$, consider

$$f(z) = -\frac{\log(1-z)}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n},$$

which is clearly holomorphic at $z = 0$.[9]

Definition 2.33. *The dilogarithm is defined for $|z| < 1$ by the integral of the power series*

$$L_2(z) := \int_0^z f(w)dw = \sum_{n=1}^{\infty} \frac{z^n}{n^2},$$

and is defined by analytic continuation in general, so that we get a function $L_{2,\gamma}(z)$ for each path γ in $\mathbb{C} - \{0,1\}$ (with beginning point 0). Thus we write

$$L_{2,\gamma}(z) = -\int_{0,\gamma}^{z} \log_\gamma(1-w)\frac{dw}{w}.$$

Let D be the simply connected set given by $D = \mathbb{C}\backslash[1,\infty)$. For $z \in D$, $f_\gamma(z)$ is independent of the path connecting 0 to z. We simply then label $f = f_\gamma$, which is clearly analytic on D. However, we are interested in the analytic continuation of $L_{2,\gamma}$ is general. For technical reasons, we will choose γ to begin at $\frac{1}{2}$, instead of 0. Thus:

$$L_{2,\gamma}(z) = -\int_{\frac{1}{2},\gamma}^{z} \log_\gamma(1-w)\frac{dw}{w}.$$

Theorem 2.34. *For $z \in \mathbb{C}\backslash\{0,1\}$, the function*

$$z \mapsto D_\gamma(z) := \operatorname{Im}\big(L_{2,\gamma}(z)\big) + \arg_\gamma(1-z)\log|z|,$$

is independent of the path γ in $\mathbb{C}\backslash\{0,1\}$.

Proof. (Outline)

I. If $\gamma \sim_{\text{hom}} \eta$ in $\mathbb{C} - \{0,1\}$, then $D_\gamma = D_\eta$. The basic idea here is that we can reduce to the local situation where γ and η are "close" to each other.

II. If η differs from γ by a small loop winding counterclockwise around 1 once, then

$$\arg_\eta(1-z) = \arg_\gamma(1-z) + 2\pi.$$

III. Thus we can reduce (up to homotopy) to $\eta = \gamma + \gamma_1$, where γ_1 is a small circle centered around 1 with winding number 1. We use the principal branch of $\frac{\log(1-z)}{z}$. Then:

$$\int_{\gamma_1} \frac{\log(1-w)}{w}dw = \int_{\gamma_1} \frac{\log(1-w)}{1-(1-w)}dw = \int_{\gamma_1} \underbrace{\log(1-w)}_{\mu} \underbrace{\sum_{n=0}^{\infty}(1-w)^n dw}_{dV}$$

[9]Details: $\frac{d}{dz}(-\log(1-z)) = \frac{1}{1-z} \overset{(|z|<1)}{=} \sum_{n=0}^{\infty} z^n$, $\Rightarrow$ $-\log(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{z^n}{n}$. Thus $-\frac{\log(1-z)}{z} = \sum_1^{\infty} \frac{z^{n-1}}{n}$ is defined on $0 < |z| < 1$, with removeable singularity at $z = 0$. $\Rightarrow$ defined on $|z| < 1$.

$$\stackrel{\text{parts}}{=} \; -\sum_{n=0}^{\infty} \frac{(1-w)^{n+1}}{n+1} \log(1-w)\Big|_{z_1,\gamma_1}^{z_1} \; + \; \underbrace{\int_{\gamma_1} \sum_{n=0}^{\infty} \frac{(1-w)^n}{n+1} dw}_{0}\,.$$

[where we use the fact that $\frac{(1-w)^n}{n+1}$ is analytic about $w = 1$.]

$$= \log z \log(1-z)\Big|_{z_1,\gamma_1}^{z_1} = 2\pi\,\mathrm{i} \log_{\mathrm{Pr}}(z_1)$$

(principal value of $\log z$).

Thus

$$-L_{2,\eta}(z) = \int_{\eta} \frac{\log(1-w)}{w} dw = \int_{\gamma} \frac{\log(1-w)}{w} dw + \int_{z_1,\gamma}^{z} 2\pi\,\mathrm{i}\,\frac{dw}{w} + 2\pi\,\mathrm{i} \log_{\mathrm{Pr}}(z_1)$$

$$= -L_{2,\gamma}(z) + 2\pi\,\mathrm{i}(\log z + 2\pi m\,\mathrm{i}) \quad (\text{some } m \in \mathbb{Z})$$

$$= -L_{2,\gamma}(z) + 2\pi\,\mathrm{i} \log z - (2\pi)^2 m.$$

Now take imaginary parts. A similar story holds if up to homotopy, η, γ differ by a loop around 0. $D(z)$ is called the Bloch-Wigner function.

§3. Several Variables

We begin with some notation. Let $\mathbb{C}$ and $\mathbb{R}$ be the fields of complex and real numbers respectively, and $z = (z_1, \dots, z_n)$ the coordinates of $\mathbb{C}^n$. If we write $z_j = x_j + \sqrt{-1}y_j$ then we can identify $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ by the $\mathbb{R}$–linear isomorphism $(z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$. Via this identification, any map $f : \mathbb{C}^n \to \mathbb{C}^m$ has a corresponding real map $f_{\mathbb{R}} : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$. We write $f \in C^k(\mathbb{C}^n)$ to mean $f_{\mathbb{R}} \in C^k(\mathbb{R}^{2n})$, i.e. of *real* differentiable class C^k. Now assume $f = (f_1, \dots, f_m) \in C^1(\mathbb{C}^n)$ and introduce the operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left\{\frac{\partial}{\partial x_j} - \mathrm{i}\,\frac{\partial}{\partial y_j}\right\} \text{ and } \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left\{\frac{\partial}{\partial x_j} + \mathrm{i}\,\frac{\partial}{\partial y_j}\right\}.$$

The complex derivative of f is given by the $m \times n$ jacobian matrix $Df(z) = (\partial f_i/\partial z_j)$. Likewise, if we write $f_j = u_j + \mathrm{i}\,v_j, u_j = u_j(x_1, y_1, \dots, x_n, y_n), v_j = v_j(x_1, y_1, \dots, x_n, y_n)$, then the corresponding real jacobian is given by

$$Df_{\mathbb{R}} = \begin{pmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial y_1 & \dots & \dots & \partial u_1/\partial x_n & \partial u_1/\partial y_n \\ \partial v_1/\partial x_1 & \partial v_1/\partial y_1 & \dots & \dots & \partial v_1/\partial x_n & \partial v_1/\partial y_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \partial v_m/\partial x_1 & \partial v_m/\partial y_1 & \dots & \dots & \partial v_m/\partial x_n & \partial v_m/\partial y_n \end{pmatrix}$$

The following diagram *does not* in general commute:

$$(**) \qquad \begin{array}{ccc} \mathbb{C}^n & \xrightarrow{Df(z)} & \mathbb{C}^m \\ \|\wr & & \|\wr \\ \mathbb{R}^{2n} & \xrightarrow{Df_{\mathbb{R}}} & \mathbb{R}^{2m} \end{array}$$

Proposition–definition 3.0. *$f : \mathbb{C}^n \to \mathbb{C}^m$ is said to be holomorphic (or complex analytic) if any of the following equivalent conditions hold:*

(1) $(**)$ commutes for all $z \in \mathbb{C}^n$, i.e. $Df_{\mathbb{R}}$ is complex linear.

(2) $\partial f_j/\partial \overline{z}_i = 0$ for all i and j, equivalently the Cauchy-Riemann equations hold:

$$\frac{\partial u_j}{\partial x_i} = \frac{\partial v_j}{\partial y_i}, \quad \frac{\partial u_j}{\partial y_i} = -\frac{\partial v_j}{\partial x_i}.$$

(3) f_j is holomorphic in z_i for all i & j. I.e. for each i,

$$\lim_{\Delta z_i \to 0} \frac{f(z_1, \dots, z_i + \Delta z_i, \dots, z_n) - f(z_1, \dots, z_n)}{\Delta z_i},$$

exists for all $z \in \mathbb{C}$.

(4) Each f_j is an absolutely convergent power series about each point $p \in \mathbb{C}^n$. [I.e. at say $p = (0, \dots, 0) \in \mathbb{C}$, then $f_j(z_1, \dots, z_n) = \sum_\alpha a_\alpha z^\alpha$ where $|a_\alpha| \leq c_1\big(c_2^{[\alpha]}\big)$, for some $c_1,\ c_2 > 0$.[10]

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftarrow$ (4) is a HW exercise. We will show how (3) $\Rightarrow$ (4). Consider $f : \mathbb{C}^n \to \mathbb{C}$ holomorphic, i.e. $f = f_j$ in above proposition. Set $p = (0, \dots, 0) \in \mathbb{C}$. Consider $r = (r_1, \dots, r_n),\ r' = (r_1', \dots, r_n') \in \mathbb{R}^n$, with $0 < r_j < r_j'\ \forall j$, and consider $p \in \overline{\Delta}_r \subset \Delta_{r'} \subset \overline{\Delta_{r'}}$, where $\Delta_r = \{z \in \mathbb{C}^n \mid |z_j| < r_j,\ \forall j\}$. We will exhibit $f(z)$ as a uniformly convergent power series on some nbhd of $\overline{\Delta}_r$ by integrating over $\overline{\Delta}_{r'}$. Then:

$$\begin{aligned} f(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{\xi_n = r_n'} \frac{f(z_1, \dots, z_{n-1}, \xi_n)}{(\xi_n - z_n)} d\xi_n \quad \text{(CIF)} \\ &= \dots = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|\xi_1| = r_1', \dots, |\xi_n| = r_n'} \frac{f(\xi_1, \dots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \cdots d\xi_n \end{aligned}$$

But

$$\begin{aligned} \frac{1}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} &= \frac{1}{\xi_1 \cdots \xi_n} \left(\frac{1}{(1 - \frac{z_1}{\xi_1}) \cdots (1 - \frac{z_n}{\xi_n})}\right) \\ &= \sum_{j_1, \dots, j_n = 0}^{\infty} \frac{z_1^{j_1} \cdots z_n^{j_n}}{\xi_1^{j_1 + 1} \cdots \xi_n^{j_n + 1}}. \end{aligned}$$

– Expand integrand as a power series
– Interchange order of integration and summation

It makes sense to restrict our focus to the case $m = 1$, viz., $f : \mathbb{C}^n \to \mathbb{C}$. Some results carry over to several variables:

Proposition 3.1. *(i) If f and g are holomorphic on a connected open set U and $f = g$ on a nonempty open subset of U, then $f = g$ on U.*[11]

(ii) The modulus of a (non-constant) holomorphic function f on an open set U has no maximum in U.

There are some striking differences between complex analysis of 1 variable and that of $n \geq 2$ variables. For example, set $U = \Delta_{r'} \backslash \Delta_r$.

[10]Notes: Choose $0 < c < c_2^{-1}$. If $|z_i| \leq c\ \forall i$, then $|z^\alpha| \leq c^{[\alpha]}$, and thus $|a_\alpha z^\alpha| \leq c_1 \cdot c^{[\alpha]} c_2^{[\alpha]} = c_1 \cdot \lambda^{[\alpha]}$ where $0 < \lambda = c \cdot c_2 < 1$. Next $\sum_\alpha \lambda^{[\alpha]} = \sum_\alpha \mathrm{e}^{-\rho[\alpha]}$, where $\rho = -\log(\lambda) > 0$. This series can be compared, via the integral test, to

$$\int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\rho t_1 - \cdots - \rho t_n} dt_1 \cdots dt_n = \left(\int_0^\infty \mathrm{e}^{-\rho t} dt\right)^n = \rho^{-n} < \infty.$$

[11]Set $U_0 = \{p \in U \mid f \equiv g \text{ in a nbhd of } p \text{ in } U\}$. Then U_0 is open by definition. Set $U_1 = U \backslash U_0$. I claim that U_1 is also open, and hence $U = U_0 \coprod U_1$ with $U_0 \neq \emptyset$ and U connected $\Rightarrow U_1 = \emptyset$. For $p \in U_1$ let $\Delta_\epsilon(p)$ be a disk centered at p and lying in U. Then we must show that $f \not\equiv g$ on any open subset of $\Delta_\epsilon(p)$ [$\Rightarrow \Delta_\epsilon(p) \subset U_1 \Rightarrow U_1$ open]. For this, we can now reduce to the case where $\Delta_\epsilon = \mathbb{C}^n$, f, g are holomorphic on $\mathbb{C}^n$ and $f \equiv g$ in a nbhd V of $(0, \dots, 0) \in \mathbb{C}^n$. But $f(z_1, 0, \dots, 0) = g(z_1, 0, \dots, 0)$ in a nbhd of $0 \in \mathbb{C} \Rightarrow$ can assume that $\mathbb{C} \times (0, \dots, 0) \subset V$. Now continue inductively to deduce that $V = \mathbb{C}^n$. Hence $p \in \mathbb{C}^n \Rightarrow p \in V$. This contradicts $p \in U_1$.

Hartog's Theorem 3.2. *($n \geq 2$) Any holomorphic function f on a nbhd of $U \cup \partial\Delta_{r'}$ extends analytically to $\Delta_{r'}$.*

Proof. For fixed $(z_1, \ldots, z_{n-1})$, the vertical slice region in U looks like the annulus $r_n < |z_n| < r'_n$ or the disk $|z_n| < r'_n$. We try to extend f in every slice by the CIF, setting

$$F(z_1, \ldots, z_n) = \frac{1}{2\pi\sqrt{-1}} \int_{w_n = r'_n} \frac{f(z_1, \ldots, z_{n-1}, w_n)}{(w_n - z_n)} dw_n.$$

F is clearly defined on $\Delta_{r'}$, is holomorphic in z_n by differentiation under integral sign; moreover since $\frac{\partial f}{\partial \bar{z}_j} = 0 \ \forall j = 1, \ldots, n-1$, it is holomorphic in $z_1, \ldots, z_{n-1}$ as well. Further, $F|_U = f$ by CIF. [Reason: Set $V := \{z \mid r_j < |z_j| < r'_j \ \forall j = 1, \ldots, n-1\} \subset U$. Then clearly $F|_V = f|_V$. But $V \neq \emptyset$ open in U implies that $F|_U = f$ by the CIF.]

Corollary 3.3. *A holomorphic function on the complement of a point in an open set $U \subset \mathbb{C}^n$ ($n > 1$) extends to a holomorphic function in all of U.*[12]

<u>Weierstrass Preparation Theorem</u>

Recall single variable representation of an analytic function

$$f(z) = (z - z_0)^d \mu(z),$$

where $\mu(z_0) \neq 0$, ($\Rightarrow$ zeros of f are isolated). In general:

Weierstrass Preparation Theorem $\equiv$ Local representation of holomorphic functions in several variables.

<u>The setting</u>

$$f(\underbrace{z_1, \ldots, z_{n-1}}_{\text{call this } z}, w) : \left\{ \begin{array}{l} \text{nbhd of} \\ 0 \text{ in } \mathbb{C}^n \end{array} \right\} \to \mathbb{C}$$

holomorphic, with:

(i) $f(0, \ldots, 0) = 0$

(ii) $f(0, \ldots, 0, w) = aw^d +$ (higher degree terms), where $a \neq 0$ (<u>Note</u> (i) $\Rightarrow d \geq 1$).

Choose $r,\ \delta, \epsilon > 0$, such that:

(a) $|f(0, w)| \geq \delta$ for $|w| = r$ (and in particular, can assume the only roots of $f(0, w)$ in $|w| \leq r$ are $w = 0$ (multiplicity d), using zeros of analytic functions being are isolated).

(b) $|f(z, w)| \geq \frac{\delta}{2}$ for $|w| = r$ & $||z|| < \epsilon$.

[12] One interpretation of this result is that topologically, $\mathbb{C}^n \backslash \{(0, \ldots, 0)\}$ is simply-connected for $n \geq 2$. Thus at the very least, there is no topological onbstruction to extending a given holomorphic function.

By Rouche's theorem[13], we can assume (for small ϵ) that for z fixed with $||z|| < \epsilon$, $f(z,w)$ has d roots in w (with $|w| < r$), namely $w = b_1(z), \ldots, b_d(z)$ (including multiplicity). [Note $b_j(0) = 0 \ \forall\ j$.] Let Δ_j = small disk in w-plane about $b_j(z)$ for z fixed, $\Rightarrow$, over Δ_j:

$$
\begin{aligned}
f(z,w) &= (w - b_j(z))^\ell f_j(z,w) \quad \text{(local representation)} \\
\Rightarrow \frac{\partial f}{\partial w} &= \ell (w - b_j(z))^{\ell-1} f_j(z,w) + (w - b_j(z))^\ell \frac{\partial f_j}{\partial w}(z,w) \\
\Rightarrow \frac{\partial f / \partial w}{f(z,w)} &= \frac{\ell}{(w - b_j(z))} + \underbrace{\left(\frac{(w - b_j(z))^\ell \partial f_j / \partial w}{f(z,w)} \right)}_{\substack{\text{removeable} \\ \text{singularity} \\ \text{in } w}}
\end{aligned}
$$

Therefore, by the residue theorem:

$$
b_1^q(z) + \cdots + b_d^q(z) = \frac{1}{2\pi\,\mathrm{i}} \int_{|w|=r} \frac{w^q (\partial f / \partial w)}{f(z,w)} dw, \quad q = 0, 1, 2, \ldots
$$

$$
\Rightarrow \sum_{j=1}^d b_j^q(z)
$$

are analytic functions of z for $||z|| < \epsilon$, and $q = 0, 1, 2, \ldots$. (Likewise, repeating the above discussion about Δ_j, b_j is analytic if it is a multiplicity 1 root of $f(z,w)$.)

Consider the elementary symmetric polynomials in $b_1(z), \ldots., b_d(z)$, given by:

$$
(w - b_1(z)) \cdots (w - b_d(z)) = w^d - \sigma_1(z) w^{d-1} + \cdots + (-1)^d \sigma_d(z),
$$

where:

$$
\begin{aligned}
\sigma_1(z) &= \sum_{j=1}^d b_j(z), \\
\sigma_2(z) &= \sum_{i<j} b_i(z) b_j(z) = \frac{\left(\sum_{j=1}^d b_j(z) \right)^2 - \left(\sum_{j=1}^d b_j^2(z) \right)}{2}, \\
&\vdots \\
\sigma_d(z) &= \prod_{j=1}^d b_j(z) = \cdots
\end{aligned}
$$

It is easy to show (exercise) that $\sigma_1(z), \ldots, \sigma_d(z)$ can be expressed as ploynomials in $\sum_{j=1}^d b_j^q(z), \quad q = 1, 2, \ldots,$

$$
\Rightarrow g(z,w) := w^d - \sigma_1(z) w^{d-1} + \cdots + (-1)^d \sigma_d(z),
$$

[13] Set $g_z(w) = f(z,w) - f(0,w)$. Then for $||z|| < \epsilon$, $|g_z(w)| < |f(0,w)|$ on $|w| = r$. Thus $f(0,w)$ and $f(z,w) = f(0,w) + g_z(w)$ have the same number of zeros in w in $|w| < r$.

(which is clearly holomorphic in w for fixed z), is holomorphic for $||z|| < \epsilon$ & $|w| < r$; moreover it vanishes on the same set as f. Therefore

$$h(z,w) := \frac{f(z,w)}{g(z,w)}$$

is defined and holomorphic for $||z|| < \epsilon$ & $|w| < r$ & outside zero set of f (same as g).

Next, for fixed z (with $||z|| < \epsilon$), $h(z,w)$ has removeable singularities in w (with $w| < r$), $\Rightarrow h(z,w)$ defined $||z|| < \epsilon$ & $|w| \le r$ and analytic in w for each fixed z, as well as the complement of the zero locus. Writing

$$h(z,w) = \frac{1}{2\pi\,\mathrm{i}} \int_{|\mu|=r} \underbrace{\frac{h(z,\mu)}{\mu - w}}_{\substack{\text{analytic} \\ \text{by construction}}} d\mu,$$

it follows that h is holomorphic in z as well (via differentiation in z under the integral sign). Note that $b_j(0) = 0 \ \forall\ j$, $\Rightarrow \sigma_j(0) = 0 \ \forall\ j$. This leads to:

Definition 3.4. *A Weierstrass polynomial in w is a polynomial of the form*

$$w^d + a_1(z)w^{d-1} + \cdots + a_d(z),$$

where $a_j(0) = 0 \ \forall\ j$.

In summary, we have:

Weierstrass Preparation Theorem 3.5. *Assume given*

$$f : \left\{ \begin{matrix} \text{nbhd of} \\ 0 \in \mathbb{C}^n \end{matrix} \right\} \to \mathbb{C}$$

holomorphic and $f \not\equiv 0$ on w-axis. Then in some neighbourhood of 0, f can be written uniquely in the form $f = g \cdot h$ where g is a Weierstrass polynomial of degree $d \ge 0$ in w & $h(0) \ne 0$.

Proof of uniqueness. In some small neighbourhood of 0 in $\mathbb{C}^n$, and for fixed z, the w-roots of g are the same as f (namely d of them); moreover the coefficients of the monic polynomial g are precisely the symmetric polynomials in these roots, hence uniqueness follows.

Riemann Extension Theorem 3.6. *Let Δ be a polydisk in $\mathbb{C}^n$, $f \not\equiv 0$ holomorphic on Δ (i.e. $f : \Delta \to \mathbb{C}$) and let $g : \overline{\Delta} \backslash \{f = 0\} \to \mathbb{C}$ be a given bounded holomorphic function. Then g extends to a holomorphic function on Δ.*

Proof. Without loss of generality, Δ centered at 0 (and say $f(0) = 0$), and we will extend g in a nbhd of 0. We can also assume the coordinates $f(z_1, \dots, z_{n-1}, w)$ with $f \not\equiv 0$ on the line $z = 0$. As before, we can find $r, \epsilon, \delta > 0$ such that $|f(0,w)| \ge \delta$ for $|w| = r$, and $|f(z,w)| \ge \delta/2$ for $||z|| < \epsilon$. Then f only has zeros (in a nbhd of $0 \in \Delta$) only in the interior of disks $z = z_0$ fixed ($||z_0|| < \epsilon$) & $|w| \le r$. By the 1-variable

Riemann extention theorem, we can extend g to a function $\tilde{g}$ in $||z|| < \epsilon$ &$|w| < r$, holomorphic away from $\{f = 0\} =: V(f)$ & holomorphic in w everywhere. As before, can write

$$\tilde{g}(z,w) = \frac{1}{2\pi \mathrm{i}} \int_{|\mu|=r} \frac{\tilde{g}(z,\mu)}{\mu - w} d\mu \quad \Rightarrow \tilde{g} \text{ holomorphic in } z \text{ as well.}$$

Further notation $\mathcal{O}_n :=$ ring of germs of holomorphic functions in a nbhd of 0 in $\mathbb{C}^n = \mathbb{C}\{z_1, \ldots, z_n\} =$ ring of convergent power series about 0.

Weierstrass Division Theorem 3.7. *Let $g(z,w) \in \mathcal{O}_{n-1}[w]$ be a Weierstrass polynomial of degree d in w. Then $\forall\ f \in \mathcal{O}_n$, we can write $f = gh + r$ where $r(z,w)$ is a polynomial of degree $< d$ in w.*

Proof. Let $\epsilon, \delta, r > 0$ be given as before (i.e. $|g(0,w)| \geq \delta$ for $|w| = r$ & $|g(z,w)| \geq \delta/2$ for $||z|| < \epsilon$ & $|w| = r$). Define

$$h(z,w) = \frac{1}{2\pi \mathrm{i}} \int_{|\mu|=r} \frac{f(z,\mu)}{g(z,\mu)} \frac{d\mu}{(\mu - w)},$$

which is clearly holomorphic in z and w for $||z|| < \epsilon$ and $|w| < r$ (and a natural guess for a candidate h). Setting $r = f - gh$, it is clear that r is holomorphic. Need to show $r(z,w)$ is a polynomial of degree $< d$ in w. But

$$\begin{aligned} f - gh &= \frac{1}{2\pi \mathrm{i}} \int_{|\mu|=r} \left[f(z,\mu) - g(z,w) \frac{f(z,\mu)}{g(z,\mu)} \right] \frac{d\mu}{(\mu - w)} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{|\mu|=r} \frac{f(z,\mu)}{g(z,\mu)} \left(\frac{g(z,\mu) - g(z,w)}{\mu - w} \right) d\mu \end{aligned}$$

But $P_{\mu,z}(w) := g(z,\mu) - g(z,w)$ is of degree d in w, with root $w = \mu$. Thus:

$$\frac{g(z,\mu) - g(z,w)}{\mu - w} = P_1(z,\mu) w^{d-1} + \cdots + P_d(z,\mu),$$

(for some $P_j(z,\mu)$), which is a polynomial in w of degree $\leq d - 1$.[14] Thus:

$$r(z,w) = a_1(z) w^{d-1} + \cdots + a_d(z),$$

where

$$a_j(z) = \frac{1}{2\pi \mathrm{i}} \int_{|\mu|=r} \frac{f(z,\mu)}{g(z,\mu)} P_j(z,\mu) d\mu.$$

Consequences of the Weierstrass Division Theorem

[14] Or use

$$\mu^d - w^d = (\mu - w)(\mu^{d-1} + w\mu^{d-2} + \cdots + w^{d-2}\mu + w^{d-1}).$$

1) Given $f \in \mathcal{O}_n$ with $f = \sum_{i=1}^n a_i z_i +$ higher order terms, with $a_1 \neq 0$ (and $w = z_1$), we can write $f = \mu \cdot g$ where $g \in \mathcal{O}_{n-1}[w = z_1]$ is a Weierstrass polynomial of degree 1, and $\mu(0) \neq 0$. Now let $k \in \mathcal{O}_n$. The Weierstrass division theorem implies that $k = \tilde{g}f + r$ where r has degree 0 in $\mathcal{O}_{n-1}[w]$, and where we use the fact that f and g agree up to $\bullet\mu$ (a unit, i.e. μ, $\mu^{-1} \in \mathcal{O}_n$). Thus $\deg_w r = 0 \Rightarrow r = r(z_2, \dots, z_n)$. [This decomposition is unique: First, $\deg_w(\ )$ is well-defined on $\mathcal{O}_{n-1}[w]$. Let g be Weierstrass of deg d. Then:

$$k = h_1 g + r_1 = h_2 g + r_2$$

$$\Rightarrow (h_1 - h_2)g = (r_2 - r_1)$$

$$\Rightarrow (h_1 - h_2)g = r_2 - r_1$$

Taking degrees, and if $h_1 - h_2 \neq 0$, then we have:

$$d \leq \deg_w(h_1 - h_2) + \deg_w g = \deg_w((h_1 - h_2)g) = \deg_w(r_2 - r_1) \leq d - 1,$$

i.e. $d \leq d - 1$, which is absurd!]

Thus, if we write $(f) := \mathcal{O}_n \cdot f$, then:

$$\frac{\mathcal{O}_n}{(f)} \simeq \mathcal{O}_{n-1}.$$

Corollary 3.8. *Suppose that $\{f_i \mid i = 1, n - r\}$ have independent linear terms at 0 and that $f_j(0) = 0\ \forall\ j$. Then:*

$$\frac{\mathcal{O}_n}{(f_1, \dots, f_{n-r})} \simeq \mathcal{O}_r,$$

where $(f_1, \dots, f_{n-r})$ is the ideal generated by $\{f_1, \dots, f_{n-r}\}$ in $\mathcal{O}_n$.

2) Given $f \in \mathcal{O}_n$ with $f = \sum_{i=1}^n a_i z_i +$ higher order terms, with $a_1 \neq 0$, we can write $z_1 = \mu \cdot f + r$, where $r = r(z_2, \dots, z_n)$, and $\mu(0) \neq 0$, hence $\mu^{-1} \in \mathcal{O}_n$ and $\mu^{-1}(0) \neq 0$. Thus

$$f = \mu^{-1} \cdot \big(z_1 - r(z_2, \dots, z_n)\big).$$

This is a special case of the implicit function theorem. In particular, locally, $f = 0 \Leftrightarrow z_1 = r(z_2, \dots, z_n)$. An inductive argument gives the following generalization:

Theorem 3.9. *Assume given $\{f_i \mid i = 1, \dots, n-r\}$, with $f_1(0) = \cdots = f_{n-r}(0) = 0$ and*

$$\det\left(\frac{\partial f_i}{\partial z_j}\bigg|_{1 \leq i,j \leq n-r}\right)(0) \neq 0.$$

Then there exists absolutely convergent power series $\{g_1, \dots, g_{n-r}\}$, with $g_j = g_j(z_1, \dots, z_r)$, & $g_j(0) = 0$, such that locally about $0 \in \mathbb{C}^n$,

$$f := (f_1, \dots, f_{n-r}) = 0 \Leftrightarrow x_j = g_j(z_1, \dots, z_r),\ \forall\ j = 1, \dots, n - r.$$

Some algebra facts

Let $\mathbb{A}$ be a commutative ring with unity $1 \in \mathbb{A}$. Recall that $\mathbb{A}$ is an integral domain $\Leftrightarrow$ the cancellation law holds for $\mathbb{A}$, equivalently, $x \cdot y = 0 \Rightarrow x = 0$ or $y = 0$. $\mu \in \mathbb{A}$ is a unit $\Leftrightarrow \exists \nu \in \mathbb{A}$ such that $\mu \cdot \nu = 1$. The group of units in $\mathbb{A}$ is denoted by $\mathbb{A}^{\times}$. A non-unit $\nu \in \mathbb{A} \backslash \{0\}$ is irreducible if for $x,\ y \in \mathbb{A}$, $\nu = x \cdot y \Rightarrow x \in \mathbb{A}^{\times}$ or $y \in \mathbb{A}^{\times}$. $\mathbb{A}$ is called a unique factorization domain (UFD), if if every non-zero $\xi \in \mathbb{A}$ can be written as a product $\xi = \nu_1 \cdots \nu_\ell$ of irreducibles, in a unique way, i.e. up to relabelling, the ν_i's are unique up to multiplication by units.

Here are some more facts:

(1) $\mathbb{A}$ a UFD $\Rightarrow \mathbb{A}[t]$ a UFD. This is due to Gauss' lemma. [Thus if k is a field, the polynomial ring $k[x_1, \ldots, x_n]$ is a UFD (induction on n).

(2) If $\mathbb{A}$ is a UFD, and $\mu, \nu \in \mathbb{A}[t]$ are relatively prime, then $\exists$ relatively prime $\alpha, \beta \in \mathbb{A}[t]$, $\gamma \in \mathbb{A}\backslash\{0\}$, such that $\alpha\mu + \beta\nu = \gamma$. We call γ the resultant of μ & ν.

Some notation. Let $p \in \mathbb{C}^n$. Put $\mathcal{O}_{\mathbb{C}^n,p} :=$ ring of convergent powers series at p. Thus $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0}$ The properties of $\mathcal{O}_{\mathbb{C}^n,p}$ are the same (via translation) as those of $\mathcal{O}_n$.

Proposition 3.10. *$\mathcal{O}_n$ is a UFD.*

Proof. By induction on n. It is obvious that $\mathcal{O}_1$ is a UFD. Let $f(z_1, \ldots, z_{n-1}, w) \in \mathcal{O}_n \backslash \{0\}$, where we can assume (for a suitable choice of coordinates $(z_1, \ldots, z_{n-1}, w)$, that $f(0, \ldots, 0, w) \not\equiv 0$. By the Weierstrass preparation theorem, can write $f = g \cdot \mu$ where $\mu \in \mathcal{O}_n^{\times}$, (unit) and $g \in \mathcal{O}_{n-1}[w]$ is a Weierstrass polynomial. By Gauss' lemma and induction, $\mathcal{O}_{n-1}[w]$ is a UFD. Thus we can write $g = g_1 \cdots g_m \in \mathcal{O}_{n-1}[w]$ where $g_1, \ldots, g_m$ are irreducible in $\mathcal{O}_{n-1}[w]$. Moreover $g_1, \ldots, g_m$ are uniquely determined (up to multiplication by units) in $\mathcal{O}_{n-1}[w]$. This implies the existence of an irreducible decomposition in $\mathcal{O}_n$. To prove uniqueness, suppose that $f = f_1 \cdots f_k$ is another irreducible decomposition. Then $f(0, \ldots, 0, w) \not\equiv 0 \Rightarrow f_j(0, \ldots, 0, w) \not\equiv 0\ \forall\ j$, hence can write $f_j = \tilde{g}_j \mu_j$, where $\tilde{g}_j$ is a Weierstrass polynomial, and μ_j a unit. Note: $\tilde{g}_j$ must be irreducible in $\mathcal{O}_{n-1}[w]$, as f_j is irreducible. Thus $f = g\mu = (\prod_j \tilde{g}_j) \cdot (\prod_j \mu_j))$, with $g = \prod_j g_j,\ \prod_j \tilde{g}_j$ both Weierstrass polynomials. By the Weierstrass preparation theorem (uniqueness part), $\prod_j g_j = \prod_j \tilde{g}_j$, and since $\mathcal{O}_{n-1}[w]$ is a UFD, this implies that up to multiplication by units, $\{\tilde{g}_j\}$, $\{g_j\}$ agree.

Proposition 3.11. *If f and g are relatively prime in $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,0}$, then for $||z|| < \epsilon$, f and g are relatively prime in $\mathcal{O}_{\mathbb{C}^n,z}$.*

Proof. We can assume that $f(0, \ldots, 0, z_n) \not\equiv 0$ and $g(0, \ldots, 0, z_n) \not\equiv 0$. Hence we can assume that f and g are both Weierstrass polynomials in $w = z_n$, by the Weierstrass preparation theorem. For $z' \in \mathbb{C}^{n-1}$ and $||z'||$ small, we have $f(z', z_n) \not\equiv 0$ in z_n. Recall that if γ is the resultant of f and g, then $\alpha f + \beta g = \gamma,\ \alpha, \beta \in \mathcal{O}_{n-1}[w = z_n],\ \gamma \in \mathcal{O}_{n-1}$; moreover this equation holds in some neighbourhood of $0 \in \mathbb{C}^n$. Suppose to contrary for small $||z_0||$ with $z_0 \in \mathbb{C}^n$, f and g have a common factor $h(z', z_n)$ in $\mathcal{O}_{\mathbb{C}^n, z_0}$ ($\Rightarrow h(z_0) = 0$ (otherwise h is a the unit)). Then $h|f$ & $h|g \Rightarrow h|\gamma$, $\Rightarrow h \in \mathcal{O}_{n-1}$. But $h(z_{0,1}, \ldots, z_{0,n-1}) \equiv 0$ in z_n, $\Rightarrow$ $f(z_{0,1}, \ldots, z_{o,n-1}, z_n) \equiv 0$, a contradiction to $f(z_{0,1}, \ldots, z_{0,n-1}, z_n) \not\equiv 0$. $\square$

As an application of Weierstrass division theorem, we prove the following:

Analytic Nullstellensatz Theorem 3.12. *If $f(z, w) \in \mathcal{O}_n$ is irreducible, and if $h \in \mathcal{O}_n$ vanishes on the set $\{f(z, w) = 0\}$ (in some neighbourhood of $0 \in \mathbb{C}^n$), then $f\big|h$ in $\mathcal{O}_n$.*

Proof. Without loss of generality, we can assume that f is a Weierstrass polynomial of degree k in w. Thus f being irreducible $\Rightarrow f$ & $\frac{\partial f}{\partial w}$ are relatively prime in $\mathcal{O}_{n-1}[w]$ (as $\deg_w f > \deg_w \frac{\partial f}{\partial w}$). Therefore we can write:

$$\alpha \cdot f + \beta \cdot \frac{\partial f}{\partial w} = \gamma, \quad \gamma \in \mathcal{O}_{n-1}, \quad \gamma \not\equiv 0.$$

Note that for a given z_0, if $f(z_0, w)$ has a multiple root μ, then

$$f(z_0, \mu) = \frac{\partial f}{\partial w}(z_0, \mu) = 0, \quad \Rightarrow \gamma(z_0) = 0.$$

Therefore $f(z, w)$ has k distinct roots in w for $\gamma(z) \neq 0$. By the division theorem, $h = f \cdot g + r$, $r \in \mathcal{O}_{n-1}[w]$, $\deg r < k$. But for z_0 outside $\{\gamma = 0\}$, $f(z_0, w)$ and hence $h(z_0, w)$ share at least k distinct roots in w. Thus $\deg_w r < k \Rightarrow r(z_0, w) = 0$ in $\mathbb{C}[w]$. Thus $r \equiv 0$ and therefore $h = f \cdot g$. $\square$

<u>The inverse and implicit function theorems</u>

We now consider $f : \mathbb{C}^n \mapsto \mathbb{C}^m$ any C^∞ map, and let $(w_1, \ldots, w_m)$ be affine cordinates for $\mathbb{C}^m$. Thus we can write $w := (w_1, \ldots, w_m) = f = (f_1, \ldots, f_m)$. $f_j = f_j(z_1, \ldots, z_n)$. Write $z_j = x_j + \mathrm{i}\, y_j$ and likewise, $f_j = w_j = \mu_j + \mathrm{i}\, \nu_j$. For $p \in \mathbb{C}^n$, and $q = f(p)$, we write for brevity, $\frac{\partial}{\partial z_i}$ for $\frac{\partial}{\partial z_i}|_p$, $\frac{\partial}{\partial w_j}$ for $\frac{\partial}{\partial w_j}|_q$. The holomorphic tangent spaces[15] of $\mathbb{C}^n$ and $\mathbb{C}^m$ at p and q respectively, are given by:

$$T_p(\mathbb{C}^n) = \mathbb{C}\frac{\partial}{\partial z_1} \oplus \cdots \oplus \mathbb{C}\frac{\partial}{\partial z_n}$$
$$T_q(\mathbb{C}^m) = \mathbb{C}\frac{\partial}{\partial w_1} \oplus \cdots \oplus \mathbb{C}\frac{\partial}{\partial w_m}$$

[15]Our point of view of tangent spaces is via derivations. A p-centered derivation is a $\mathbb{C}$-linear map $D : \mathcal{O}_{\mathbb{C}^n, p} \to \mathbb{C}$ satisfying Leibniz' rule, viz., $D(f \cdot g) = g(p)D(f) + f(p)D(g) \in \mathbb{C}$. For any $f \in \mathcal{O}_{\mathbb{C}^n, p}$, we can write $f(z) = f(p) + \sum_{j=1}^n \frac{\partial f}{\partial z_j}(p)(z_j - p_j) + h$, where h involves higher order terms in $z - p = (z_1 - p_1, \ldots, z_n - p_n)$. By Leibniz' rule and linearity, it is clear that $D(f(p)) = D(h) = 0$, and hence $D(f) = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(p)D(z_j) = \left(\sum_{j=1}^n a_j \frac{\partial}{\partial z_j}\Big|_p\right)(f)$, where $a_j := D(z_j) \in \mathbb{C}$. Note that $\left(\sum a_j \frac{\partial}{\partial z_j}\right)\Big|_p = 0 \Rightarrow a_\ell = \left(\sum a_j \frac{\partial}{\partial z_j}\right)\Big|_p (z_\ell) = 0$, hence $\left\{\frac{\partial}{\partial z_1}\Big|_p, \ldots, \frac{\partial}{\partial z_1}\Big|_p\right\}$ are independent and span the space of p-derivations $\mathrm{Der}_p(\mathcal{O}_{\mathbb{C}^n, p})$. One puts $T_p(\mathbb{C}^n) = \mathrm{Der}_p(\mathcal{O}_{\mathbb{C}^n, p})$. This provides a coordinate free definition of the tangent space. Given a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^m$ with $f(p) = q$, one has an induced linear map $df(p) : T_p(\mathbb{C}^n) \to T_q(\mathbb{C}^m)$ defined as follows. Let $g \in \mathcal{O}_{\mathbb{C}^m, q}$. Then $f^*(g) := g \circ f \in \mathcal{O}_{\mathbb{C}^n, p}$. For any derivation $\xi \in \mathrm{Der}_p(\mathcal{O}_{\mathbb{C}^n, p}$, $df(p)(\xi) \in \mathrm{Der}_q(\mathcal{O}_{\mathbb{C}^m, q}$ is given by the formula $\big(df(p)(\xi)\big)(g) = \xi(f^*(g))$. If we write $w = (w_1, \ldots, w_m) = f = (f_1, \ldots, f_m)$, $f_j = f_j(z_1, \ldots, z_n)$, and where $\left\{\frac{\partial}{\partial z_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p\right\}$, $\left\{\frac{\partial}{\partial w_1}\Big|_q, \ldots, \frac{\partial}{\partial w_m}\Big|_q\right\}$

Using the identifications $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, $\mathbb{R}^{2m} \simeq \mathbb{C}^m$, we have the corresponding real tangent spaces $T_p(\mathbb{C}^n)_\mathbb{R}$ and $T_q(\mathbb{C}^m)_\mathbb{R}$, with corresponding complexifications given by:

$$\begin{aligned} T_p(\mathbb{C}^n)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} &= \mathbb{C}\frac{\partial}{\partial x_1} \oplus \mathbb{C}\frac{\partial}{\partial y_1} \oplus \cdots \oplus \mathbb{C}\frac{\partial}{\partial x_n} \oplus \mathbb{C}\frac{\partial}{\partial y_n} \\ &= \mathbb{C}\frac{\partial}{\partial z_1} \oplus \mathbb{C}\frac{\partial}{\partial \overline{z}_1} \oplus \cdots \oplus \mathbb{C}\frac{\partial}{\partial z_n} \oplus \mathbb{C}\frac{\partial}{\partial \overline{z}_n} \\ T_q(\mathbb{C}^m)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} &= \mathbb{C}\frac{\partial}{\partial \mu_1} \oplus \mathbb{C}\frac{\partial}{\partial \nu_1} \oplus \cdots \oplus \mathbb{C}\frac{\partial}{\partial \mu_m} \oplus \mathbb{C}\frac{\partial}{\partial \nu_m} \\ &= \mathbb{C}\frac{\partial}{\partial w_1} \oplus \mathbb{C}\frac{\partial}{\partial \overline{w}_1} \oplus \cdots \oplus \mathbb{C}\frac{\partial}{\partial w_m} \oplus \mathbb{C}\frac{\partial}{\partial \overline{w}_m} \end{aligned}$$

With respect to the bases

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}, \quad \left\{\frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \nu_1}, \ldots, \frac{\partial}{\partial \mu_m}, \frac{\partial}{\partial \nu_m}\right\},$$

the derivative

$$(D(f)_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C} : T_p(\mathbb{C}^n)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \to T_q(\mathbb{C}^m)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C},$$

is given by:

(I) $$(D(f)_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C} = \begin{bmatrix} \frac{\partial \mu_1}{\partial x_1} & \frac{\partial \mu_1}{\partial y_1} & \cdots & \frac{\partial \mu_1}{\partial x_n} & \frac{\partial \mu_1}{\partial y_n} \\ \frac{\partial \nu_1}{\partial x_1} & \frac{\partial \nu_1}{\partial y_1} & \cdots & \frac{\partial \nu_1}{\partial x_n} & \frac{\partial \nu_1}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \mu_m}{\partial x_1} & \frac{\partial \mu_m}{\partial y_1} & \cdots & \frac{\partial \mu_m}{\partial x_n} & \frac{\partial \mu_m}{\partial y_n} \\ \frac{\partial \nu_m}{\partial x_1} & \frac{\partial \nu_m}{\partial y_1} & \cdots & \frac{\partial \nu_m}{\partial x_n} & \frac{\partial \nu_m}{\partial y_n} \end{bmatrix}$$

Next, with respect to the bases

$$\left\{\frac{\partial}{\partial z_1}, \ldots, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\}, \quad \left\{\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_m}, \frac{\partial}{\partial \overline{w}_1}, \ldots, \frac{\partial}{\partial \overline{w}_m}\right\},$$

the derivative is given by

(II) $$(D(f)_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C} = \begin{bmatrix} \left(\frac{\partial w_i}{\partial z_j}\right) & | & \left(\frac{\partial w_i}{\partial \overline{z}_j}\right) \\ ---& & --- \\ \left(\frac{\partial \overline{w}_i}{\partial z_j}\right) & | & \left(\frac{\partial \overline{w}_i}{\partial \overline{z}_j}\right) \end{bmatrix}$$

---

are the respective bases of $T_p(\mathbb{C}^n)$ and $T_q(\mathbb{C}^m)$, then

$$D(f)_{ij}(p) = \sum_{\ell=1}^{n} D(f)_{ij}(p) \frac{\partial}{\partial w_\ell}\Big|_q (w_i) = df(p)\Big(\frac{\partial}{\partial z_j}\Big|_p\Big)(w_i) = \frac{\partial}{\partial z_j}\Big|_p (w_i \circ f) = \frac{\partial f_i}{\partial z_j}(p).$$

In particular, if f is holomorphic, the (II) becomes:

$$\text{(III)} \qquad (D(f)_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} = \begin{bmatrix} (\frac{\partial w_i}{\partial z_j}) & | & 0 \\ --- & & --- \\ 0 & | & \overline{(\frac{\partial w_i}{\partial z_j})} \end{bmatrix}$$

Now assume that $m = n$. Note that both derivative realizations (I) and (II) agree up to conjugate, hence have the same determinant. Thus in this case f holomorphic implies

$$\det(D(f)_{\mathbb{R}}) := \det((D(f)_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}) = \left|\det\left(\frac{\partial f_i}{\partial z_j}\right)\right|^2,$$

where

$$Df = \left(\frac{\partial f_i}{\partial z_j}\right) : T_p(\mathbb{C}^n) \to T_q(\mathbb{C}^n),$$

is the <u>holomorphic</u> derivative with respect to the bases:

$$\left\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right\}, \quad \left\{\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_n}\right\},$$

and

$$D(f)_{\mathbb{R}} : \begin{bmatrix} \frac{\partial \mu_1}{\partial x_1} & \frac{\partial \mu_1}{\partial y_1} & \dots & \frac{\partial \mu_1}{\partial x_n} & \frac{\partial \mu_1}{\partial y_n} \\ \frac{\partial \nu_1}{\partial x_1} & \frac{\partial \nu_1}{\partial y_1} & \dots & \frac{\partial \nu_1}{\partial x_n} & \frac{\partial \nu_1}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \mu_n}{\partial x_1} & \frac{\partial \mu_n}{\partial y_1} & \dots & \frac{\partial \mu_n}{\partial x_n} & \frac{\partial \mu_n}{\partial y_n} \\ \frac{\partial \nu_n}{\partial x_1} & \frac{\partial \nu_n}{\partial y_1} & \dots & \frac{\partial \nu_n}{\partial x_n} & \frac{\partial \nu_n}{\partial y_n} \end{bmatrix} : T_p(\mathbb{C}^n)_{\mathbb{R}} \to T_q(\mathbb{C}^n)_{\mathbb{R}}$$

is the corresponding <u>real</u> derivative, with respect to the bases

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}, \quad \left\{\frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \nu_1}, \dots, \frac{\partial}{\partial \mu_n}, \frac{\partial}{\partial \nu_n}\right\},$$

We now prove:

Inverse Function Theorem 3.13. *Let U, V be open sets in $\mathbb{C}^n$, $p \in U$ and $f : U \to V$ be a given holomorphic map with $(\frac{\partial f_i}{\partial z_j})(p)$ non-singular. Then f is $1-1$ in a neighbourhood of p and f^{-1} is holomorphic at $q := f(p)$.*

Proof. We have $\det\big(D(f)_{\mathbb{R}}\big) = \big|\det\big(\frac{\partial f_i}{\partial z_j}\big)\big|^2 \neq 0$, hence by the C^∞ inverse function theorem [real version], f has a C^∞ inverse f^{-1} near q. Thus $f^{-1}(f(z)) = z$ for z in a neighbourhood of p. Let $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$ be respective complex coordinates about U and V respectively, and write $z = (z_1, \dots, z_n) = f^{-1} = (\tilde{f}_1^{-1}, \dots, \tilde{f}_n^{-1})$ for the (complex) coordinates of f^{-1}. (Likewise $w = (w_1, \dots, w_n) = (f_1, \dots f_n) = f$.) Then by the chain rule and $\forall\ i$ & j:

$$0 = \frac{\partial z_i}{\partial \overline{z}_j} = \sum_k \frac{\partial \tilde{f}_i^{-1}}{\partial w_k} \underbrace{\frac{\partial f_k}{\partial \overline{z}_j}}_{=0} + \sum_k \frac{\partial \tilde{f}_i^{-1}}{\partial \overline{w}_k} \frac{\partial \overline{f}_k}{\partial \overline{z}_j}$$

Thus

$$\left(\frac{\partial \tilde{f}_i^{-1}}{\partial \overline{w}_k}\right) \overline{\left(\frac{\partial f_k}{\partial z_j}\right)} = 0, \quad \text{hence } \left(\frac{\partial \tilde{f}_i^{-1}}{\partial \overline{w}_k}\right) = 0.$$

Thus f^{-1} is holomorphic in a neighbourhood of q. $\square$

Implicit Function Theorem 3.14. *Assume given $f_1, \dots, f_k \in \mathcal{O}_n$ with $f_j(0) = 0$ $\forall\ j$, and such that*

$$\det\left(\frac{\partial f_i}{\partial z_j}(0)\right)_{1\le i,j\le k} \neq 0.$$

Then $\exists\ w_1, \dots, w_k \in \mathcal{O}_{n-k}$, with $w_i(0) = 0\ \forall\ i$, such that in a neigbourhood of $0 \in \mathbb{C}^n$:

$$f_1(z) = \cdots = f_k(z) = 0 \Leftrightarrow z_i = w_i(z_{k+1}, \dots, z_n),\ \ \forall\ 1 \le i \le k.$$

Proof. Again, by the C^∞ implicit function theorem, C^∞ functions $\{w_1, \dots, w_k\}$ exist. To check holomorphicity, write $z = (z_{k+1}, \dots, z_n)$. Then for $k+1 \le j \le n$, and all i, we have $f_i(w(z), z) = 0$, hence:

$$0 = \frac{\partial}{\partial \overline{z}_j}\big(f_i(w(z), z)\big) = \underbrace{\frac{\partial f_i}{\overline{z}_j}(w(z), z)}_{=0} + \sum_{q=1}^{k} \frac{\partial f_i}{\partial w_q}(w(z), z)\frac{\partial w_q}{\partial \overline{z}_j}$$

$$+ \sum_{q=1}^{k} \underbrace{\frac{\partial f_i}{\partial \overline{w}_q}(w(z), z)}_{=0}\ \frac{\partial \overline{w}_q}{\partial z_i}$$

Thus

$$0 = \sum_{q=1}^{k} \frac{\partial f_i}{\partial w_q}\frac{\partial w_q}{\partial \overline{z}_j}, \quad \text{hence } \frac{\partial w_q}{\partial \overline{z}_j} = 0\ \forall\ q\ \&\ j.$$

□

Analytic sets

Definitions 3.15. (1) *Let $U \subset \mathbb{C}^n$ be an open subset. A closed subset $V \subset U$ is an analytic variety in U if for any $p \in U$, there exists a neighbourhood $U' \ni p$ in U, such that $V \cap U'$ is cut out by the zero locus of a finite number of holomorphic functions $\{f_1, \dots, f_k\}$ on U'.*

(2) *An analytic variety V is called an analytic hypesurface if V is locally the zero locus of a single holomorphic function, i.e. $k = 1$ in (1) above, $\forall p \in V$.*

(3) *An analytic variety $V \subset U \subset \mathbb{C}^n$ is said to be irreducible on U, if V cannot be written as a union of 2 analytic varieties $V_1,\ V_2 \subset U$, where $V_j \neq V$ for $j = 1, 2$.*

(4) *An analytic variety $V \subset U \subset \mathbb{C}^n$ is said to be irreducible at $p \in V$ if $V \cap U'$ is irreducible on U' for small neighbourhoods $p \in U' \subset U$.*

Warning: Irreducible $\not\Rightarrow$ local irreducibility. For example, let $V = \{z_2^2 = z_1^3 + z_1^2 = 0\} \subset \mathbb{C}^2$, and $p = (0, 0) \in V$. Then V is an irreducible analytic variety in $\mathbb{C}^2$ and yet it is not irreducible at p. For instance in a neighbourhood of $(0, 0)$ we have two branches $z_2 = \pm z_1\sqrt{z_1 - 1}$, corresponding to the two branches of $\sqrt{z_1 - 1}$ about $z_1 = 0$.

Remarks 3.16. (1) Let $f \in \mathcal{O}_n$ be irreducible. I claim that $V := \{f = 0\}$ is irreducible at $0 \in \mathbb{C}^n$. [Proof. If $V = V_1 \cup V_2$ with $V_j \neq V$, then $\exists\ f_1,\ f_2$ such that $f_j \equiv 0$ on V_j, $j = 1, 2$, and that $f_1 \not\equiv 0$ on V_2, $f_2 \not\equiv 0$ on V_1. But $f_1 \cdot f_2 \equiv 0$ on V. Hence by the Nullstellensatz, $f | f_1 \cdot f_2$. Since f is irreducible and $\mathcal{O}_n$ is a UFD, it follows that either $f|f_1$ or $f|f_2$. In other words, either $V_1 \supset V$ ($\Rightarrow V_1 = V$), or $V_2 \supset V$ ($\Rightarrow V_2 = V$), a contradiction.]

(2) Suppose that $V := \{f = 0\}$ is an analytic hypersurface in some neighbourhood of $0 \in \mathbb{C}^n$, where $f \in \mathcal{O}_n$. Since $\mathcal{O}_n$ is a UFD, we can write $f = \prod_{j=1}^m f_j$, where f_j is irreducible in $\mathcal{O}_n$. Set $V_j = \{f_j = 0\}$. Then we have $V = V_1 \cup \cdots \cup V_m$ with V_j irreducible at 0. A a consequence, we deduce that for any analytic hypersurface V, and $p \in V$, then V can be expressed uniquely in some neighbourhood of p as a union of a finite number of irreducible analytic hypersurfaces through p.

(3) Regarding (2), the general result in this direction is the following (proof omitted): Any analytic variety X can be decomposed uniquely in the form $X = \bigcup_{j \in I} X_j$, where X_j is irreducible, and the union is locally finite, and where $X_i \not\subset X_j$ $\forall\ i \neq j$.

(4) One can also show that any ireducible analytic variety X has an open dense subset $X_{\text{smooth}} \subset X$ (a manifold), and a "singular set" $X_\Sigma \subset X$; moreover X_Σ is a proper analytic subvariety of X.

§4. Complex Manifolds

Definition 4.0. *A complex manifold X of dimension n is a Hausdorff and second countable topological space, together with coordinate charts $\{(U_j, h_j)\}_{j\in J}$ where:*

(1) $\{U_j\}_{j\in J}$ is an open cover of X.

(2) $h_j : U_j \xrightarrow{\approx} V_j$ is a homeomorphism onto an open set $V_j \subset \mathbb{C}^n$.

(3) the transition functions $h_i \circ h_j^{-1} : V_j \to V_i$ are holomorphic (for all i and j) wherever defined.

$$\begin{tikzcd} & U_j \arrow[dl, "h_j"'] & & U_i \arrow[dr, "h_i"] & \\ V_j \arrow[rrrr, "h_i\circ h_j^{-1}"] & & & & V_i \end{tikzcd}$$

Remarks 4.1. (1) It is customary to maximize the family $\{(h_j, U_j)_{j\in J}\}$ satisfying (1), (2), (3) in (4.0) above, and call the resulting data a complex structure on X.

(2) A variant of the above definition is a real differentiable manifold, where the transition functions $h_i \circ h_j^{-1}$ are required to be C^∞ (instead of holomorphic), and where $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$.

(3) Every complex manifold of dimension n has an underlying structure of a real differentiable manifold of real dimension $2n$.

(4) The conditions X Hausdorff and second countable imply that X is paracompact, i.e. every open cover of X admits a locally finite refinement. This implies the existence of partitions of unity subordinate to a given open cover of X.

(5) Let $X,\ Y$ be complex manifolds with respective coordinate charts

$$\{(h_j, U_j)_{j\in J}\} \quad \text{and} \quad \{(k_i, W_i)_{i\in I}\}.$$

A continuous map $F : X \to Y$ is said to be holomorphic if $k_i \circ F \circ h_j^{-1}$ is holomorphic (wherever defined) for all i and j.

Examples of manifolds

(**I**) $X =$ open subset of $\mathbb{C}^n$ and $(h_j, U_j) = (\text{Identity}, X)$.

(**II**) $X,\ Y$ manifolds of dimensions n and m respectively and with respective coordinate charts $\{(h_j, U_j)_{j\in J}\}$, $\{(k_i, W_i)_{i\in I}\}$. Then $X\times Y$ with the product topology, is a manifold of dimension $n+m$, with coordinate charts $\{(h_j\times k_i, U_j\times W_i)_{(j,i)\in J\times I}\}$.

(**III**) *Real and Complex tori.* Let $\{v_1, \dots, v_n\}$ be independent vectors in $\mathbb{R}^n$. The abelian group $L \simeq \mathbb{Z}^n$ generated by $\{v_1, \dots, v_n\}$ is called a lattice. $T = \mathbb{R}^n/L$ is called a real torus, with quotient topology via the quotient map $\pi : \mathbb{R}^n \to T$. It is clear that π is open since $\pi^{-1}(\pi(V)) = \cup_{\omega \in L} V + \omega$. If $V \subset \mathbb{R}^n$ is an open set satisfying $^\sharp\{V + p \ \cap \ L\} \leq 1$ for all $p \in \mathbb{R}^n$, then $(\pi|_V)^{-1} : \pi(V) \xrightarrow{\approx} V$ defines local coordinates on T. The corresponding transition functions are given by translations by elements in L. There is a diffeomorphism $T \simeq (S^1)^n$ where the unit circle is identified with $\mathbb{R}/\mathbb{Z}$. If $L \simeq \mathbb{Z}^{2n}$ is a lattice in $\mathbb{C}^n$, then the resulting $T = \mathbb{C}^n/L$ is called a complex torus. The complex tori are comprised of all the compact, connected, complex analytic Lie groups (using some basic properties of the exponential map on Lie groups together with the maximum-modulus principle [to show Exp is a homomorphism]).

(**IV**) *Compact Riemann surfaces ['curves'].* A compact Riemann surface X is a 1-dimensional compact complex manifold. The underlying differentiable structure is a real *oriented* compact manifold of dimension 2. Conversely, given any real 2-dimensional oriented manifold X with Riemannian metric ds^2, one can construct (locally) an oriented *isothermal* coordinate system (x, y) so that the metric takes the form $ds^2 = \mu^2(dx^2 + dy^2)$. If (u, v) is another oriented isothermal coordinate system, then it easily follows that $w = u + \sqrt{-1}v$ depends holomorphically on $z = x + \sqrt{-1}y$, i.e. X has a complex structure. The differentiable classification of compact real orientable 2-dimensional manifolds X is trivial, i.e. there is only one differentiable structure on X up to diffeomorphism. Topologically, X is classified by the genus g, which is the numbers of handles attached to the 2-sphere S^2. On the other hand for $g > 0, X$ will have a family of distinct complex structures inducing the same differentiable structure (cf. the case $g = 1$ below).

<u>Case</u> $g = 0$: This is the Riemann sphere S^2 and by stereographic projection,

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can be viewed as the extended plane $\mathbb{R}^2 \cup \infty$. In this case there is only one complex structure on S^2 which can be constructed by viewing $S^2 = \mathbb{C} \cup \infty$, with $z =$ coordinate of $\mathbb{C}$ and $w = 1/z =$ coordinate about ∞. Equivalently $S^2 = \mathbb{P}^1 =$ complex projective 1-space (cf. $\mathbb{P}^n$ below). $\mathbb{P}^1$ is also called a rational curve. The result that there is only one complex structure on S^2 can be deduced from the

Riemann-Roch theorem, a theorem which can be viewed as the solution of the classical Mittag-Leffler problem for compact Riemann surfaces!

<u>Case</u> $g = 1$: A sphere with one handle is given by the torus below:

torus $S^1 \times S^1$
(elliptic curve)

We can describe all such elliptic curves as the Riemann surface associated to $w = \sqrt{(z - z_1)(z - z_2)(z - z_3)}$, where $\{z_1, z_2, z_3\}$ are distinct, and a family of complex structures on $S^1 \times S^1$ is obtained by varying the z_j's. The domain for each of the 2 branches of w is given below:

to ∞

$\nearrow$

z_2 z_3 $\leftarrow$ 2 copies, 1 for each of the 2 branches of $\sqrt{\prod_1^3 (z - z_j)}$

z_1

$\mathbb{C} \cup \infty$ − slits

By analytic continuation, the domains for the 2 branches can be glued together along the respective slits to form a torus. Thus:

z_3 ∞

z_1 z_2

S^2 − slits

We remark in passing that if $\{z_1, z_2, z_3\}$ are not distinct, the resulting analytic space will have singularities. For example in:

E.g. $z_1 = z_2 \neq z_3$

(rational elliptic curve)

the corresponding analytic space (with nodal singularity) can be viewed topologically as a sphere with 2 distinct points glued together, therefore (by desingularization) the genus of this space is zero, hence the name *rational* elliptic curve.

Note that by varying the z_j's, it is possible to construct a family of elliptic curves degenerating to a rational elliptic curve with nodal singularity. We also remark that the Riemann surface associated to $w = \sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$, for $\{z_1, z_2, z_3, z_4\}$ distinct, is also a torus, however the compactified solution set to $w^2 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$ must be desingularized at ∞ in order to arrive at the torus.

For convenience of notation, we will express $w^2 = \prod(z-z_j)$ in normal Weierstrass form, namely $w^2 = h(z)$, where $h(z) = z^3 + bz + c$ has distinct roots, and set $f(z,w) = w^2 - h(z)$. If we let p_0 be the point of ∞ of the Riemann surface X associated to $w = \sqrt{h(z)}$, then it turns out that p_0 is a point of inflexion. There is a globally defined 1-form $dz/w = dz/\sqrt{h(z)}$ on X obtained via a residue calculation[16], which is holomorphic everywhere, even at ∞ $(= p_0)$. There is also the classical elliptic integral: $p \mapsto \int_{p_0}^{p} dz/w$, which by Stoke's theorem is well defined modulo *periods* $\{\int_\beta dz/w \mid \beta \in H_1(X,\mathbb{Z})\} \simeq \mathbb{Z}^2$, as can be easily seen in the diagram below:

$p_0 \;\cdot \qquad\qquad\qquad \cdot\; p$

From this, we arrive at a mapping $\Phi : X \to T$, where $T \simeq \mathbb{C}/\mathbb{Z}^2$ is a complex torus, called the *Abel-Jacobi* map. It is well known using Riemann-Roch that Φ is biholomorphic; moreover there is a theorem of Abel, which translates into:[17]

Theorem 4.2. *Let $\{P, Q, R\}$ be points on X. Then P, Q, R are collinear $\Leftrightarrow$ $\Phi(P) + \Phi(Q) + \Phi(R) = 0$ in the group law on T [$\Rightarrow$ 9 points of inflexion on X].*

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[16] First, we identify X with $f = 0$. Taking differentials, we have $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial w}dw$. Hence $dz \wedge df = \frac{\partial f}{\partial w} dz \wedge dw$ and $dw \wedge df = -\frac{\partial f}{\partial z} dz \wedge dw$. Thus

$$\frac{dz \wedge dw}{f} = \frac{dz}{\partial f/\partial w} \wedge \frac{df}{f} = -\frac{dw}{\partial f/\partial w} \wedge \frac{df}{f}.$$

The residue of the meromorphic 2-form along $X = \{f = 0\}$ is given by

$$\left.\frac{dz}{\partial f/\partial w}\right|_X = -\left.\frac{dw}{\partial f/\partial w}\right|_X.$$

For example

$$2\left.\frac{dz}{\partial f/\partial w}\right|_X = \frac{dz}{w}.$$

By analogy with the ordinary residue, if $f(z)$ is analytic at $p \in \mathbb{C}$, then $\mathrm{Res}_{z-p=0}\left(f(z)\frac{dz}{z-p}\right) = \mathrm{Res}_{z-p=0}\left(f(z)\frac{d(z-p)}{z-p}\right) = f(z)\Big|_{z-p=0} = f(p)$.

[17] The proof of this theorem goes as follows. Firstly, proving $\Rightarrow$ implies the converse. Thus we will assume that $\{P, Q, R\} \subset X$ are collinear. One constructs a "pencil" of complex lines $\{\mathbb{P}^1_t\}_{t\in\mathbb{P}^1}$ such that if we write $\mathbb{P}_t \cap X = p_1(t) + p_2(t) + p_3(t)$, then $\mathbb{P}_0 \cap X = P + Q + R$ and $\mathbb{P}_\infty \cap X = 3p_0$, where it is observed that $p_0 \in X$ is a point of inflexion. If we compose with Φ and add in T, viz., $t \in \mathbb{P}^1 \mapsto \Phi(p_1(t)) + \Phi(p_2(t)) + \Phi(p_3(t))$, we end up with a holomorphic map $\mathbb{P}^1 \to T$. But $\mathbb{P}^1$ and $\mathbb{C}$ are simply-connected, and there is the homotopy lifting property that

E.g. $X:\ w^2 = z^3 - z = z(z+1)(z-1)$

∞ $\mathbb{R}^2 \cap X$

R

Q

P

-1 0 1 ∞

$P+Q$

∞

There is a converse result, namely if $L \simeq \mathbb{Z}^2$ is a lattice in $\mathbb{C}$ with corresponding torus $T = \mathbb{C}/L$, then T is biholomorphic to a Riemann surface associated to $w = \sqrt{h(z)}$, $h(z)$ given above, i.e. T is biholomorphic to a [compactified] non-singular degree 3 plane curve. To see this, we consider the Weierstrass $\wp$ function $\wp(z) = z^{-2} + \sum_{\mu \in L-0}((z-\mu)^{-2} - \mu^{-2})$, a doubly periodic non-constant meromorphic function on $\mathbb{C}$, hence a non-constant meromorphic function on T. It is well known that $\wp(z)$ satisfies the non-linear differential equation $[\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\wp(z) - g_3$, where $g_2 = 60G_4$, $g_3 = 140G_6$ and $G_n = \sum_{\mu \in L-0} \mu^{-n}$. Consider the dictionary $\wp'(z) \leftrightarrow w,\ \wp(z) \leftrightarrow z,\ -g_2 \leftrightarrow b,\ -g_3 \leftrightarrow c$, and the Riemann surface X associated to $w = \sqrt{h(z)}$, $h(z) = 4z^3 + bz + c$. One checks that the map $k : \mathbb{C} \to X$ given by $(z,w) = k(t) = \begin{cases} (\wp(t), \wp'(t)) & \text{if } t \notin L \\ \infty & \text{if } t \in L \end{cases}$ induces a biholomorphism $T \xrightarrow{\sim} X$.

In summary, and to be more precise (cf. definition of $\mathbb{P}^n$ below), there is a correspondence:

$$\left\{ \begin{matrix} \text{smooth degree 3 plane} \\ \text{curves in } \mathbb{P}^2 \end{matrix} \right\} \underset{\text{Weierstrass } \wp \text{ function}}{\overset{\text{Abel-Jacobi map } \Phi}{\rightleftarrows}} \left\{ \begin{matrix} \text{complex tori} \\ T \simeq \mathbb{C}/\mathbb{Z}^2 \end{matrix} \right\}$$

which is a bijection on analytic isomorphism classes. We also conclude that the non-constant meromorphic function $\wp(z)$ provides us with an explicit way of *algebraizing* the complex torus T.

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implies that the map $\mathbb{P}^1 \to T$ factors into:

$$\begin{array}{ccc} & & \mathbb{C} \\ & \nearrow & \downarrow \\ \mathbb{P}^1 & \to & T, \end{array}$$

where $\mathbb{P}^1 \to \mathbb{C}$ is *a priori* continuous, albeit analytic, observing that local coordinates on T come from $\mathbb{C}$. By the maximum modulus principle, this map is constant. Thus $\Phi(P+Q+R) = 3\Phi(p_0) = 0$.

Now let $L_1,\ L_2$ be lattices in $\mathbb{C}$ and define $T_j = \mathbb{C}/L_j$. If $f : T_1 \to T_2$ is an analytic isomorphism with $f(0) = 0$, then since $\mathbb{C}$ is the universal cover of the T_j's and by the description of local coordinates on T_j in terms of $\mathbb{C}$, it follows that there exists an analytic lifting F of f:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F\ \sim} & \mathbb{C} \\ \downarrow & & \downarrow \\ T_1 & \xrightarrow{f\ \sim} & T_2 \end{array}$$

Therefore $F(z + w_1) = F(z) + w_2$ for $w_1 \in L_1, z \in \mathbb{C}$, and some $w_2 \in L_2$, and hence $F'(z)$ is periodic with respect to L_1. Now $F'(z)$ induces a holomorphic map $T_1 \to \mathbb{C}$, and by the maximum-modulus principle, $F'(z)$ must be constant. We may assume therefore that $F(z) = az$ for some $a \in \mathbb{C}^\times$ satisfying $a \cdot L_1 = L_2$. Since $F(z)$ is conformal, the angles between lattice vectors are preserved. Now set $L(\tau) = \mathbb{Z} \oplus \mathbb{Z}\tau$, with say $Im(\tau) > 0$ and $T(\tau) = \mathbb{C}/L(\tau)$, and note that by rotation and dilation, every one dimensional complex torus is biholomorphic to some $T(\tau)$.

τ

θ

$L(\tau)$: $\to$ 1

By varying τ, we obtain a family of complex structures on $T(\tau) = S^1 \times S^1$

Case $g = 2$: All such Riemann surfaces can be described as the Riemann surface associated to $\sqrt{(z - z_1) \cdots (z - z_k)}$ where $k = 5$ or 6 and $\{z_1, \ldots , z_k\}$ are distinct. Again a family of complex structures is obtained by varying the z_j's.

$g = 2$

Case $g \geq 3$: In the previous two cases, we could describe the Riemann surfaces as those associated to $w = \sqrt{(z - z_1) \cdots (z - z_k)}$, where $3 \leq k \leq 6$. In general, a Riemann surface is called hyperelliptic if it is of this form for some k, equivalently, can be expressed as a double cover of $\mathbb{P}^1$. This double cover will be branched at $\{z_1, \ldots , z_k\}$ for k even, and at $\{z_1 \ldots , z_k, \infty\}$ for k odd. In general there are $2g+2$ such points, and they are also the so-called Weierstrass points of the Riemann surface. In general, for $g \geq 3$, 'most' Riemann surfaces are *not* hyperelliptic.

$g \geq 3$

(V) *Complex projective space.* [Notation: S^n is the unit n-sphere in $\mathbb{R}^{n+1}$]

Proposition-Definition 4.3. *Define $\mathbb{P}^n$ in any 3 equivalent ways:*

(1) As a point set $\mathbb{P}^n = \{1$-dimensional $\mathbb{C}$-subspaces in $\mathbb{C}^{n+1}\}$.

(2) $S^{2n+1}/\sim$ where $z \sim w$ if $e^{\mathrm{i}\,t}z = w$ for some $t \in \mathbb{R}$, with the quotient topology.

(3) $\{\mathbb{C}^{n+1} - 0\}/\sim$ where $z \sim w$ if $a \cdot z = w$ for some $a \in \mathbb{C}^{\times}$ (again with the quotient topology).

Remarks 4.4. With regard to the above definition, (1) implies $\mathbb{P}^n$ is a special case of a *Grassmannian* (cf. below), (2) implies $\mathbb{P}^n$ is compact, and (3) is the most useful for doing calculations. We will think of $\mathbb{P}^n = \{\mathbb{C}^{n+1} - 0\}/\mathbb{C}^{\times}$ in the sense of (3) in (1.6) with quotient map $\pi : \mathbb{C}^{n+1} - 0 \to \mathbb{P}^n$. If $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} - 0$, we will write $[z] = [z_0, \dots, z_n]$ instead of $\pi(z)$. $[z]$ are called homogeneous coordinates on $\mathbb{P}^n$. We can cover $\mathbb{P}^n$ by coordinate charts $\{(U_j, h_j) \mid j = 0, \dots, n\}$ where $U_j = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}$ and $h_j : U_j \xrightarrow{\approx} \mathbb{C}^n$ is given by $h_j([z_0, \dots, z_n]) = (z_0/z_j, \dots, \widehat{z_j/z_j}, \dots, z_n/z_j) \in \mathbb{C}^n$, and where $\widehat{\ }$ means delete. By a simple calculation, $h_i \circ h_j^{-1}$ is holomorphic over $h_j(U_j \cap U_i)$, *a fortiori* $\mathbb{P}^n$ is a complex manifold of dimension n. E.g. $n = 1$: set $z = z_0/z_1$, $w = z_1/z_0$. Then $\mathbb{P}^1 = \{z-$plane $\mathbb{C}\} \cup \{[1, 0]\} = \mathbb{C} \cup \infty$, i.e. where $z =$ coordinate of $\mathbb{C}$ and $w =$ coordinate of ($w-$)plane $\mathbb{C}$ about ∞.

The cellular decomposition of $\mathbb{P}^n$: We can express $\mathbb{P}^n = \{[z] \in \mathbb{P}^n \mid z_n \neq 0\} \coprod \{[z] \in \mathbb{P}^n \mid z_n = 0\} = \mathbb{C}^n \coprod \mathbb{P}^{n-1}$. $\mathbb{P}^{n-1}$ is called the hyperplane at infinity. Proceeding inductively, we arrive at $\mathbb{P}^n = \mathbb{C}^n \coprod \mathbb{C}^{n-1} \coprod \cdots \coprod \mathbb{C}^1 \coprod \{\infty\}$. As a consequence of this decomposition, we can read off the generators for integral homology $\overline{\mathbb{C}^k} = \mathbb{P}^k$ (and $\partial \mathbb{P}^k = 0$). In particular:

$$H_j(\mathbb{P}^n, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } j = 2k \text{ and } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}.$$

(**VI**) *Affine varieties (and Stein manifolds).*

(A) Affine varieties. Let $\mathbb{C}[x_1, \dots, x_n]$ be a polynomial ring in n-letters,

$$f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n].$$

We set $V(f_1, \dots, f_m) = \{p \in \mathbb{C}^n \mid f_j(p) = 0$ for all $j = 1, \dots, m$, an algebraic subset of $\mathbb{C}^n$. It is easy to see that $V(f_1, \dots, f_m) = V(f_1) \cap \cdots \cap V(f_m)$.

Definition 4.5. *$X = V(f_1, \dots, f_m) \subset \mathbb{C}^n$ is called an affine variety.*

Tangent space $T_p(X)$. Let $X = V(f_1, \dots, f_m)$ be a variety in $\mathbb{C}^n$ and $p \in X$. Any $v \in T_p(X) \subset T_p(\mathbb{C}^n) \simeq \mathbb{C}^n$ should have the property that $v \cdot \nabla f(p) = 0$ for all $f \in (f_1, \dots, f_m)$, where $(f_1, \dots, f_m)$ is the ideal generated by $\{f_1, \dots, f_m\}$.

Proposition-definition 4.6. *The tangent space $T_p(X)$ is given by:*

$$\{v+p \mid v\cdot\nabla f_j(p) = 0,\ j = 1, \dots, m\} = \{v+p \mid v\cdot\nabla f(p) = 0,\ \forall\, f \in (f_1, \dots, f_m)\}.$$

$X = V(y^2 - x^3 - x^2)$
(real zeros)

$X = V(y^2 - x^3)$
(real zeros)

$p = (0,0)$ (Node), $T_p(X) \simeq \mathbb{C}^2$ $\qquad$ $p = (0,0)$ (Cusp), $T_p(X) \simeq \mathbb{C}^2$

Theorem 4.7. *Let $X = V(f_1, \ldots, f_m) \subset \mathbb{C}^n$ be an affine variety. Suppose that for any $p \in X$, there exists $\{f_{i_1}, \ldots, f_{i_r}\} \subset \{f_1, \ldots, f_m\}$ such that the matrix $\big(\partial f_{i_\ell}(p)/\partial x_j(p)\big)$ has rank r and that about $p \in \mathbb{C}^n$, X is locally described by $V(f_{i_1}, \ldots, f_{i_r})$. Then X is a [closed] submanifold of $\mathbb{C}^n$ of dimension $n-r$, called a smooth affine variety.*

It therefore follows that locally about p, X is cut out by r polynomials with independent differentials, and hence by the implicit function theorem, is a submanifold of $\mathbb{C}^n$. For example, if in (4.7) above, we have

$$\det \begin{bmatrix} \frac{\partial f_{i_1}}{\partial x_1} & \cdots & \frac{\partial f_{i_1}}{\partial x_r} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_{i_r}}{\partial x_1} & \cdots & \frac{\partial f_{i_r}}{\partial x_r} \end{bmatrix} (p) \neq 0,$$

then there exists open sets $\underline{U} \subset \mathbb{C}^n$, $V_a \subset \mathbb{C}^r$, $V_b \subset \mathbb{C}^{n-r}$, where $p = (a,b)$, $\underline{U} = V_a \times V_b$, $p \in \underline{U}$, $a \in V_a$, $b \in V_b$, and a holomorphic function $g(x_{r+1}, \ldots, x_n) : V_b \to V_a$, such that $g(b) = a$ and where for $(x_1, \ldots, x_n) \in U$,

$$f_{i_\ell}(x_1, \ldots, x_n) = 0 \ \forall\ \ell = 1, \ldots, r$$

$$\Leftrightarrow f_{i_\ell}(g(x_{r+1}, \ldots, x_n), x_{r+1}, \ldots, x_n) = 0 \ \forall\ \ell = 1, \ldots, r.$$

Now set $U = \underline{U} \cap X$, $\varphi = \mathrm{Pr}_{V_b}\Big|_U : U \to V_b$, where $\mathrm{Pr}_{V_b} : \underline{U} = V_a \times V_b \to V_b$ is the projection. Then φ is a homeomorphism of U onto V_b, with inverse map $(x_{r+1}, \ldots, x_n) \mapsto (g(x_{r+1}, \ldots, x_n), x_{r+1}, \ldots, x_n)$, which also defines a holomorphic graph-map from $V_b \to \mathbb{C}^n$. It follows that one can cover X by coordinate charts made up of projections, whose composites with the graph-maps are obviously holomorphic. In other words, the transition functions are holomorphic.

We remark that X in (4.7) is an example of a *Stein* manifold.

(B) Stein manifolds. These are the complex manifolds X for which there is a holomorphic embedding $h : X \hookrightarrow \mathbb{C}^N$ such that $h(X)$ is closed in $\mathbb{C}^N$, and are characterized by the cohomological condition: $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$ and for all coherent sheaves $\mathcal{F}$ on X. [$\mathcal{F}$ coherent means for any $p \in X$ there is an open neighbourhood U of p and finite presentation $\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{F}|_U$, where $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U.] By the maximum-modulus principle, X cannot be compact.[18]

(**VII**) *Projective varieties.* We first introduce some notation. $z = (z_0, \ldots, z_n)$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and if $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}_+^{n+1}$ then $[\alpha] = \sum \alpha_i$, $z^\alpha = z_0^{\alpha_0} \cdots z_n^{\alpha_n}$. $f \in \mathbb{C}[z_0, \ldots, z_n]$ is said to be homogeneous of degree d if $f(az) = a^d f(z)$ for all $a \in \mathbb{C}$, equivalently $f = \sum_{[\alpha]=d} b_\alpha z^\alpha$ ($b_\alpha \in \mathbb{C}$).

[18] Another formulation of Stein is this: A Stein manifold X is a complex manifold with a strictly plurisubharmonic exhaustion function. I.e. $\exists$ a C^2 $\tau : X \to \mathbb{R}_+$ such that $\tau^{-1}[0, \delta]$ is compact $\forall\ \delta > 0$, and τ is strictly plurisubharmonic. This means that with regard to lo-

Definition 4.8. *Let $f_1, \dots, f_m$ be homogeneous. Then*

$$X = V(f_1, \dots, f_m) = \{[z] \in \mathbb{P}^n \mid f(z) = 0 \text{ for all } f \in \mu\}$$

is called a projective variety.

Remarks 4.9. If f is homogeneous of degree d, then on $U_j = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}$, we can consider the non-homogeneous polynomial $f_a = f/z_j^d = f(z_0/z_j, \dots, 1 = z_j/z_j, \dots, z_n/z_j)$ (called the affinization) in the variables

$$\{x_1, \dots, x_n\} = \{z_0/z_j, \dots, \widehat{z_j/z_j}, \dots, z_n/z_j\}.$$

Conversely if $g(x_1, \dots, x_n)$ is any polynomial of degree d (over U_j), then the homogenization of g is given by $g_h(z) = z_j^d g(z_0/z_j, \dots, \widehat{z_j/z_j}, \dots, z_n/z_j)$. As a consequence of these operations (f_a, g_h) it is easy to show the following. Let $X = V(f_1, \dots, f_m)$ be a projective variety. Then $X_a = X \cap U_j$ is the affine variety corresponding to $V((f_1)_a, \dots, (f_m)_a)$.

Proposition-definition 4.10. *A variety $X \subset \mathbb{P}^n$ is said to be smooth (or non-singular) if either of the two equivalent conditions hold:*

(i) $X \cap U_j \subset \mathbb{C}^n$ is smooth for $j = 0, \dots, n$.

(ii) $C(X) - \{0\}$ is smooth in $\mathbb{C}^{n+1}$, where $C(X) = V(\mu) \subset \mathbb{C}^{n+1}$ (an affine variety called the cone of X).

$\nwarrow$

X

$C(X) \subset \mathbb{C}^{n+1}$

$\bullet\ 0$

---

cal holomorphic cordinates $(z_1, \dots, z_n)$ on X, the matrix $\left(\frac{\partial^2 \tau}{\partial z_i \partial \bar{z}_j}\right)$ is positive definite. Note that if $w = (w_1 \dots, w_n) = w(z)$ is a holomorphic change of coordinates, then $\left(\frac{\partial^2 \tau}{\partial z_i \partial \bar{z}_j}\right) = {}^{\mathrm{T}}\left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial^2 \tau}{\partial z_i \partial \bar{z}_j}\right)\overline{\left(\frac{\partial w}{\partial z}\right)}$. Thus strictly plurisubharmonic is independent of local holomorphic coordinates, and hence is well-defined on a complex manifold. It is immediate from the definition that closed submanifolds of Stein manifolds are themselves Stein. On $\mathbb{C}^n$, one sets $\tau(z_1, \dots, z_n) = \sum_j |z_j|^2$. Hence $\mathbb{C}^n$ is Stein, and therefore any closed submanifold of $\mathbb{C}^n$ is likewise Stein.

Definition 4.11. *A smooth projective variety X is called a projective algebraic manifold. We define* $\dim X = \dim C(X) - 1$.

Examples of projective algebraic manifolds of dimension n:

(i) $X = V(z_0^d + \cdots + z_{n+1}^d) \subset \mathbb{P}^{n+1}$ (Fermat hypersurface of degree d).

(ii) $X = V(z_0 z_1^{d-1} + \cdots + z_n z_{n+1}^{d-1} + z_{n+1} z_0^{d-1}) \subset \mathbb{P}^{n+1}$ (hypersurface of 'Klein' type, of degree d), provided $d \neq 2$.

<u>Example.</u> *(Elliptic curve)* Let $X = V\big(z_2^2 z_0 - (z_1^3 + bz_1 z_0^2 + cz_0^3)\big) \subset \mathbb{P}^2$. In the affine coordinates $(x, y) = (z_1/z_0, z_2/z_0)$ of $\mathbb{C}^2 \approx U_0 \subset \mathbb{P}^2$, we have $X \cap U_0 = V\big(f(x, y) := y^2 - h(x)\big) \subset \mathbb{C}^2$, where $h(x) = x^3 + bx + c$. [Here

$$f(x, y) = \frac{z_2^2 z_0 - (z_1^3 + bz_1 z_0^2 + cz_0^3)}{z_0^3}.]$$

It is easy to check that x is smooth $\Leftrightarrow$ $h(x)$ has (3) distinct roots. To see this, note that $df = -h'(x)dx + 2ydy = 0$ on $U_0 \cap X \Leftrightarrow y = 0$ & $h(x) = h'(x) = 0$. Note that the line $V(z_0)$ meets X at infinity, in this case at $V\big(z_0, z_2^2 z_0 - (z_1^3 + bz_1 z_0^2 + cz_0^3)\big) = V\big(z_0, z_1^3\big) = 3[0, 0, 1]$. $p_0 := [0, 0, 1]$ is called the point at infinity. It is obviously a point of inflexion. At infinity, we introduce affine coordinates $(\mu, \nu) = (z_0/z_2, z_1/z_2)$. Thus $X \cap U_2 = V(g(\mu, \nu))$, where

$$g(\mu, \nu) = \frac{z_2^2 z_0 - (z_1^3 + bz_1 z_0^2 + cz_0^3)}{z_2^3} = \mu - (\nu^3 + b\nu\mu^2 + c\mu^3).$$

In these coordinates, p_0 corresponds to $(0, 0)$. But $dg(0, 0) = d\mu \neq 0$. Hence X is smooth at infinity. Thus X is smooth $\Leftrightarrow$ $h(x)$ has distinct roots.

We pose the following basic

(4.12) <u>Question</u> When is a compact complex manifold a projective algebraic manifold?

In order to answer this question we recall the definition of the following.

Definition 4.13. *A closed subset $V \subset \mathbb{C}^n$ is called an analytic subset if for any $p \in V$, there exists a neighbourhood U_p of p (in the classical topology) such that $V \cap U_p$ is cut out by finitely many analytic functions $g_1(x_1, \ldots, x_n) = \cdots = g_k(x_1, \ldots, x_n) = 0$ on U_p. $V \subset \mathbb{P}^n$ analytic means an analytic set locally, i.e. on each coordinate patch $U_j \subset \mathbb{P}^n$.*

Now a consequence of Remmert's proper mapping theorem and Hartog's removeable singularity theorem is the very important:

Chow's Theorem 4.14. *On $\mathbb{P}^n$, analytic $\Rightarrow$ algebraic.*

Corollary 4.15. *Let X be a compact complex manifold, and suppose there is a holomorphic embedding $h : X \hookrightarrow \mathbb{P}^N$. Then X is a projective algebraic manifold.*

Theorem 4.16. *Let $T = \mathbb{C}^n/L$ be a complex torus. Then T is abelian $\Leftrightarrow$ there is a positive definite hermitian form $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that $Im(H)|_L$ is integral. (Such an H determines a polarization $E = Im(H)|_L$ on T.)*

(iii) *Grassmannians.* We set $G = G(k,n) = \{k\text{-dimensional subspaces } \mathbb{C}^k \subset \mathbb{C}^n\}$. For example $G(1,n+1) = \mathbb{P}^n$. Alternatively, viewing $\mathbb{P}^k = \pi(\mathbb{C}^{k+1} - 0) \subset \mathbb{P}^n$, we have $G(k+1,n+1) = \{\mathbb{P}^k\text{'s} \subset \mathbb{P}^n\}$. Let $V \subset \mathbb{C}^n$ be a k-dimensional subspace with basis $\{v_1, \dots, v_k\}$, with corresponding $k \times n$ matrix Δ with rows $\{v_1, \dots, v_k\}$ and columns $\{w_1, \dots, w_n\}$. For $J = \{j_1 < \cdots < j_k\} \subset \{1, \dots, n\}$, we set $\Delta_J =$ corresponding $k \times k$ matrix with minor $|\Delta_J| = \det(\Delta_J) = w_{j_1} \wedge \cdots \wedge w_{j_k}$. Suppose $\{v_1', \dots, v_k'\}$ is another basis of V with corresponding Δ', and write $v_i' = \sum_j a_{ij} v_j$, $A = (a_{ij})$. Then $\det(A) \neq 0$, $\Delta' = A\Delta$, and in particular the property $|\Delta_J| \neq 0$ for a given J depends only on V. We can cover G by open sets of the form $\{U_J = \{V \in G \mid |\Delta_J| \neq 0\}_{J \subset \{1,\dots,n\}}\}$. If $V \in U_J$, then V has a unique representative matrix of the form $\Delta' = \Delta_J^{-1}\Delta$, i.e. where the corresponding $k \times k$ matrix determined by J is the identity. In particular $U_J \simeq \mathbb{C}^{k(n-k)}$ is described by the $k(n-k)$ remaining coordinates, and coupled with the fact that $\Delta_I^{-1}\Delta_J$ (for $I,\ J \subset \{1, \dots, n\}$) varies analytically with respect to these coordinates over $U_I \cap U_J$, it follows that the U_J's form coordinate patches on G with holomorphic transition functions, i.e. $G(k,n)$ is a $k(n-k)$ dimensional complex manifold. By restricting to a unitary basis for a given $V \in G$, it follows that the unitary group $U(n)$ acts continuously and transitively on G, and hence G is compact and connected. Some additional facts about G are:

(i) Like $\mathbb{P}^n$, G has a cell decomposition by various $\mathbb{C}^m$'s, and in particular the homology is generated by algebraic cycles.

(ii) Let $N = \binom{n}{k} - 1$. Then G is cut out by quadratic polynomials in $\mathbb{P}^N$, hence is projective algebraic.

Another way to see that G is projective algebraic is to consider the well-defined map $P : G \to \mathbb{P}^N$ given by $P(V) = [\cdots, |\Delta_J|, \cdots]$. If $V \in U_J$, then working on the coordinate patch $z_J = |\Delta_J| \neq 0$ on $\mathbb{P}^N$, it follows that the coordinates of V in $\mathbb{C}^{k(n-k)} \simeq U_J$ will appear among the coordinates of $P(V)$, hence P is an embedding. P is called the *Plücker* embedding, and by (4.15), G is projective algebraic.

(**VIII**) *Hopf manifolds.* Our desire to include this class of manifolds in our list of examples is to provide a more complete picture of complex manifolds. Consider the group action $\mathbb{Z} \times \{\mathbb{C}^n \backslash 0\} \to \mathbb{C}^n \backslash 0$ given by $(m,z) \mapsto 2^m z$. This action is free and properly discontinuous, so that the resulting quotient space $X = \{\mathbb{C}^n \backslash 0\}/\mathbb{Z}$ is a complex manifold with local coordinates obtained from $\mathbb{C}^n \backslash 0$, called a Hopf manifold. I claim X is diffeomorphic to $S^{2n-1} \times S^1$ (and hence X is compact). To see this, we observe that the composite $S^{2n-1} \times [1/2, 1] \to \mathbb{C}^n \backslash 0 \to X$ given by $(v,t) \mapsto tv$ is surjective, with $(v,1)$ and $(v,1/2)$ mapping to the same point in X, inducing $S^{2n-1} \times S^1 \xrightarrow{\simeq} X$. To see that it is a diffeomorphism, one need only check that the differential of this map is of maximal rank, and apply the inverse function theorem. For the case $n = 2$, Kodaira proved that every complex structure on $S^3 \times S^1$ corresponds to a given free and properly discontinuous action of some group G on $\mathbb{C}^2 \backslash 0$ such that $\{\mathbb{C}^2 \backslash 0\}/G \approx S^3 \times S^1$. By considering the cell decomposition of X (or by applying the Künneth formula), one can easily check that $H^2(X, \mathbb{Z}) = 0$ for $n \geq 2$. By a well known condition on projective algebraic manifolds, X is not projective algebraic for $n \geq 2$.

In conclusion to this lecture, is the following picture describing the types of manifolds we encountered thus far. Our interests will be focussed on the compact Kähler manifolds, and more particularly on the projective algebraic manifolds.

$$\{\text{complex manifolds e.g. Stein}\}$$

$$\bigcup$$

$$\{\text{compact complex manifolds e.g. Hopf}\}$$

$$\bigcup$$

$$\{\text{compact complex Kähler e.g. complex tori}\}$$

$$\bigcup$$

$$\{\text{projective algebraic}\}$$

§5. Meromorphic maps

Definition 5.0. *(i) A proper holomorphic map $f : X \to Y$ between complex manifolds is called a proper modification if there exists proper analytic subsets A of X and B of Y such that f is a biholomorphic map of $X\backslash A$ onto $Y\backslash B$.*

(ii) A meromorphic map from X to Y is given by an irreducible analytic subset $\Sigma \subset X \times Y$ such that $Pr_1|_\Sigma : \Sigma \to X$ is a proper modification.

Example 5.1. *The blow-up of a point.* Let $[z] = [z_0, \dots, z_n]$, $w = (w_0, \dots, w_n)$ be the respective homogeneous and affine coordinates of $\mathbb{P}^n$ and $\mathbb{C}^{n+1}$. Define $E = \{([z], w) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid w = \mu \cdot z \text{ for some } \mu \in \mathbb{C}\}$. We first show that E has the structure of a 'quasi-projective' algebraic variety. The equation $w = \mu \cdot z \Rightarrow w_i = \mu \cdot z_i$ for all i, in particular $z_j w_i = z_j \mu \cdot z_i = z_i \mu \cdot z_j = z_i w_j$, i.e. $E \subset E' \overset{def}{=} V(\{w_i z_j - z_i w_j \mid 0 \leq i, j \leq n\})$. To show equality, let $([z], w) \in E'$. Then $[z] \in \mathbb{P}^n \Rightarrow z_{j_0} \neq 0$ for some j_0. But $w_i z_{j_0} = z_i w_{j_0}$ for all i, hence $w_i = \mu \cdot z_i$ for all i, where $\mu = w_{j_0}/z_{j_0} \in \mathbb{C}$. To determine the geometric properties of E (e.g. irreducibility, etc.), it is best to analyze E via the projections $Pr_1 : \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{P}^n$, $Pr_2 : \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Set $\pi_j = Pr_j|_E$, and note that the π_j's are surjective:

$$\begin{array}{ccc} E & \xrightarrow{\pi_2} & \mathbb{C}^{n+1} \\ \pi_1 \downarrow & & \\ \mathbb{P}^n & & \end{array}$$

For $w \in \mathbb{C}^{n+1}$ and $w \neq 0$, $\pi_2^{-1}(w) = ([w], w)$; moreover $\pi_2^{-1}(0) = (\mathbb{P}^n, 0)$, and where $(\mathbb{P}^n, 0)$ is called an *exceptional* divisor in E. Since $\pi_2^{-1} : \mathbb{C}^{n+1}\backslash 0 \to E \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ is holomorphic, it follows that $\pi_2 : E\backslash(\mathbb{P}^n, 0) \xrightarrow{\sim} \mathbb{C}^{n+1}\backslash 0$ is a biholomorphism. Turning to the projection π_1, we have $\pi_1^{-1}([z]) = \{([z], \mu \cdot z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid \mu \in \mathbb{C}\}$. We recall the quotient map $\pi : \mathbb{C}^{n+1}\backslash 0 \to \mathbb{P}^n$. Then $\pi_2 \circ \pi_1^{-1}([z]) = \{\mu \cdot z \mid \mu \in \mathbb{C}\} = \overline{\pi^{-1}([z])}$ is the line corresponding to $[z] \in \mathbb{P}^n$, i.e. the fibers of π_1 are the lines corresponding to points in $\mathbb{P}^n$. $\pi_1 : E \to \mathbb{P}^n$ is also called the *tautological* line bundle over $\mathbb{P}^n$. By construction it follows that E is obtained from $\mathbb{C}^{n+1}$ by replacing the origin by all limiting secants, i.e. by $\mathbb{P}^n$. To be more precise, let $\gamma(t) : \{t \in \mathbb{C} \mid |t| < \epsilon\} \to \mathbb{C}^{n+1}$ be a holomorphic curve with the properties that $\gamma(0) = 0$, $\gamma(t) \neq 0$ for $0 < |t| < \epsilon$, and $\gamma'(0) = w \neq 0$. [E.g. $\gamma(t) = t \cdot w$.] Then for $t \neq 0$, $\pi_2^{-1}(\gamma(t)) = ([\gamma(t)], \gamma(t)) = ([(\gamma(t) - \gamma(0))/t], \gamma(t))$ and hence $\lim_{t\to 0} \pi_2^{-1}(\gamma(t)) = ([\gamma'(0)], 0) \in (\mathbb{P}^n, 0)$.

In $\mathbb{C}^{n+1}$ In E

$$\begin{array}{ccccc} & & & | \ | \ | \ | \ | \ | \ | \ | & \\ \bullet & | & \to & - - - - - - - - - - - & (\mathbb{P}^n, 0) \\ & & & | \ | \ | \ | \ | \ | \ | \ | & \end{array}$$

$\bullet =$ origin

Definition 5.2. *E is called the blow-up of $\mathbb{C}^{n+1}$ at the origin.*

Remarks 5.2. (1) The notation for the blow-up is $B_0(\mathbb{C}^{n+1})$ and is also called (since E is cut out by quadrics) the quadratic transform of $\mathbb{C}^{n+1}$ at 0. Using local coordinates together with $\pi_2 : B_0(\mathbb{C}^{n+1})\backslash(\mathbb{P}^n, 0) \xrightarrow{\sim} \mathbb{C}^{n+1}\backslash 0$, the same construction can be used to blow up any point p on a complex manifold X, i.e. to arrive at the manifold $B_p(X)$. If X is projective algebraic, then so is $B_p(X)$. More generally, we can also define blow-ups $B_D(X)$ where D is a holomorphically embedded submanifold of X, and of codimension ≥ 2.

(2) While it is generally easy to blow up manifolds X along submanifolds $D \subset X$, the converse question "when can an analytic subset $D \subset X$ be blown down holomorphically via $h : X \to Y$ such that $h : X\backslash D \xrightarrow{\sim} Y\backslash h(D)$ is biholomorphic?", is a very non-trivial problem!

We now return to our discussion of E and first prove:

Claim: E is smooth and irreducible. *E is irreducible:* Let E' = closure of $\pi_2^{-1}(\mathbb{C}^{n+1}\backslash 0)$ in E (in either the classical or Zariski topology, the closures are well known to be the same), and note that E' is irreducible since $\mathbb{C}^{n+1}\backslash 0$ is likewise irreducible. For $[z] \in \mathbb{P}^n$, set $\ell_{[z]} = \pi_1^{-1}([z])$ and $\ell^*_{[z]} = \ell_{[z]}\backslash(\mathbb{P}^n, 0) \cap \ell_{[z]} \simeq \{t \cdot z \in \mathbb{C}^{n+1} \mid t \in \mathbb{C}^\times\}$. Then $\ell^*_{[z]} \subset E'$ and since E' is closed, we have $\overline{\ell^*_{[z]}} = \ell_{[z]} \subset E'$, i.e. $E' = E$. Note that $\dim E = n + 1$. *E is smooth:* We already know that $E\backslash(\mathbb{P}^n, 0) \simeq \mathbb{C}^{n+1}\backslash 0$ is smooth, so it suffices to check the smoothness of E along $(\mathbb{P}^n, 0)$, i.e. along $w = 0$. On $w = 0$, we have $d(z_i w_j - w_i z_j) = z_i dw_j - z_j dw_i$. But $[z] \in \mathbb{P}^n \Rightarrow z_{j_0} \neq 0$ for some j_0, and therefore $\{z_i dw_{j_0} - z_{j_0} dw_i \mid i = 0, \ldots, n$ and $i \neq j_0\}$ is a set of n linearly independent differentials cutting out $T_{([z],0)}(E)$ in $T_{([z],0)}(\mathbb{P}^n \times \mathbb{C}^{n+1}) \simeq \mathbb{C}^{2n+1}$. Smoothness now follows from $\dim T_{([z],0)}(E) = n + 1 = \dim E$.

Another way to establish the smoothness of E is to compute the "local trivializations" of the line bundle $\pi_1 : E \to \mathbb{P}^n$ with respect to the standard affine open cover $\{U_0, \ldots, U_n\}$ of $\mathbb{P}^n$. Define $h_j : \pi_1^{-1}(U_j) \xrightarrow{\sim} U_j \times \mathbb{C} \simeq \mathbb{C}^{n+1}$ by the formula $h_j([z], w) = ([z], w_j) = (z_0/z_j, \ldots, \widehat{z_j/z_j}, \ldots, z_n/z_j, w_j)$ via $U_j \simeq \mathbb{C}^n$. It then follows that the inverse $h_j^{-1} : \mathbb{C}^{n+1} \xrightarrow{\sim} \pi_1^{-1}(U_j)$ is the holomorphic map given by the formula $h_j^{-1}(x_1, \ldots, x_n, t) = ([x'], t \cdot x')$, where $x' = (x_1, \ldots, x_j, 1, x_{j+1}, \ldots, x_n)$. Therefore $h_j : \pi_1^{-1}(U_j) \xrightarrow{\sim} \mathbb{C}^{n+1}$ is biholomorphic for all j, hence again E is smooth with coordinate charts $\{(h_j, \pi_1^{-1}(U_j)) \mid j = 0, \ldots, n\}$. We also remark that $h_j|_{\pi_1^{-1}([z])} : \pi_1^{-1}([z]) \xrightarrow{\sim} \{[z]\} \times \mathbb{C}$ is *linear* in the 2nd factor. To compute the transition functions, we note that for $[z] \in U_i \cap U_j$, $([z], w_i) = h_i([z], w) = h_i \circ h_j^{-1} \circ h_j([z], w) = h_i \circ h_j^{-1}([z], w_j) = ([z], g_{ij}([z]) w_j)$ where $g_{ij} : U_i \cap U_j \to GL(1, \mathbb{C}) = \mathbb{C}^\times$ are the transition functions. But $w_i z_j - z_i w_j = 0$ on E, i.e. on $U_i \cap U_j$, $z_i z_j \neq 0$, and therefore $w_i = (z_i/z_j) w_j$, *a fortiori* $g_{ij}([z]) = z_i/z_j$.

A final remark concerning E: Let $\tilde{f} : \mathbb{P}^n \to E$ be a holomorphic section of the tautological bundle $\pi_1 : E \to \mathbb{P}^n$. Then $\pi_2 \circ \tilde{f} : \mathbb{P}^n \to \mathbb{C}^{n+1}$ is holomorphic, and hence constant by the maximum-modulus principle. It follows that $\tilde{f}$ must be the zero section, which gets blown down to a point. In general, we say that a holomorphic line bundle L over a compact, complex manifold X is (weakly) negative if the only holomorphic section of L over X is the zero section and which gets blown down to a point. By restricting the tautological line bundle to submanifolds of $\mathbb{P}^n$, it follows that every projective algebraic manifold has a (weakly) negative line bundle; in fact the existence of a negative line bundle on X is equivalent to X being projective algebraic.

<u>Example 5.3</u>. *Cremona transformation*

The 'map' given by

$$[w_0, w_1, w_2] = \left[\frac{1}{z_0}, \frac{1}{z_1}, \frac{1}{z_2}\right] \ : \mathbb{P}^2 \to \mathbb{P}^2,$$

can be equivalently described by

$$(w_0, w_1, w_2) = \left(\frac{\lambda}{z_0}, \frac{\lambda}{z_1}, \frac{\lambda}{z_2}\right),$$

for some $\lambda \in \mathbb{C}^\times$. This leads to the relations

$$\lambda = z_0 w_0 = z_1 w_1 = z_2 w_2.$$

Taking the closure of the graph, the subvariety

$$Z := V(z_0 w_0 - z_1 w_1, z_1 w_1 - z_2 w_2) \subset \mathbb{P}^2 \times \mathbb{P}^2,$$

defines a meromorphic map $Z : \mathbb{P}^2 \to \mathbb{P}^2$, called the *Cremona transformation.* Let $\ell_i = V(z_i)$, $\underline{\ell}_j = V(w_j)$, $\Delta = \ell_0 \cup \ell_1 \cup \ell_2 = V(z_0 z_1 z_2)$, $\underline{\Delta} = \underline{\ell}_0 \cup \underline{\ell}_1 \cup \underline{\ell}_2 = V(w_0 w_1 w_2)$. Then $Z : \mathbb{P}^2 \backslash \Delta \xrightarrow{\sim} \mathbb{P}^2 \backslash \underline{\Delta}$ is an analytic isomorphism; moreover it is obvious that $Z \circ Z =$ Identity on Δ. Let $P_0 = [1, 0, 0] = \underline{P}_0$, $P_1 = [0, 1, 0] = \underline{P}_1$, $P_2 = [0, 0, 1] = \underline{P}_2$, and note that $\ell_0 = \overleftrightarrow{P_1 P_2}$, $\ell_1 = \overleftrightarrow{P_0 P_2}$, $\ell_2 = \overleftrightarrow{P_0 P_1}$, and a similar story for $\underline{\ell}_j$. It is easy to check that

$$Z[P_i] = \underline{\ell}_i, \ i = 0, 1, 2,$$

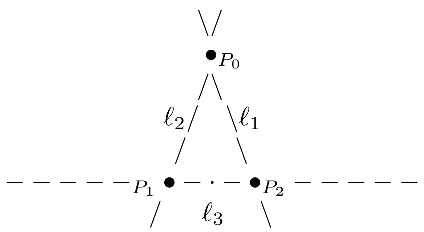
and that for i, j, k distinct:

$$Z[\ell_i \backslash \{P_j, P_k\}] = \underline{P}_i, \ i = 1, 2, 3.$$

Next, Z defines the blow-up at P_j exactly as in the earlier blow-up example, up to isomorphism. For example at P_0, and if $(a, b) \neq (0, 0)$

$$\lim_{t \to 0} Z[1, ta, tb] = [0, b, a].$$

In fact via $\mathrm{Pr}_1 : Z \to \mathbb{P}^2$, $Z \xrightarrow{\sim} B_{P_0 + P_1 + P_2}(\mathbb{P}^2)$.



Addendum: The Dolbeault invariant and Stein Manifolds

Dolbeault invariant: Let $D \subset \mathbb{C}^n$ be an open subset. For any C^∞ function $f : D \to \mathbb{C}$, recall the operator

$$\overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

The vector space of all $\overline{\partial} f$ for all such f, will be denoted by $B_{\overline{\partial}}(D)$. Now consider the vector space of differentials of the form $\eta := \sum_{j=1}^{n} g_j d\overline{z}_j$, where $g_j : D \to \mathbb{C}$ is C^∞. We put

$$Z_{\overline{\partial}}(D) = \{\eta \mid \overline{\partial}(\eta) := \sum_j \overline{\partial} g_j \wedge d\overline{z}_j = 0\}.$$

The condition $\overline{\partial}\eta = 0$ is equivalent to saying that

$$\frac{\partial g_i}{\partial \overline{z}_j} = \frac{\partial g_j}{\partial \overline{z}_i} \ \forall i \ \& \ j.$$

Exercise. Show that $\overline{\partial}^2 f = 0$ for any C^∞ function $f : D \to \mathbb{C}$.

From the exercise, it follows that $B_{\overline{\partial}}(D) \subset Z_{\overline{\partial}}(D)$.

Definition. *The Dolbeault cohomology group of D is given by*

$$H_{\overline{\partial}}(D) = \frac{Z_{\overline{\partial}}(D)}{B_{\overline{\partial}}(D)}.$$

It is well-known that $H^{\overline{\partial}}(D)$ can be identified with the cohomology of a certain coherent sheaf, namely $H^1(D, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of convergent power series on D. For Stein manifolds, D, $H^1(D, \mathcal{O}) = 0$. Thus

$$D \text{ Stein } \Rightarrow H_{\overline{\partial}}(D) = 0.$$

<u>Example 1</u>. Let $D \subset \mathbb{C}$ be an open set. It can be shown that the equation

$$\frac{\partial g}{\partial \overline{z}} = f,$$

has a C^∞ solution, for any C^∞ complex-valued f. This implies that $H_{\overline{\partial}}(D) = 0$. It turns out that D is Stein as well.

<u>Example 2</u>. Let

$$D = \left\{ z = (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, 0 < |z_1|^2 + |z_2|^2 < 1 \right\}.$$

Claim. *$H_{\overline{\partial}}(D) \neq 0$, hence D is not Stein.*

Proof. For $z_1 z_2 \neq 0$, write

$$\frac{1}{z_1 z_2} = \frac{\overline{z}_2}{z_1(|z_1|^2 + |z_2|^2)} + \frac{\overline{z}_1}{z_2(|z_1|^2 + |z_2|^2)}.$$

We define ω by the prescription:

$$\omega = \begin{cases} \overline{\partial}\big(\frac{\overline{z}_2}{z_1(|z_1|^2+|z_2|^2)}\big) & \text{if } z_1 \neq 0 \\ \\ -\overline{\partial}\big(\frac{\overline{z}_1}{z_2(|z_1|^2+|z_2|^2)}\big) & \text{if } z_2 \neq 0 \end{cases}$$

Note that ω is defined on D, since $\overline{\partial}\big(\frac{1}{z_1 z_2}\big) = 0$; moreover $\overline{\partial}^2 = 0 \Rightarrow$ is $\overline{\partial}\omega = 0$. Now suppose that $\omega = \overline{\partial} f$, where $f : D \to \mathbb{C}$ is C^∞, and set

$$g := z_1 f \;\; - \;\; \left(\frac{\overline{z}_2}{|z_1|^2 + |z_2|^2}\right)$$

Note that $\overline{\partial} z_1 = 0 \Rightarrow \overline{\partial}\big(\frac{1}{z_1} g\big) = \frac{1}{z_1}\overline{\partial} g$ if $z_1 \neq 0$. Hence, if $z_1 \neq 0$:

$$\begin{aligned} \frac{1}{z_1}\overline{\partial} g &= \overline{\partial} f - \frac{1}{z_1}\overline{\partial}\left(\frac{\overline{z}_2}{|z_1|^2 + |z_2|^2}\right) \\ &= \overline{\partial} f - \overline{\partial}\left(\frac{\overline{z}_2}{z_1(|z_1|^2 + |z_2|^2)}\right) \\ &= \omega - \omega = 0 \end{aligned}$$

Thus g is holomorphic if $z_1 \neq 0$. Therefore we can write:

$$g(z_1, z_2) = \sum_{k=-\infty}^{\infty} g_k(z_2) z_1^k,$$

where by contour integration, one can verify that $g_k(z_2)$ is holomorphic $\forall k$. But g is bounded near $z_1 = 0$, hence by Riemann extension, $g_k(z_2) = 0$ for all $k < 0$. Therefore $g(z_1, z_2)$ is holomorphic in $\{|z_1|^2 + |z_2|^2 < 1\}$, hence $g(0, z_2)$ is holomorphic at $z_2 = 0$. But $g(0, z_2) = -\frac{1}{z_2}$, which is not holomorphic at $z_2 = 0$, a contradiction. Thus $\omega \neq \overline{\partial} f$, i.e. $H_{\overline{\partial}}(D) \neq 0$.

Appendix A. Sheaves in Complex Analysis

Presheaves and sheaves. Unless otherwise specified, we will assume in this section that X is a 'nice' space, e.g. second countable and Hausdorff. Let K be a commutative ring with 1.

Definition A.0. *(1) A presheaf $\mathcal{S}$ of K-modules over X is given by the following data:*

(i) A K-module $\mathcal{S}(U)$ for every open set $U \subset X$.

(ii) whenever V, U are open with $V \subset U$, there is a K-module homomorphism $\rho_{VU} : \mathcal{S}(U) \to \mathcal{S}(V)$, called the restriction map satisfying:

(iii) whenever $V \subset U \subset W$ are inclusions of open sets, $\rho_{VW} = \rho_{VU} \circ \rho_{UW}$.

$$\begin{tikzcd} \mathcal{S}(W) \arrow[rr, "\rho_{VW}"] \arrow[dr, "\rho_{UW}"'] & & \mathcal{S}(V) \\ & \mathcal{S}(U) \arrow[ur, "\rho_{VU}"'] & \end{tikzcd}$$

and (iv) $\rho_{UU} = Id_{\mathcal{S}(U)}$ for all open $U \subset X$.

(2) A homomorphism $\tilde{f} : \mathcal{S}_1 \to \mathcal{S}_2$ of presheaves (of K-modules) is given by the following data:

(i) for every open set $U \subset X$, a K-module homomorphism $\tilde{f}_U : \mathcal{S}_1(U) \to \mathcal{S}_2(U)$ satisfying:

(ii) whenever $V \subset U$ are open, there is a commutative diagram:

$$\begin{tikzcd} \mathcal{S}_1(U) \arrow[r, "\tilde{f}_U"] \arrow[d, "\rho^1_{VU}"'] & \mathcal{S}_2(U) \arrow[d, "\rho^2_{VU}"] \\ \mathcal{S}_1(V) \arrow[r, "\tilde{f}_V"] & \mathcal{S}_2(V) \end{tikzcd}$$

Proposition A.1. *The following definitions are tautologies.*

(A) [Complete presheaves] A presheaf $\mathcal{S}$ is said to be complete if the following two conditions are satisfied for any open set $U \subset X$ and any open cover $\{U_j\}_{j \in J}$ of U:

(C1) If $f \in \mathcal{S}(U)$ and $\rho_{U_j U}(f) = 0$ for all $j \in J$, then $f = 0$.

(C2) Given $\{f_j \in \mathcal{S}(U_j)\}_{j \in J}$ satisfying $\rho_{U_i \cap U_j, U_j}(f_j) = \rho_{U_i \cap U_j, U_i}(f_i)$ for all i and j, then there exists $f \in \mathcal{S}(U)$ such that $\rho_{U_j U}(f) = f_j$ for all $j \in J$.

(B) [Sheaves] A sheaf $\underline{\mathcal{S}}$ of K-modules over X is given by a topological space $\underline{\mathcal{S}}$, a continuous map $\pi : \underline{\mathcal{S}} \to X$ such that:

(i) π is a local homeomorphism.

(ii) $\pi^{-1}(p)$ is a K-module for every $p \in X$.

(iii) the laws of composition (addition and scalar multiplication by $k \in K$) are continuous.

Remarks A.2. (1) The definition of a sheaf homomorphism is the 'obvious' one.

(2) $\underline{\mathcal{S}}_p \overset{def}{=} \pi^{-1}(p)$ is called the stalk of $\underline{\mathcal{S}}$ over p.

Proof of proposition (outline). Given $\underline{\mathcal{S}}$ in (B) and U open in X, we define the associated complete presheaf by the prescription $\underline{\mathcal{S}}(U) = K$-module of continuous sections of $\underline{\mathcal{S}}$ over U, and if $V \subset U$, then ρ_{VU} is given by restriction. Conversely, given $\mathcal{S}$ in (A) and any $p \in X$, we define $\underline{\mathcal{S}}_p = \lim\limits_{\substack{\to \\ U \ni p}} \mathcal{S}(U)$ [= so-called K-module of germs at p], and set $\underline{\mathcal{S}} = \coprod_{p \in X} \underline{\mathcal{S}}_p$ with projection map $\pi : \underline{\mathcal{S}} \to X$. We topologize $\underline{\mathcal{S}}$ as follows. For U open and $p \in U$, we denote by $\rho_{p,U} : \mathcal{S}(U) \to \underline{\mathcal{S}}_p$ the corresponding 'germ' map. A basis for the topology is given by $O_{f,U}$, where $U \subset X$ open, $f \in S(U)$ and $O_{f,U} = \{\rho_{p,U}(f) \mid p \in U\}$. It is clear that in this topology, π is a local homeomorphism, moreover one can easily verify the continuity of the laws of composition. The correspondence between presheaves and sheaves will be denoted by μ, β viz:

$$\left\{ \begin{matrix} \text{presheaves } \mathcal{S} \text{ of} \\ K - \text{modules over } X \end{matrix} \right\} \begin{matrix} \xrightarrow{\mu} \\ \xleftarrow{\beta} \end{matrix} \left\{ \begin{matrix} \text{sheaves } \underline{\mathcal{S}} \text{ of } K- \\ \text{modules over } X \end{matrix} \right\}$$

It is a simple exercise to check that $\mu \circ \beta(\underline{\mathcal{S}}) \simeq \underline{\mathcal{S}}$ and that if $\mathcal{S}$ is *complete* then $\beta \circ \mu(\mathcal{S}) \simeq \mathcal{S}$.

Examples of presheaves/sheaves. In the examples below, the maps ρ_{VU} will be defined by restriction.

(1) Let X be a complex manifold. The presheaf $\mathcal{O}_X$ given by

$U \subset X$ open $\mapsto \mathcal{O}_X(U) = \{$ holomorphic functions $f : U \to \mathbb{C}\}$ is a sheaf (i.e. a complete presheaf).

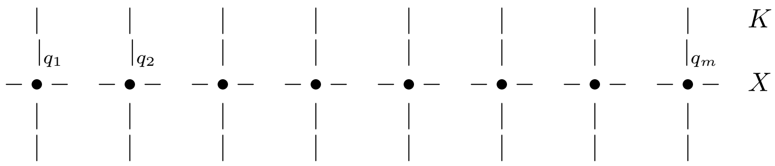
(2) Let X be a compact connected complex manifold. We define the presheaf $\mathcal{S}$ by the formula $\mathcal{S}(U) = \{f/g \mid f,\ g$ are holomorphic functions on U and $g \not\equiv 0$ on each connected component of $U\}$. Let $\mathcal{M}_X$ be the sheaf associated to this presheaf. Then $\mathcal{M}_X$ is called the sheaf of germs of meromorphic functions on X. The meromorphic function field Mer_X is by definition the global sections of $\mathcal{M}_X$. By the maximum-modulus principle, $\mathcal{S}(X) = \mathbb{C}$, and therefore if $Mer_X \neq \mathbb{C}$ (e.g. if X is projective algebraic), then $\mathcal{S}$ is not complete.

(3) Let G be any K-module. We define the constant sheaf $\mathcal{G} = X \times G$ with projection $\pi = Pr_1 : \mathcal{G} \to X$. The topology on $\mathcal{G}$ is the product topology, where G has the discrete topology. Note that $\mathcal{G}(X) = G^N$, where $N =$ number of connected components of X.

(4) Let $\Sigma = \{q_1, \dots, q_m\} \subset X$. Introduce the presheaf $\mathcal{S}$ by the prescription

$$U \subset X \text{ open } \mapsto \mathcal{S}(U) = \begin{cases} K & \text{if } U \cap \Sigma \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

[$\rho_{VU} = Id$ if $V \cap \Sigma \neq \emptyset$ and 0 otherwise.] The skyscraper sheaf is the associated sheaf $\underline{\mathcal{S}}$.



Note that $\underline{\mathcal{S}}(X) = K^m$, whereas $\mathcal{S}(X) = K$ ($\Rightarrow \mathcal{S}$ not complete for $m \geq 2$).

(5) Let $\tilde{f} : \mathcal{S}_1 \to \mathcal{S}_2$ be a homomorphism of complete presheaves. While it is easy to check that $\ker \tilde{f}$ is a complete (sub)presheaf of $\mathcal{S}_1$, it is not in general the case that the image presheaf Im $\tilde{f}$ is complete. For example if $X = \mathbb{C}^*$, $\mathcal{O}_X$ (respectively $\mathcal{O}_X^*$) = complete presheaf of holomorphic (respectively nowhere vanishing holomorphic) functions over $\mathbb{C}^*$, then the exponential function $\exp z$ defines a presheaf homomorphism $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$. If we set $\mathcal{S} =$ image presheaf $\exp(\mathcal{O}_X)$, then it is easy to see that $\beta \circ \mu(\mathcal{S}) = \mathcal{O}_X^*$. Note that z defines a global section of $\mathcal{O}_X^*$ over $\mathbb{C}^*$ and yet $z \notin \mathcal{S}(\mathbb{C}^*)$, otherwise $\exp(f(z)) = z$ for some holomorphic function $f(z)$ on $\mathbb{C}^*$, *a fortiori* $\log z$ would be holomorphic on $\mathbb{C}^*$, contradiction.

Remarks A.3. The algebraic operations on sheaves (e.g. image sheaves, quotient sheaves, tensor products, etc.) are best defined on the presheaf level via β, and then converted to the associated sheaves via μ. This relieves us of the burden of having to redefine a sheaf topology for each algebraic operation. For example, $\underline{\mathcal{S}}_1 \otimes_K \underline{\mathcal{S}}_2 =$ sheaf associated to the presheaf $U \mapsto \underline{\mathcal{S}}_1(U) \otimes_K \underline{\mathcal{S}}_2(U)$. An exact sequence of sheaves $\cdots \to \underline{\mathcal{S}}_{j-1} \to \underline{\mathcal{S}}_j \to \underline{\mathcal{S}}_{j+1} \to \cdots$ means the image of the previous morphism is the kernel of the next, and is equivalent to exactness on the stalk level.

Appendix B. The Origins of Sheaf Cohomology Theory: Mittag-Leffler Problem

Consider a s.e.s. of sheaves over a space M:

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{F} \to 0.$$

This leads to a LES:

$$0 \to \Gamma(M, \mathcal{R}) \to \Gamma(M, \mathcal{S}) \xrightarrow{\psi} \Gamma(M, \mathcal{F}) \xrightarrow{\delta} H^1(M, \mathcal{R}) \to H^1(M, \mathcal{S}) \to \cdots$$

Problems What is the image of ψ? If we know this, then we can say something about both $\Gamma(M, \mathcal{S})$ and $\Gamma(M, \mathcal{F})$. If we know $\ker \delta$, then we know $\mathrm{Im}\psi$. In particular, if $H^1(M, \mathcal{R}) = 0$, then ψ is onto.

Example- Cousin's first problem.

Let M be a complex manifold. $\{U_\alpha\}$ an open cover. $\{m_\alpha\}$ collection of meromorphic functions such that

(i) $m_\alpha \in \Gamma(U_\alpha, \mathcal{M})$

(ii) $m_\beta - m_\alpha \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O})$

Does $\exists m \in \Gamma(M, \mathcal{M})$ such that $m - m_\alpha \in \Gamma(U_\alpha, \mathcal{O}), \forall \alpha$.

Consider the s.e.s.

$$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0,$$

and the LES:

$$0 \to \Gamma(M, \mathcal{O}) \to \Gamma(M, \mathcal{M}) \xrightarrow{\psi} \Gamma(M, \mathcal{M}/\mathcal{O}) \xrightarrow{\delta} H^1(M, \mathcal{O}) \to \cdots$$

Conditions (i) and (ii) on the collection $\{m_\alpha\}$ $\Leftrightarrow$ to saying we have a section $s \in \Gamma(M, \mathcal{M}/\mathcal{O})$.

Problem. Does $\exists\ f \in \Gamma(M, \mathcal{M})$ such that $\psi(f) = s$?

Answer. YES, if $H^1(M, \mathcal{O}) = 0$. We invoke:

Theorem B. *[Cartan] Let M be a Stein manifold (= a closed analytic submanifold of $\mathbb{C}^N$). Further, let $\mathcal{R}$ be a coherent sheaf of $\mathcal{O}$-modules.*[19] *Then $H^q(M, \mathcal{R}) = 0$ $\forall q \geq 1$.*

In particular, M Stein $\Rightarrow H^1(M, \mathcal{O}) = 0$. Thus Cousin's first problem has an affirmative answer for all Stein manifolds. This includes all open (i.e. non-compact) Riemann surfaces. Hence this result includes the classical Mittag-Leffler theorem.

[19] $\mathcal{R}$ coherent means tat localy, about any point $p \in M$, there is a finite presentation, viz., an exact sequence $\mathcal{O}^p \to \mathcal{O}^q \to \mathcal{R} \to 0$. For example, $\mathcal{O}$ is coherent, and more generally, any locally free sheaf of $\mathcal{O}$-modules (e.g. sheaves of germs of holomorphic sections of vector bundles) is coherent.

Cousin's second problem.

Let M be a complex manifold, and let $\{U_\alpha\}$ be an open cover of M. Further, let $\{m_\alpha\}$ be a collection of meromorphic functions satisfying:

(i) $m_\alpha \in \Gamma(U_\alpha, \mathcal{M}^\times)$

(ii) $m_\beta/m_\alpha \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^\times)$

Recall the s.e.s.

$$0 \to \mathcal{O}^\times \to \mathcal{M}^\times \to \mathcal{D} \to 0,$$

where $\mathcal{D} := \mathcal{M}^\times/\mathcal{O}^\times$ is the sheaf of [Cartier] divisors. Thus there is a LES

$$0 \to \Gamma(M, \mathcal{O}^\times) \to \Gamma(M, \mathcal{M}^\times) \xrightarrow{\psi} \Gamma(M, \mathcal{D}) \xrightarrow{\delta} H^1(M, \mathcal{O}^\times) \to \cdots$$

Conditions (i) and (ii) on the data $\{m_\alpha\}$, is equivalent to $D \in \Gamma(M, \mathcal{D})$. $\Gamma(M, \mathcal{D})$ is called the group of Cartier divisors on M.

Problem. Does $\exists\ f \in \Gamma(M, \mathcal{M}^\times)$ such that $\psi(f) = D$, i.e. $\mathrm{div}(f) = D$, viz., D a principal divisor.

Answer. YES if $Pic(M) := H^1(M, \mathcal{O}^\times) = 0$.

It turns out that $D\psi(f)$ is purely topological condition:

We consider the following "exponential" s.e.s.:

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi\,\mathrm{i}\,z)} \mathcal{O}^\times \to 0.$$

This yields the LES:

$$\cdots \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^\times) \xrightarrow{\beta} H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}) \to \cdots$$

Now assume that M is Stein. Then $H^q(M, \mathcal{O}) = 0\ \forall q \geq 1,\ \Rightarrow \beta$ is an isomorphism. In this case, we have the picture:

$$0 \to \Gamma(M, \mathcal{O}^\times) \to \Gamma(M, \mathcal{M}^\times) \to \Gamma(M, \mathcal{D}) \xrightarrow{\delta} H^1(M, \mathcal{O}^\times) \overset{\beta}{\simeq} H^2(M, \mathbb{Z}),$$

Put $c = \beta \circ \delta$ (called the first Chern class map). Then $D = \psi(f) \Leftrightarrow c(D) = 0$. In particular, Cousin's second problem always has an affirmative answer if M is Stein and $H^2(M, \mathbb{Z}) = 0$. This is satisfied by every open Riemann surface, and therefore the result includes the classical Weierstrass theorem.