Computing the Limit Points of Quasi-components of Regular Chains in Dimension One

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2. Motivation
3. An introductory example (informal)
4. A more advanced example (informal)
5. Limit points and Puiseux expansions of an algebraic curve
6. Puiseux expansions of a regular chain and $\lim(W(T))$
7. Computation of $\lim(W(T))$
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Specification of the problem

Input

- Let $R \subset \mathbb{C}[X_1, \ldots, X_s]$ be a regular chain.
- Let $h_R$ be the product of initials of polynomials of $R$.
- Let $W(R)$ be the quasi-component of $R$, that is $V(R) \setminus V(h_R)$.

Output

The non-trivial limit points of $W(R)$, that is $\overline{W(R)^Z} \setminus W(R)$, denoted by lim($W(R)$).
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Motivation (I): the Ritt problem

The Ritt problem

Given the characteristic sets of two prime differential ideals $\mathcal{I}_1$ and $\mathcal{I}_2$, determine whether $\mathcal{I}_1 \subseteq \mathcal{I}_2$ holds or not:

- No algorithm is known,
- Equivalent to other key problems, see (O. Golubitsky et al., 2009).

The algebraic counterpart of the Ritt problem

Given regular chains $R_1$ and $R_2$, determine whether $\text{sat}(R_1) \subseteq \text{sat}(R_2)$ holds or not, without computing a basis for $\text{sat}(R_1)$ or $\text{sat}(R_2)$:

- No algorithm is known,
- Such an algorithm could be used to solve the differential problem.

Our strategy for the algebraic version

\[
\sqrt{\text{sat}(R_1)} \subseteq \sqrt{\text{sat}(R_2)} \iff W(R_2)^Z \subseteq W(R_1)^Z
\]

\[
W(R)^Z = W(R) \cup \text{lim}(W(R))
\]
Motivation (II): from Kalkbrener to Wu-Lazard decompositions

Specification (in the case of an irreducible variety)

**Input:** An irreducible algebraic set $V(F)$ and a regular chain $R$ s.t. $V(F) = W(R)^Z$

**Output:** Regular chains $R_1, \ldots, R_e$ s.t. $V(F) = W(R_1) \cup \cdots \cup W(R_e)$

Wu’s trick

- Compute a triangular decomposition of $F \cup \{h_R\}$.
- The trick generalizes to the case where $V(F)$ is not irreducible.
- In practice, this process is very inefficient (many repeated calculations).

Our proposed strategy

- Compute $V(F) \setminus W(R)$ directly as the set $\lim(W(R))$. 
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Example one

The variable order is $x < y < z$. The regular chain is:

$$\begin{align*}
  xz - y^2 &= 0 \\
  y^5 - x^2 &= 0
\end{align*}$$

What are the limits of $y$ and $z$ when $x$ approaches 0?
Example one

The variable order is $x < y < z$. The regular chain is:

\[
\begin{align*}
    xz - y^2 &= 0 \\
    y^5 - x^2 &= 0
\end{align*}
\]

What are the limits of $y$ and $z$ when $x$ approaches 0?

Figure: No limit points at $x = 0$. 
Example two

The variable order is $x < y < z$. The regular chain is:

$$\begin{cases} 
    xz - y^2 = 0 \\
    y^5 - x^3 = 0 
\end{cases}$$

What are the limits of $y$ and $z$ when $x$ approaches 0?
Example two

The variable order is $x < y < z$. The regular chain is:

$$\begin{cases} 
   xz - y^2 = 0 \\
   y^5 - x^3 = 0 
\end{cases}$$

What are the limits of $y$ and $z$ when $x$ approaches 0?

**Figure:** One limit point at $x = 0$. 
How to compute the limit points

The variable order is $x < y < z$.

$$R_1 := \begin{cases} xz - y^2 = 0 \\ y^5 - x^2 = 0 \end{cases}$$

(1) solve $y^5 - x^2 = 0$, we get $y = x^{\frac{2}{5}}$

(2) substitute $y = x^{\frac{2}{5}}$ into $xz - y^2 = 0$, we get $xz - x^{\frac{4}{5}} = 0$

(3) since $x \neq 0$, we have $z = x^{-\frac{1}{5}}$

(4) so there are no limit points

$$R_2 := \begin{cases} xz - y^2 = 0 \\ y^5 - x^3 = 0 \end{cases}$$

(1) $y = x^{\frac{3}{5}}$

(2) $z = x^{\frac{1}{5}}$

(3) the limit point is $(x = 0, y = 0, z = 0)$
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The problem

- Input: the regular chain \( R \) below with \( X_1 < X_2 < X_3 \)

\[
R := \begin{cases} 
  r_2 & = (X_1 + 2)X_1X_2^2 + (X_2 + 1)(X_3 + 1) \\
  r_1 & = X_1X_2^2 + X_2 + 1
\end{cases}
\]

The product of the initials of its polynomials is \( h_R := X_1(X_1 + 2) \).

- Output: Limit points of \( W(R) \) at \( h_R = 0 \).

Puiseux series expansions of \( r_1 \) at \( X_1 = 0 \)

- The two Puiseux expansions of \( r_1 \) at \( X_1 = 0 \) are:

\[
[X_1 = T, X_2 = -1 - T + O(T^2)],
\]

\[
[X_1 = T, X_2 = -T^{-1} + 1 + T + O(T^2)].
\]

- The second expansion cannot result in a limit point while the first one might.
Limit points of $W(R)$ at $X_1 = 0$

- After substituting the first expansion into $r_2$, we have:
  \[ r'_2 = (T + 2)TX_3^2 + (-T + O(T^2))(X_3 + 1) \]

- Now, we compute Puiseux series expansions of $r'_2$ which are
  \[
  [T = T, X_3 = 1 - 1/3T + O(T^2)], \\
  [T = T, X_3 = -1/2 + 1/12 T + O(T^2)].
  \]

- So the regular chains
  \[
  \begin{align*}
  &\begin{cases}
    X_3 - 1 = 0 \\
    X_2 + 1 = 0 \\
    X_1 = 0
  \end{cases}, \\
  &\begin{cases}
    X_3 + 1/2 = 0 \\
    X_2 + 1 = 0 \\
    X_1 = 0
  \end{cases}
  \]

  give the limit points of $W(R)$ at $X_1 = 0$. 
Limit points of $W(R)$ at $X_1 = -2$

- Puiseux series expansions of $r_1$ at the point $X_1 = -2$:

  \[
  \begin{align*}
  X_1 &= T - 2, 
  X_2 &= 1 + \frac{1}{3}T + O(T^2), \\
  X_1 &= T - 2, 
  X_2 &= -\frac{1}{2} - \frac{1}{12}T + O(T^2).
  \end{align*}
  \]

- After substitution into $r_2$, we obtain:

  \[
  \begin{align*}
  r'_{12} &= (T - 2)TX_3^2 + \left(2 + \frac{1}{3}T + O(T^2)\right)(X_3 + 1) \\
  r'_{22} &= (T - 2)TX_3^2 + \left(\frac{1}{2} - \frac{1}{12}T + O(T^2)\right)(X_3 + 1).
  \end{align*}
  \]

- Puiseux expansions of $r'_{12}$ and $r'_{22}$ at $T = 0$ resulting in limit points:
  
  i) for $r'_{12}$: $[T = T, X_3 = -1 + T + O(T^2)]$
  
  ii) for $r'_{22}$: $[T = T, X_3 = -1 + 4T + O(T^2)]$

- The limit points of $W(R)$ at $X_1 = -2$ are represented by the regular chains \{ $X_1 + 2, X_2 - 1, X_3 + 1$ \} and \{ $X_1 + 2, X_2 + 1/2, X_3 + 1$ \}. 
Visualizing the limit points of $W(R)$

The limit points are:

\[
\begin{align*}
&\begin{cases}
X_3 - 1 = 0 \\
X_2 + 1 = 0 \\
X_1 = 0
\end{cases}, \quad 
\begin{cases}
X_3 + 1/2 = 0 \\
X_2 + 1 = 0 \\
X_1 = 0
\end{cases}, \quad 
\begin{cases}
X_3 + 1 = 0 \\
X_2 - 1 = 0 \\
X_1 + 2 = 0
\end{cases}, \quad 
\begin{cases}
X_3 + 1 = 0 \\
X_2 + 1/2 = 0 \\
X_1 + 2 = 0
\end{cases},
\end{align*}
\]
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Zariski topology

Zariski closure

- Let $k$ be an algebraically closed field, like $\mathbb{C}$.
- We denote by $\mathbb{A}^s$ the **affine $s$-space** over $k$.
- An **affine algebraic variety** of $\mathbb{A}^s$ is the set of common zeroes of a collection $F \subseteq k[X_1, \ldots, X_s]$ of polynomials.
- The **Zariski topology** on $\mathbb{A}^s$ is the topology whose **closed sets** are the affine algebraic varieties of $\mathbb{A}^s$.
- The **Zariski closure** of a subset $W \subseteq \mathbb{A}^s$ is the intersection of all affine algebraic varieties containing $W$.

The set $\{y = 0, x \neq 0\}$ and its Zariski closure $\{y = 0\}$. 
Zariski topology and the Euclidean topology

The relation between the two topologies

- With $k = \mathbb{C}$, the affine space $\mathbb{A}^s$ is endowed with both topologies.
- The basic open sets of the Euclidean topology are the open balls.
- The basic open sets of Zariski topology are the complements of hypersurfaces.
- Thus, a Zariski closed (resp. open) set is closed (resp. open) in the Euclidean topology on $\mathbb{A}^s$.
- That is, Zariski topology is coarser than the Euclidean topology.

Theorem (The relation between two closures (D. Mumford))

- Let $V \subseteq \mathbb{A}^s$ be an irreducible affine variety.
- Let $U \subseteq V$ be nonempty and open in the Zariski topology induced on $V$.

Then, $U$ has the same closure in both topologies. In fact, we have

$$V = \overline{U}^Z = \overline{U}^E.$$
Limit points

Let \((X, \tau)\) be a topological space and \(S \subseteq X\) be a subset.

A point \(p \in X\) is a limit point of \(S\) if every neighborhood of \(p\) contains at least one point of \(S\) different from \(p\) itself.

If \(X\) is a metric space, the point \(p\) is a limit point of \(S\) if and only if there exists a sequence \((x_n, n \in \mathbb{N})\) of points of \(S \setminus \{p\}\) such that \(\lim_{n \to \infty} x_n = p\).

The limit points of \(S\) which do not belong to \(S\) are called non-trivial, denoted by \(\lim(S)\).

Example

Consider the interval \(S := [1, 2) \subset \mathbb{R}\). The point 2 is a non-trivial limit point of \(S\).
Limit points of the quasi-component of a regular chain

Recall Mumford’s Theorem

- Let $V \subseteq \mathbb{A}^s$ be an irreducible affine variety.
- Let $U \subseteq V$ be nonempty and open in the Zariski topology induced on $V$.

Then $V = \overline{U}^Z = \overline{U}^E$.

Corollary

Let $R$ be a regular chain. Recall that $\text{sat}(R) := \langle R \rangle : \text{init}(R)\infty$ is its saturated ideal and $W(R) = V(R) \setminus V(\text{init}(R))$ is its quasi-component. Then, we have

$$V(\text{sat}(R)) = \overline{W(R)}^Z = \overline{W(R)}^E.$$ 

We use $\overline{W(R)}$ to denote this common closure.

$\lim(W(R)) := \overline{W(R)} \setminus W(R)$ denotes the limit points of $W(R)$. 
Field of Puiseux series

- Let $T$ be a symbol.
- $\mathbb{C}[[T]]$ : ring of formal power series.
- $\mathbb{C}\langle T \rangle$ : ring of convergent power series.
- $\mathbb{C}[[T^*]] = \bigcup_{n=1}^{\infty} \mathbb{C}[[T^{1/n}]]$ : ring of formal Puiseux series.
- $\mathbb{C}\langle T^* \rangle = \bigcup_{n=1}^{\infty} \mathbb{C}\langle T^{1/n} \rangle$ : ring of convergent Puiseux series.
- $\mathbb{C}((T^*))$ : quotient field of $\mathbb{C}[[T^*]]$, or the field of Puiseux series.
- $\mathbb{C}(\langle T^* \rangle)$ : quotient field of $\mathbb{C}\langle T^* \rangle$, or the field of convergent Puiseux series.

We have
- $\mathbb{C}[[T]] \subset \mathbb{C}[[T^*]] \subset \mathbb{C}((T^*))$; $\mathbb{C}\langle T \rangle \subset \mathbb{C}\langle T^* \rangle \subset \mathbb{C}(\langle T^* \rangle)$
- $\mathbb{C}\langle T \rangle \subset \mathbb{C}[[T^*]]$; $\mathbb{C}\langle T^* \rangle \subset \mathbb{C}[[T^*]]$; $\mathbb{C}(\langle T^* \rangle) \subset \mathbb{C}((T^*))$

Example

We have $\sum_{i=0}^{\infty} T^i \in \mathbb{C}(\langle T \rangle)$, $\sum_{i=0}^{\infty} T^{i/2} \in \mathbb{C}(\langle T^* \rangle)$ and $\sum_{i=-3}^{\infty} T^{i/2} \in \mathbb{C}(\langle T^* \rangle)$. 
Theorem (Puiseux)

*Both* $\mathbb{C}((T^*))$ and $\mathbb{C}(\langle T^* \rangle)$ *are algebraically closed fields.*

Puiseux expansions

- Let $k = \mathbb{C}((X^*))$ or $\mathbb{C}(\langle X^* \rangle)$.
- Let $f \in k[Y]$, where $d := \deg(f, Y) > 0$.
- There exist $\varphi_i \in k$, $i = 1, \ldots, d$, such that

$$
\frac{f}{\text{lc}(f, Y)} = (Y - \varphi_1) \cdots (Y - \varphi_d).
$$

- We call $\varphi_1, \ldots, \varphi_d$ the *Puiseux expansions* of $f$ at the origin.

Example

- $(Y^2 - X) = (Y - X^{\frac{1}{2}})(Y + X^{\frac{1}{2}})$.
- Puiseux expansions of $Y^2 - XY - X$:

$$
Y - (X^{\frac{1}{2}} + \frac{1}{2} X + \frac{1}{8} X^{\frac{3}{2}} + O(X^2)),
Y - (-X^{\frac{1}{2}} + \frac{1}{2} X - \frac{1}{8} X^{\frac{3}{2}} + O(X^2)).
$$
Puiseux parametrizations

Let $f \in \mathbb{C}\langle X \rangle[Y]$. A Puiseux parametrization of $f$ is a pair $(\psi(T), \varphi(T))$ of elements of $\mathbb{C}\langle T \rangle$ for some new variable $T$, such that

- $\psi(T) = T^\varsigma$, for some $\varsigma \in \mathbb{N}_{>0}$.
- $f(X = \psi(T), Y = \varphi(T)) = 0$ holds in $\mathbb{C}\langle T \rangle$,
- there is no integer $k > 1$ such that both $\psi(T)$ and $\varphi(T)$ are in $\mathbb{C}\langle T^k \rangle$.

The index $\varsigma$ is the ramification index of the parametrization $(T^\varsigma, \varphi(T))$.

Relation to Puiseux expansions

- Let $z_1, \ldots, z_\varsigma$ denote the primitive roots of unity of order $\varsigma$ in $\mathbb{C}$. Then $\varphi(z_i X^{1/\varsigma})$, for $i = 1, \ldots, \varsigma$, are $\varsigma$ Puiseux expansions of $f$.
- For a Puiseux expansion $\varphi$ of $f$, let $c$ minimum s.t. $\varphi = g(T^{1/c})$ and $g \in \mathbb{C}\langle T \rangle$. Then $(T^c, g(T))$ is a Puiseux parametrization of $f$.

Example

Puiseux parametrization of $Y^2 - XY - X$: 

$$(X = T^2, Y = T + \frac{1}{2} T^2 + \frac{1}{8} T^3 + O(T^4))$$
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Puiseux expansions of a regular chain

Notation

- Let \( R := \{ r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s) \} \subset \mathbb{C}[X_1 < \cdots < X_s] \) be a 1-dim regular chain.
- Assume \( R \) is strongly normalized, that is, \( \text{init}(R) \in \mathbb{C}[X_1] \).
- Let \( k = \mathbb{C}(\langle X_1^* \rangle) \).
- Then \( R \) generates a zero-dimensional ideal in \( k[X_2, \ldots, X_s] \).
- Let \( V^*(R) \) be the zero set of \( R \) in \( k^{s-1} \).

Definition

We call *Puiseux expansions* of \( R \) the elements of \( V^*(R) \).

Remarks

- The *strongly normalized assumption* is only for presentation ease.
- Generically, The 1-dim assumption extends to \( d\text{-dim } d \leq 2 \).
- Higher dimension requires the Jung-Abhyankar theorem.
An example

A regular chain $R$

$$R := \begin{cases} X_1 X_3^2 + X_2 \\ X_1 X_2^2 + X_2 + X_1 \end{cases}$$

Puiseux expansions of $R$

$$\begin{align*}
\begin{cases}
X_3 &= 1 + O(X_1^2) \\
X_2 &= -X_1 + O(X_1^2)
\end{cases} & \quad \begin{cases}
X_3 &= -1 + O(X_1^2) \\
X_2 &= -X_1 + O(X_1^2)
\end{cases} \\
\begin{cases}
X_3 &= X_1^{-1} - \frac{1}{2} X_1 + O(X_1^2) \\
X_2 &= -X_1^{-1} + X_1 + O(X_1^2)
\end{cases} & \quad \begin{cases}
X_3 &= -X_1^{-1} + \frac{1}{2} X_1 + O(X_1^2) \\
X_2 &= -X_1^{-1} + X_1 + O(X_1^2)
\end{cases}
\end{align*}$$
Relation between $\lim_0(W(R))$ and Puiseux expansions of $R$

**Theorem**

For $W \subseteq \mathbb{C}^s$, denote

$$\lim_0(W) := \{ x = (x_1, \ldots, x_s) \in \mathbb{C}^s | x \in \lim(W) \text{ and } x_1 = 0 \},$$

and define

$$V^*_\geq 0(R) := \{ \Phi = (\Phi^1, \ldots, \Phi^{s-1}) \in V^*(R) | \text{ord}(\Phi^j) \geq 0, j = 1, \ldots, s-1 \}.$$

Then we have

$$\lim_0(W(R)) = \bigcup_{\Phi \in V^*_\geq 0(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

$$V^*_\geq 0(R) := \begin{cases} 
X_3 = 1 + O(X_1^2) \\
X_2 = -X_1 + O(X_1^2)
\end{cases} \cup \begin{cases} 
X_3 = -1 + O(X_1^2) \\
X_2 = -X_1 + O(X_1^2)
\end{cases}$$

Thus the limit points are $\lim_0(W(R)) = \{(0, 0, 1), (0, 0, -1)\}$. 
Puiseux parametrizations of a regular chain

**Idea**

- Let $\Phi_i = (\Phi_i^1, \ldots, \Phi_i^{s-1}) \in V_{\geq 0}^*(R)$ be a Puiseux expansion, $1 \leq i \leq M := |V_{\geq 0}^*(R)|$. Recall that $\Phi_i^1, \ldots, \Phi_i^{s-1} \in \mathbb{C}(\langle X_1^* \rangle)$.
- $\Phi_i$ can be associated with a Puiseux parametrization $(X_1 = T^{\varsigma_i}, X_2 = g_i^1(T), \ldots, X_s = g_i^{s-1}(T))$ with $g_i^j \in \mathbb{C}\langle T \rangle$.

**Details**

- Note: $\Phi_i^j$ is an expansion of $r_j(X_1, X_2 = \Phi_i^1, \ldots, X_j = \Phi_i^{j-1}, X_{j+1})$.
- Let $(T^{\varsigma_{i,j}}, X_j = \varphi_i^j(T))$ be the corresponding Puiseux parametrization of $\Phi_i^j$, where $\varsigma_{i,j}$ is the ramification index of $\Phi_i^j$.
- Let $\varsigma_i$ be the l.c.m. of $\{\varsigma_{i,1}, \ldots, \varsigma_{i,s-1}\}$ and $g_i^j := \varphi_i^j(T = T^{\varsigma_i/\varsigma_{i,j}})$.

**Definition**

$\mathfrak{G}_R := \{(X_1 = T^{\varsigma_i}, X_2 = g_i^1(T), \ldots, X_s = g_i^{s-1}(T)), i = 1, \ldots, M\}$ is a system of Puiseux parametrizations of $R$. 
Relation between $\lim_0(W(R))$ and Puiseux parametrizations of $R$

**Notation (recall)**

Let $\mathcal{G}_R := \{(X_1 = T^{s_i}, X_2 = g^1_i(T), \ldots, X_s = g^{s-1}_i(T)), i = 1, \ldots, M\}$ be a system of Puiseux parametrizations of $R$.

**Theorem**

We have

$$\lim_0(W(R)) = \mathcal{G}_R(T = 0).$$
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7. **Computation of \( \lim(W(T)) \)**
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10. Conclusion
Let $f \in \mathbb{C}[X][Y]$. Assume that $f$ is primitive in $Y$. Then
\[
\lim_{0} (W(f)) = \{(0, y) \mid f(0, y) = 0\}.
\]

Let $f \in \mathbb{C}[X][Y]$. Assume that $f$ is general in $Y$, that is $f(0, Y) \neq 0$. Then,
\[
\lim_{0} (W(f)) = \{(0, y) \mid f(0, y) = 0\}.
\]

Let $f \in \mathbb{C}\langle X \rangle [Y]$.
Assume that $f$ is general in $Y$.
Let $\rho > 0$ be small enough such that $f$ converges in $|X| < \rho$.
Let $V_\rho(f) := \{(x, y) \mid 0 < |x| < \rho, f(x, y) = 0\}$.

Then, we have
\[
\lim_{0} (V_\rho(f)) = \{(0, y) \mid f(0, y) = 0\}.
\]
From algebra to computer: what is the challenge?

Algebra

Let $\mathcal{G}_R$ be a system of Puiseux parametrizations of $R$. Recall that we have

$$\lim_{0}(W(R)) = \mathcal{G}_R(T = 0).$$

When Walker's theorem applies or when the $T$ is a primitive regular chain, we do not need to compute $\mathcal{G}_R(T = 0)$. However, those are criteria only!

How to compute $\mathcal{G}_R$ when the previous criteria do not apply?

- We shall not compute $\mathcal{G}_R$.
- We need to compute $\mathcal{G}_R(T = 0)$.
- In fact, we compute a truncation (approximation) of $\mathcal{G}_R$. 
The back-substitution process for computing $\mathcal{G}_R$

Specifications

**Input:** $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\}$ a 1-dim strongly normalized regular chain.

**Output:** $\mathcal{G}_R$: a system of Puiseux parametrizations of $R$.

Algorithm

<table>
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<tr>
<th>Polynomial</th>
<th>Substitution</th>
<th>Puiseux parametrization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1(X_1, X_2)$</td>
<td>N/A</td>
<td>$(X_1 = T_1^{s_1}, X_2 = \varphi_1(T_1))$</td>
</tr>
<tr>
<td>$r_2(X_1, X_2, X_3)$</td>
<td>$r_2(T_1^{s_1}, \varphi_1(T_1), X_3)$</td>
<td>$(T_1 = T_2^{s_2}, X_3 = \varphi_2(T_2))$</td>
</tr>
<tr>
<td>$r_3(X_1, X_2, X_3, X_4)$</td>
<td>$r_3(T_2^{s_1s_2}, \varphi_1(T_2^{s_2}), \varphi_2(T_2), X_4)$</td>
<td>$(T_2 = T_3^{s_3}, X_4 = \varphi_3(T_3))$</td>
</tr>
</tbody>
</table>

More generally, for $i = 2, \ldots, s - 1$, we define:

- $f_i := r_i(X_1 = T_1^{s_1}, X_2 = \varphi_1(T_1), \ldots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1}) \in \mathbb{C}\langle T_{i-1}\rangle[X_{i+1}]$,

- $(T_i := T_{i-1}^{s_i}, X_{i+1} := \varphi_i(T_i))$.

**New problem:** compute Puiseux parametrizations of $f_i$ of given accuracy.
Puiseux parametrizations of $f \in \mathbb{C}\langle X \rangle [Y]$ of finite accuracy

**Definition**

- Let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathbb{C}[[X]]$.
- For any $\tau \in \mathbb{N}$, let $f^{(\tau)} := \sum_{i=0}^{\tau} a_i X^i$.
- We call $f^{(\tau)}$ the polynomial part of $f$ of accuracy $\tau + 1$.

**Definition**

- Let $f \in \mathbb{C}\langle X \rangle [Y]$, $\deg(f, Y) > 0$.
- Let $\sigma, \tau \in \mathbb{N}_{>0}$ and $g(T) = \sum_{k=0}^{\tau-1} b_k T^k$.
- Let $\{T^{k_1}, \ldots, T^{k_m}\}$ be the support of $g(T)$.
- The pair $(T^\sigma, g(T))$ is called a Puiseux parametrization of $f$ of accuracy $\tau$ if there exists a Puiseux parametrization $(T^{\varsigma}, \varphi(T))$ of $f$ such that
  
  (i) $\sigma$ divides $\varsigma$.
  (ii) $\gcd(\sigma, k_1, \ldots, k_m) = 1$.
  (iii) $g(T^{\varsigma/\sigma})$ is the polynomial part of $\varphi(T)$ of accuracy $(\varsigma/\sigma)(\tau - 1) + 1$. 

Computing Puiseux parametrizations of $f \in \mathbb{C}\langle X \rangle[Y]$ of finite accuracy

**Theorem**

- Let $f = \sum_{i=0}^{d} \sum_{j=0}^{\infty} a_{i,j}Y^i \in \mathbb{C}\langle X \rangle[Y]$.
- Then we can compute $m \in \mathbb{N}$ such that the Puiseux parametrizations of $f$ of accuracy $\tau$ are exactly the Puiseux parametrizations of $\sum_{i=0}^{d} \sum_{j=0}^{m-1} a_{i,j}Y^i$ of accuracy $\tau$.

**Lemma**

- Let $f = a_d(X)Y^d + \cdots + a_0(X) \in \mathbb{C}\langle X \rangle[Y]$.
- Let $\delta := \text{ord}(a_d(X))$.
- Then “generically”, we can choose $m = \tau + \delta$. 
Recall the back-substitution process for computing $\mathcal{G}_R$

### Algorithm

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Substitution</th>
<th>Puiseux parametrisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1(X_1, X_2)$</td>
<td>N/A</td>
<td>$(X_1 = T_1^{s_1}, X_2 = \varphi_1(T_1))$</td>
</tr>
<tr>
<td>$r_2(X_1, X_2, X_3)$</td>
<td>$r_2(T_1^{s_1}, \varphi_1(T_1), X_3)$</td>
<td>$(T_1 = T_2^{s_2}, X_3 = \varphi_2(T_2))$</td>
</tr>
<tr>
<td>$r_3(X_1, X_2, X_3, X_4)$</td>
<td>$r_3(T_2^{s_2}, \varphi_1(T_2^{s_2}), \varphi_2(T_2), X_4)$</td>
<td>$(T_2 = T_3^{s_3}, X_4 = \varphi_3(T_3))$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

More generally, for $i = 2, \ldots, s - 1$, we define:

- $f_i := r_i(X_1 = T_1^{s_1}, X_2 = \varphi_1(T_1), \ldots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1}) \in \mathbb{C}\langle T_{i-1}\rangle[X_{i+1}]$,
- $(T_i := T_{i-1}^{s_i}, X_{i+1} := \varphi_i(T_i))$. 
Putting everything together

Let \( R := \{ r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s) \} \subset \mathbb{C}[X_1 < \cdots < X_s] \). For \( 1 \leq i \leq s - 1 \), let

- \( h_i := \text{init}(r_i) \)
- \( d_i := \deg(r_i, X_{i+1}) \)
- \( \delta_i := \text{ord}(h_i) \).

**Theorem**

**One can compute positive integer numbers** \( \tau_1, \ldots, \tau_{s-1} \) **such that, in order to compute** \( \lim_0(W(R)) \), **it suffices to compute Puiseux parametrizations of** \( f_i \) **of accuracy** \( \tau_i \), **for** \( i = 1, \ldots, s - 1 \). **Moreover, generically, we can choose** \( \tau_i, i = 1, \ldots, s - 1 \), **as follows**

- \( \tau_{s-1} := 1 \)
- \( \tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k)\delta_{s-1} + 1 \)
- \( \tau_i = (\prod_{k=1}^{s-2} \varsigma_k)(\sum_{k=2}^{s-1} \delta_i) + 1, \ i = 1, \ldots, s - 3. \)

**Moreover, the indices** \( \varsigma_k \) **can be replaced with** \( d_k, \ k = 1, \ldots, s - 2. \)
Plan

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4. A more advanced example (informal)
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7. Computation of \( \lim(W(T)) \)
8. Experimentation
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Maple packages used: RegularChains and algcurves:-puiseux.

- **T**: timings of Triangularize
- **#(T)**: number of regular chains returned by Triangularize
- **d-1, d-0**: number of one and zero dimensional components
- **R**: timings spent on removing redundant components
- **#(R)**: number of irredundant components

**Table**: Removing redundant components in Kalkbrener decompositions.

<table>
<thead>
<tr>
<th>Sys</th>
<th>T</th>
<th>#(T)</th>
<th>d-1</th>
<th>d-0</th>
<th>R</th>
<th>#(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f-744</td>
<td>14.360</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>432.567</td>
<td>1</td>
</tr>
<tr>
<td>Liu-Lorenz</td>
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<td>3</td>
<td>3</td>
<td>0</td>
<td>216.125</td>
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<tr>
<td>MontesS3</td>
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<td>2</td>
<td>2</td>
<td>0</td>
<td>0.064</td>
<td>2</td>
</tr>
<tr>
<td>Neural</td>
<td>0.296</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>1.660</td>
<td>5</td>
</tr>
<tr>
<td>Solotareff-4a</td>
<td>0.632</td>
<td>7</td>
<td>7</td>
<td>0</td>
<td>32.362</td>
<td>7</td>
</tr>
<tr>
<td>Vermeer</td>
<td>1.172</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>75.332</td>
<td>2</td>
</tr>
<tr>
<td>Wang-1991c</td>
<td>3.084</td>
<td>13</td>
<td>13</td>
<td>0</td>
<td>6.280</td>
<td>13</td>
</tr>
</tbody>
</table>
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Concluding remarks

- We proposed an algorithm for computing the limit points of the quasi-component of a regular chain in dimension one.
- To this end, we make use of the *Puiseux series expansions* of a regular chain.
- In addition, we have sharp bounds on the degree of truncations that are required to compute *approximate Puiseux series expansions* from which the desired limit points can be obtained.
- Our experimental results show that this is a useful tool for dealing with triangular decompositions of polynomial systems.
- For instance, for testing inclusion between saturated ideals of regular chains in a direct manner (i.e. without computing a basis).
- Computing limit points in higher dimension may require the help of the Abhyankar-Jung theorem. This is work in progress.